cse 101 4-27-21

continue .. asymptotic growth of fcns.

<u>Defn</u>

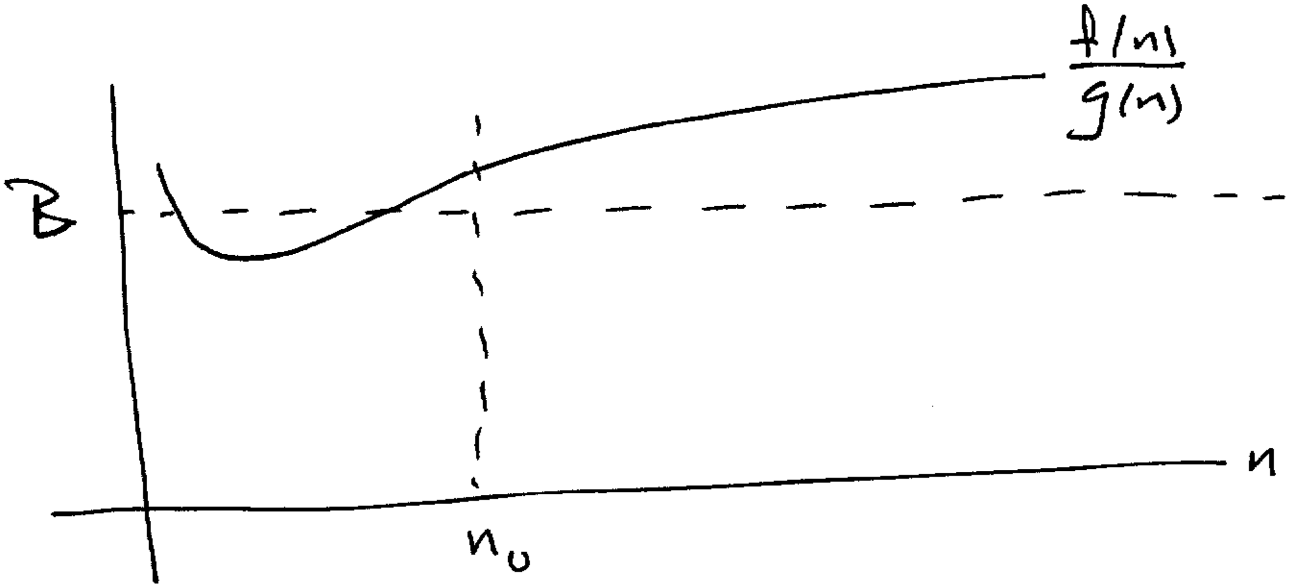
$f(n) = \Omega(g(n))$ iff there exist

$B > 0$, $n_0 \geq 0$ s.t.

$$B \leq \frac{f(n)}{g(n)}$$

for all $n \geq n_0$. we say $g(n)$ is an <u>asymptotic lower bound</u> for $f(n)$.

means: $f(n)$ grows no slower than $g(n)$.

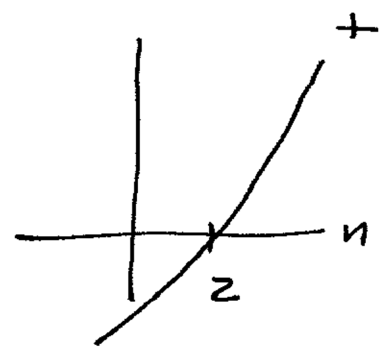


<u>Ex.</u> $f(n) = 6n^3 + 4n$, $g(n) = 2n^2$. Then

$$\frac{f(n)}{g(n)} = 3n + \frac{2}{n} \geq 7 \text{ for all } n \geq 2$$

⌈aside: why? $3n + \frac{2}{n} \geq B$

$$3n + \frac{2}{n} \geq 7$$
$$3n^2 + 2 \geq 7n$$
$$3n^2 - 7n + 2 \geq 0$$



Observe:

$$\frac{f(n)}{g(n)} \le B_1 \quad \text{iff} \quad \frac{g(n)}{f(n)} \ge B_2$$

where $B_2 = \frac{1}{B_1}$. Therefore

$$f(n) = O(g(n)) \quad \text{iff} \quad g(n) = \Omega(f(n))$$

Analogy:

$$a \le b \qquad \text{iff} \quad b \ge a$$

Apply to last 2 examples

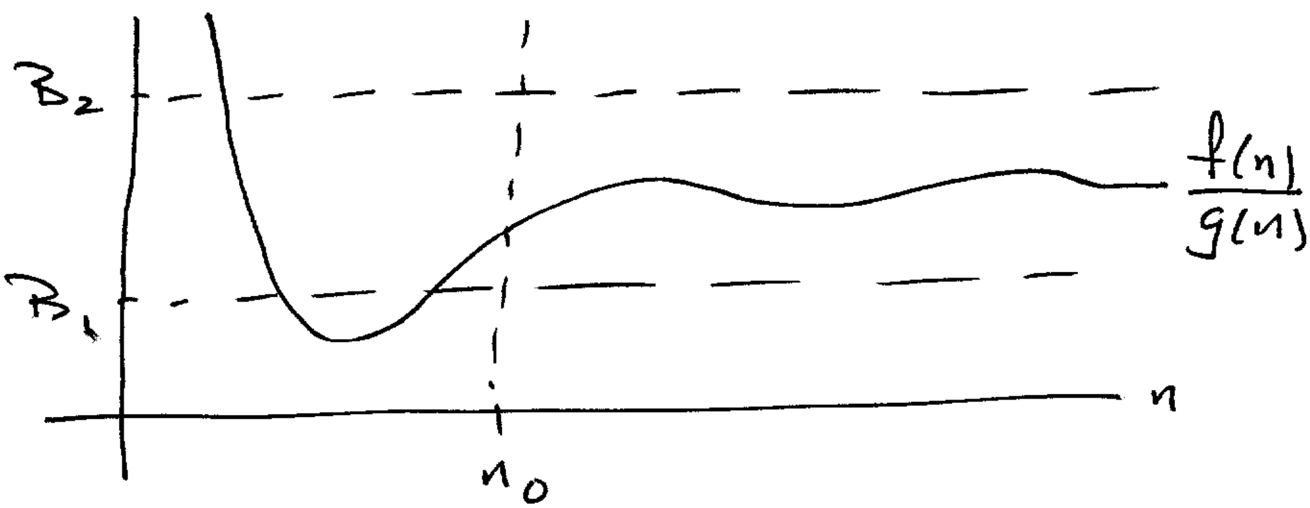
- $n = \Omega(2n+5)$
- $2n^2 = O(6n^3 + 4n)$

Defn

$f(n) = \Theta(g(n))$ iff there exist $B_1, B_2 > 0$ and $n_0 > 0$ s.t.

$$B_1 \le \frac{f(n)}{g(n)} \le B_2$$

for all $n \ge n_0$. We say $g(n)$ is a tight asymptotic bound for $f(n)$.



Equivalent: $f(n) = \Theta(g(n))$ iff both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Analogy:

$$a = b$$

iff both $a \le b$ and $a \ge b$.

so $f(n) = \Theta(g(n))$ means $f(n)$ and $g(n)$ have same asymptotic growth rate

Recall $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$

It follows that

$$f(n) = \Theta(g(n)) \text{ iff } g(n) = \Theta(f(n))$$

Analogy

$$a = b \text{ iff } b = a$$

Ex. $f(n) = 5n^2 + 11n - 24$, $g(n) = n^2$

let $B_1 = 4$, $B_2 = 6$, $n_0 = 8$. check that

$$4 \le \frac{5n^2 + 11n - 24}{n^2} \le 6$$

holds for all $n \ge 8$. so

$$5n^2 + 11n - 24 = \Theta(n^2).$$

Exercise show for any $a, b, c \in \mathbb{R}$ with $a > 0$, that

$$an^2 + bn + c = \Theta(n^2)$$

Defn

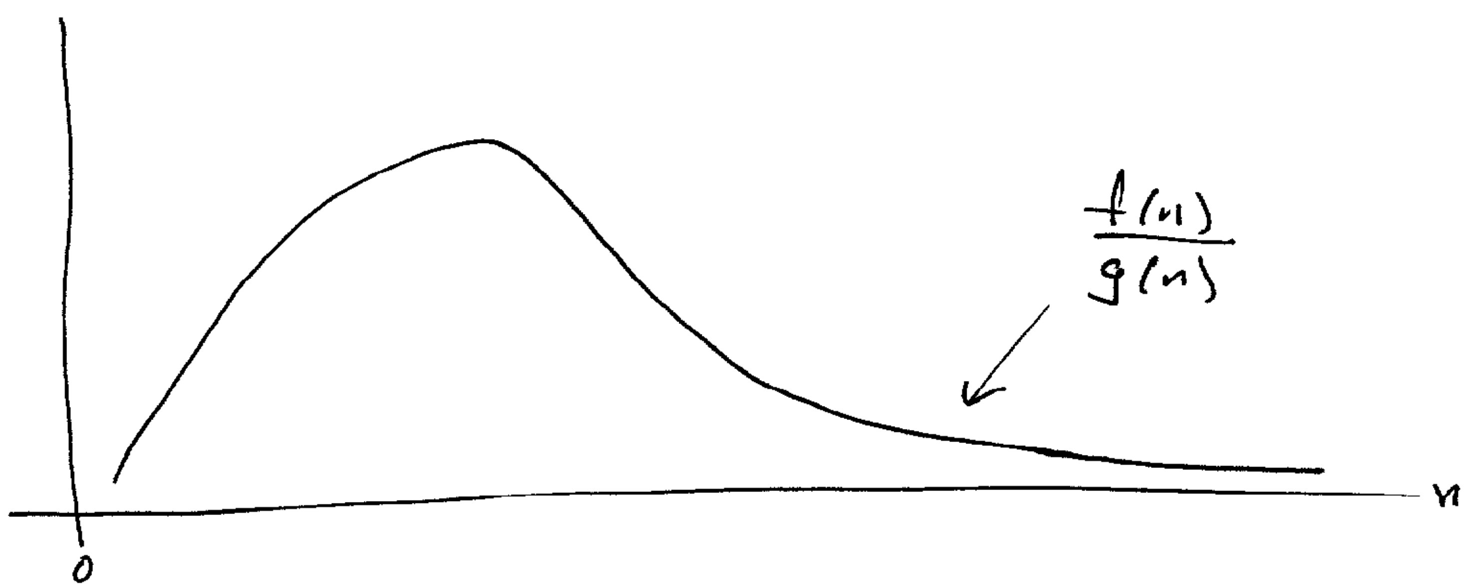
$f(n) = o(g(n))$ iff $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0$.

we say $g(n)$ is a strict asymptotic upper bound for $f(n)$.

means: $f(n)$ has a strictly slower growth rate than $g(n)$

Analogy: $f(n) = o(g(n))$

$$a < b$$



Observe: $f(n) = o(g(n)) \Rightarrow f(n) = O(g(n))$
$\not\Leftarrow$

Analogy: $a < b \Longrightarrow a \le b$
$\not\Leftarrow$

<u>Ex</u>. $f(n) = \ln(n)$, $g(n) = n$

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right) \left\{\text{l'Hopital's rule}\right.$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

$\therefore \ln(n) = o(n)$

<u>Exercise</u>: show $\log_b(n) = o(n^k)$

for any $b > 1$ and $k > 0$.

<u>Hint</u>: use $\lceil k \rceil$ applications of l'Hop.

<u>Exercise</u> show $(\log_b(n))^m = o(n^k)$

for any $b>1$, $m>0$, $k>0$.

<u>Ex.</u> $f(n) = n^k$ ($k>0$, integer) and

$g(n) = e^n$. Then k applications

of l'Hop. gives

$$\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) = \lim_{n\to\infty}\left(\frac{k n^{k-1}}{e^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{k(k-1) n^{k-2}}{e^n}\right)$$

$$\vdots$$

$$= \lim_{n\to\infty}\left(\frac{k!}{e^n}\right) = 0$$

$\therefore$ $n^k = o(e^n)$.

<u>Exercise</u>: show $n^k = o(b^n)$

for all $k \in \mathbb{R}$ and all $b > 1$

(use $\lceil k \rceil$ app. of l'Hop.).

we have:

$$(\text{all logs}) \underset{\substack{\uparrow \\ = o(\ )}}{<} (\text{all polys}) \underset{\substack{\uparrow \\ = o(\ )}}{<} (\text{all exp.})$$

<u>Ex</u> $n^2 = o(n^3)$ why?

$$\frac{n^2}{n^3} = \frac{1}{n} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

<u>Exercise</u> $n^k = o(n^m)$ iff $k < m$

Ex. $2^n = o(3^n)$ why?

$$\frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \longrightarrow 0 \quad \text{as } n \to \infty.$$

Exercise $a^n = o(b^n)$ iff $a < b$.

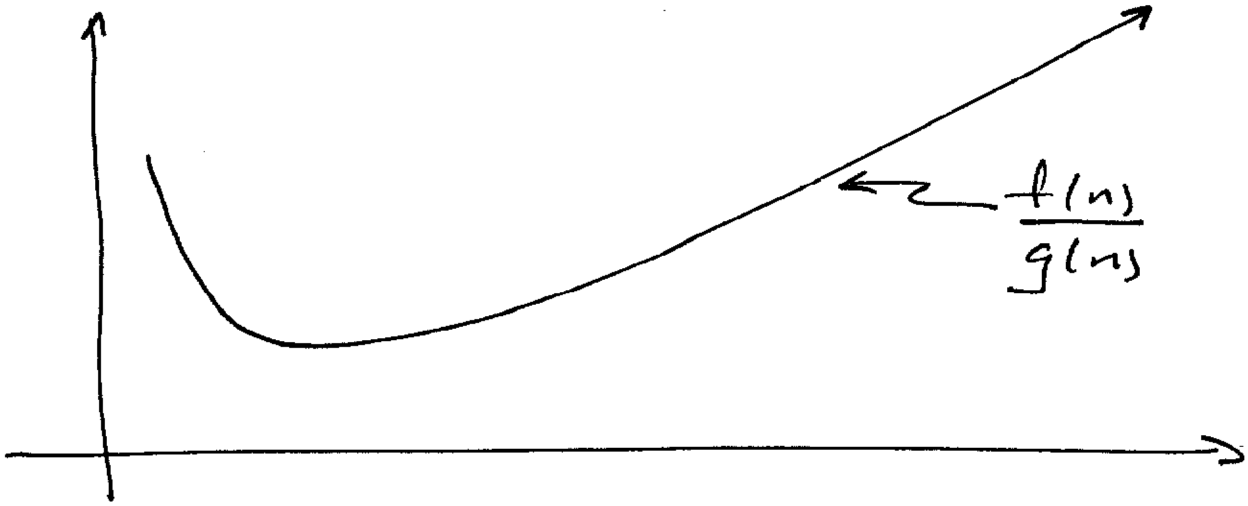
Defn
$f(n) = \omega(g(n))$ iff $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = \infty$

we say $g(n)$ is a strict asymptotic lower bound for $f(n)$.

meaning: $f(n)$ grows strictly faster than $g(n)$

Analogy: $f(n) = \omega(g(n))$

$a > b$



observe $f(n) = \omega(g(n)) \Rightarrow f(n) = \Omega(g(n))$
$\not\Leftarrow$

Analogy: $a > b \Rightarrow a \geq b$
$\not\Leftarrow$

Also observe: $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$

why?

$\frac{f(n)}{g(n)} \rightarrow 0$ as $n \rightarrow \infty$ iff $\frac{g(n)}{f(n)} \rightarrow \infty$ as $n \rightarrow \infty$

Full analogy:

$$f(n) = O(g(n)) \longleftrightarrow a \le b$$

$$f(n) = \Omega(g(n)) \longleftrightarrow a \ge b$$

$$f(n) = \Theta(g(n)) \longleftrightarrow a = b$$

$$f(n) = o(g(n)) \longleftrightarrow a < b$$

$$f(n) = \omega(g(n)) \longleftrightarrow a > b$$

Theorem:

all of these relations are <u>transitive</u>.

For example

$$f(n) = o(g(n)) \text{ and } g(n) = o(h(n))$$

$$\Longrightarrow f(n) = o(h(n))$$

Proof for o:

$$f(n) = o(g(n)) \overset{\text{def.}}{\Longrightarrow} \frac{f(n)}{g(n)} \to 0 \quad \text{as } n \to \infty$$

$$g(n) = o(h(n)) \overset{\text{def.}}{\Longrightarrow} \frac{g(n)}{h(n)} \to 0 \quad \text{" " " "}$$

Therefore

$$\frac{f(n)}{h(n)} = \frac{f(n)}{g(n)} \cdot \frac{g(n)}{h(n)} \to 0 \cdot 0 = 0 \quad \text{as } n \to \infty.$$

Therefore $f(n) = o(h(n))$. ∎

Exercise

do same for other relations.