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MODALITY AS A METALINGUISTIC PREDICATE *

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Philosophers generally use the idea of necessity in two ways. One way of looking at necessity is to construe it as a sentential operator. Necessity would operate on sentences in much the same way that the sentential operator of negation operates on sentences. Another way of looking at necessity is to construe it as a metalinguistic predicate. Upon this construal necessity would be a predicate that is applied to terms designating propositions or sentences to form new sentences. Some philosophers, such as Quine, have claimed that the use of necessity as a sentential operator can be reduced to the use of necessity as a metalinguistic predicate. But Kurt Gödel, M. H. Löb, and Richard Montague have all published articles critical of the metalinguistic conception of modality.[1] In response to them, Brian Skyrms has attempted to show that necessity as a metalinguistic predicate can be defined in terms of necessity as a sentential operator. In this paper I will state Montague's theorem and discuss Skyrms' treatment of it in an attempt to elicit the philosophical significance of each.

Montague actually proves four theorems that purport to show that necessity cannot be construed as a metalinguistic predicate. Suppose our object language T is an extension of Robinson's Arithmetic; then the metalanguage of T is expressible in T by the use of Gödel numbering. Furthermore, let us suppose necessity is a metalinguistic predicate. Corresponding to the concept of necessity will be a predicate, say N, which will range over the Gödel numbers of sentences of T. Let us denote the Gödel number of P by $G(P)$. Then $N(G(P))$ is a sentence of T that corresponds to the proposition that the sentence designated by $G(P)$ is necessary. Montague's most interesting theorem is that if for all P, Q of T,

(1) $\vdash_T N(G(P)) \rightarrow P$

(2) $\vdash_T N(G(N(G(P)) \rightarrow P))$

(3) $\vdash_T N(G(P \rightarrow Q)) \rightarrow (N(G(P)) \rightarrow N(G(Q)))$

(4) $\vdash_T N(G(P))$, if P is a logical axiom,

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then T is inconsistent.[2] One should immediately notice that conditions (1)–(4) correspond to theorems in most systems of modal logic. Thus Montague claims to have shown that if we wish to insist that necessity is a metalinguistic predicate, almost all of modal logic must be sacrificed. It is this claim that Skyrms disputes.

Skyrms attacks Montague's conclusions by constructing a consistent object language and metalanguage in which necessity is treated as a metalinguistic predicate. Skyrms lets the base language, called 'L_O', be any language as long as it contains the propositional calculus, contains only sentences of finite length, and in every model each sentence is either true or false. From this base language, Skyrms constructs a modal language, L_M, which is our object language. L_M is constructed from L_O be letting L_M be the closure of L_O under truth functions and $\Box$ as a sentential operator:

L_M: The sentences of L_M are the smallest set satisfying the following conditions:
(1) If S is a sentence of L_O, then S is a sentence of L_M.
(2) If S and T are sentences of L_M, then $(S \vee T)$, $(S \wedge T)$, $(S \supset T)$, $\sim S$, and $\Box S$ are in L_M.[3]

This is the object language Skyrms is working with.

Skyrms then constructs a metalanguage, L_ω, by means of a hierarchy of languages. L_ω will be the union of many metalanguages, the L_ns, which are constructed from L_O in the following way.

L_ns: The sentences of L_{n+1} are the smallest set satisfying the following conditions:
(1) If S is a sentence of L_n, then S and $*Q(S)$ are sentences of L_{n+1}.
(2) If S and T are sentences of L_{n+1}, then $(S \vee T)$, $(S \wedge T)$, $(S \supset T)$, $\sim S$, and $\sim T$ are sentences of L_{n+1}.[4]

Again, the metalanguage L_ω is the union of all the L_ns, $n \in \omega$. In L_ω, Skyrms wants us to interpret $Q(S)$ to be a name of S, and * to be interpreted as the necessity predicate. Thus $*Q(S)$ says that the sentence designated by $Q(S)$ is necessary. Skyrms then connects the object language and metalanguage with a mapping C:

C is a mapping from the sentences of L_M to the sentences of L_ω such that:
(i) if S is free of modalities, $C(S) = S$,
(ii) if S is $\Box R$, then $C(S) = {}^*Q(C(R))$,
(iii) if S is $(R \vee T)$, $(R \wedge T)$, $(R \supset T)$, or $\sim R$, then $C(S)$ is $(C(R) \vee C(T))$, $(C(R) \wedge C(T))$, $(C(R) \supset C(T))$, or $\sim C(R)$, respectively.[5]

The function C gives us the metalinguistic counterpart of a member of the object language. With this, Skyrms has given us an object language, a metalanguage, and the mapping between them.

Skyrms remarks that there are two concepts of necessity – validity (truth in all models) and provability – and he discusses them separately. To simplify our discussion, I wil confine my attention to his section on validity. Basically a model is a function which assigns every sentence a value of 1 or 0, taken to represent truth and falsehood respectively. The models for the base language L_O will determine the models for the L_ns. Letting f_0 denote a model of L_0, we can define a corresponding model for L_ω as follows:

The model f_{n+1} of L_{n+1} induced by a model f_0 of L_0 is the smallest extension of the model f_n of L_n induced by f_0 such that:
(i) $f(Q(S)) = S$
(ii) $f({}^*X) = 1$ if X is $Q(S)$ and S is true in all models of L_n and $f({}^*X) = 0$ otherwise.
(iii) The sentential connectives: $\vee, \wedge, \supset, \sim$, are interpreted as denoting truth functions in the usual way.

The model f of L_ω induced by a model f_0 of L_0 is the union of the models f_n induced by f_0, $(n \in \omega)$.[6]

Thus each model of L_O will generate a corresponding model of L_ω. Let us define V_ω to be the set of all sentences true in all models of L_ω. We will find it convenient to write '$\vdash_{V_\omega} Q$' for 'Q is a theorem of V_ω', or 'Q is in V_ω'.

At this point let us consider how Skyrms handles Montague's theorems. The first thing to notice is that Skyrms does not really discuss Montague's theorems directly; he discusses theorems by Löb. To simplify matters, I will discuss Montague's third theorem:

Suppose T is an extension of Robinson's Arithmetic and for all sentences P, Q of T,

(i) $\vdash_T N(G(P)) \rightarrow P$,

(ii) $\vdash_T N(G(P))$, whenever P is a sentence such that $\vdash_T P$,

then T is inconsistent.[7]

Comparing this with our theory V_ω, Skyrms establishes:

(i′) $\vdash_{V_\omega} {}^*Q(P) \rightarrow P$,

(ii′) $\vdash_{V_\omega} {}^*Q(P)$, whenever P is a sentence such that $\vdash_{V_\omega} P$.

Furthermore, we can let V_ω be an extension of Robinson's Arithmetic. The similarity between (i), (ii) and (i′), (ii′) lead one to think that V_ω is inconsistent. But there is a slight difference in that Montague's theorem uses Gödel numbers as names, whereas Skyrms is using the function Q as a naming function; different naming functions are employed. One would not expect this to make a difference, but the question must be investigated.

When we examine the connection between V_ω and Montague's theorem we see that conditions (i) and (ii) of Montague's theorem are not necessarily satisfied in V_ω. If we let V_ω be rich enough, standard constructions will yield a provability predicate \$, such that if S is in V_ω, then $\$(G(S))$ is in V_ω. Thus condition (ii) of Montague's theorem will be satisfied. However, condition (i) may still fail to be satisfied.[8] Thus we have no reason to believe that V_ω is inconsistent.

Since there is no obvious connection between Gödel numbering and the Q function, we might try to prove Montague's theorem using Q as the naming function instead of Gödel numbering. To begin, let us prove a modified version of Tarski's diagonal lemma, which uses Q as a naming function instead of Gödel numbering.

DIAGONAL LEMMA. If M is a metalanguage, O an object language, O contains the predicate calculus and the function Q, and the metalinguistic function $\Delta(\psi(x)) = \psi(Q(\psi(x)))$ is weakly represented in O by δ[9], then for any ψ there is a θ such that $\vdash_o \psi(Q(\theta)) \equiv \theta$.

Proof:

Let $\theta = \psi(\delta(Q(\psi(\delta(x)))))$. Then

$\vdash_M \Delta(\psi(\delta(x))) = \psi(\delta(Q(\psi(\delta(x))))) = \theta$

since δ weakly represents Δ in O,

$\vdash_O \delta(Q(\psi(\delta(x)))) = Q(\theta)$
since $\theta \equiv \theta$ is a theorem,
$\vdash_O \psi(\delta(Q(\psi(\delta(x))))) \equiv \psi(\delta(Q(\psi(\delta(x)))))$
by substitution,
$\vdash_O \psi(Q(\theta)) \equiv \psi(\delta(Q(\psi(\delta(x)))))$, and
$\vdash_O \psi(Q(\theta)) \equiv \theta$.

With this lemma we can prove a theorem very similar to Montague's third theorem. We will interpret ψ in the diagonal lemma as $\sim *$ (not necessary).

THEOREM. If M and O are the same as in the diagonal lemma, and for all formulas P in O

(i) $\vdash_O *Q(P) \rightarrow P$.
(ii) $\vdash_O *Q(P)$, whenever P is a sentence such that $\vdash_O P$.

then O is inconsistent.

Proof:

(1)	$\sim * Q(P) \equiv P$	diagonal lemma
(2)	$*Q(P) \rightarrow \sim P$	1
(3)	$*Q(P) \rightarrow P$	assumption (i)
(4)	$\sim * Q(P)$	2, 3
(5)	P	1, 4
(6)	$*Q(P)$	5, assumption (ii)
(7)	$*Q(P) \& \sim *Q(P)$	4, 6

Since Skyrms repeatedly claims his metalanguage is consistent, it will profit us to see if he has truly escaped the results of the previous theorem. The object langauge in question will be V_ω. First of all, V_ω contains both the predicate calculus and a naming function Q. We also have for all formulas P in V_ω:

(i) $\vdash_{V_\omega} *Q(P) \rightarrow P$,
(ii) $\vdash_{V_\omega} *Q(P)$, whenever P is such that $\vdash_{V_\omega} P$.

Thus conditions (i) and (ii) of the above theorem are satisfied. The only requirement left is that the metalinguistic function $\Delta(\psi(x)) = \psi(Q(\psi(x)))$ be weakly represented by some δ in V_ω. Let us take a closer look at this requirement. δ weakly represents Δ in V_ω if whenever $\vdash_M R = \Delta(P)$ is true, $\vdash_{V_\omega} Q(R) = \delta(Q(P))$ is true. However it is impossible for $\vdash_{V_\omega} Q(R) =$

$\delta(Q(P))$ to ever be true. The formation rules of the language L_ω require that every occurrence of a 'Q' be preceded by an asterisk; thus $Q(R) = \delta(Q(P))$ is not well formed in L_ω. In our proof of the diagonal lemma, by assuming that δ weakly represented Δ, we assumed that $\vdash_O \delta(Q(\psi(\delta(x)))) = Q(\theta)$ was true. But when our object language is L_ω, we see that the above sentence is not well formed because the occurrences of 'Q' are not preceded by asterisks; hence it cannot be a theorem of V_ω. Thus we see that because of the way the naming function Q is restricted in L_ω, Δ is not weakly represented by δ in V_ω. A function δ that would weakly represent Δ in V_ω is not a well formed formula in V_ω.

Given that Δ is not weakly representable in Skyrms' metalanguage, it is not surprising that his metalinguistic conception of necessity is consistent. Montague's theorems showed that if Gödel numbers are considered to be names of sentences and necessity is a predicate of names of sentences, then the language would be inconsistent if it were a sufficiently strong langauge. The preceding theorem showed that if instead of Gödel numbers, we take the naming function to be a quotation function, the same results hold, given that the language is sufficiently strong. This could be generalized to say that for any naming function, if the language is sufficiently strong it is inconsistent to let necessity be a metalinguistic predicate. Skyrms' language is consistent only because it is weakened by restricting the naming function.

Skyrms has shown that in severely restricted languages necessity can be treated as a metalinguistic predicate. Montague's theorem and my extension of it have shown that if the restrictions are taken off of these languages, then necessity cannot consistently be treated as a metalinguistic predicate. Thus we must side with Montague and claim that necessity cannot be a metalinguistic predicate for a reasonably rich language. Let us take L_ω^+ to be the metalanguage for L_M that we get my loosening the various restrictions that Skyrms has placed on Q. All of the theorems of L_ω will be contained in L_ω^+; V_ω is a subset of V_ω^+. What my extension of Montague's theorem shows is that V_ω^+ is inconsistent, and Skyrms has shown that V_ω is consistent. It appears that Skyrms' results do not really bear upon the question whether the metalinguistic conception of necessity is inconsistent for rich languages. The preceding extension of Montague's theorem appears to show that it must be.

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NOTES

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[1] Kurt Gödel, 'Eine Interpretation des intuitionistischen Aussagenkalküls', Ergebnisse eines mathematischen Kolloquiums, Vol. 4 (for 1931–32, published 1933), pp. 39–40.

M.H. Löb, 'Solution of a problem of Leon Henkin', Journal of Symbolic Logic 20 (1955), pp. 115–118.

Richard Montague, 'Syntactical treatments of modality, with corollaries, on reflexion principles and finite axiomatizability', Acta Philosophica Fennica xvi (Helsinki, 1963), pp. 153–167.

[2] Montague, p. 159.

[3] Brian Skyrms, 'An immaculate conception of modality or how to confuse use and mention', The Journal of Philosophy lxxv (1978), p. 369.

[4] Skyrms, p. 369.

[5] Skyrms, p 370.

[6] Skyrms, p. 370.

[7] Montague, p. 160.

[8] See George Boolos and Richard Jeffrey, Computability and Logic (Cambridge University Press, 1974), Chapter 16, and Skyrms, pp. 384–385 for a discussion of this.

[9] The metalinguistic function Δ is weakly represented by the function symbol δ in O if and only if for all sentences φ_1 and φ_2, if $\Delta(\varphi_1) = \varphi_2$ is true in M, then $\delta(Q(\varphi_1)) = Q(\varphi_2)$ is true in O.