Philosophers generally use the idea of necessity in two ways. One way of looking at necessity is to construe it as a sentential operator. Necessity would operate on sentences in much the same way that the sentential operator of negation operates on sentences. Another way of looking at necessity is to construe it as a metalinguistic predicate. Upon this construal necessity would be a predicate that is applied to terms designating propositions or sentences to form new sentences. Some philosophers, such as Quine, have claimed that the use of necessity as a sentential operator can be reduced to the use of necessity as a metalinguistic predicate. But Kurt Gödel, M.H. Löb, and Richard Montague have all published articles critical of the metalinguistic conception of modality. In response to them, Brian Skyrms has attempted to show that necessity as a metalinguistic predicate can be defined in terms of necessity as a sentential operator. In this paper I will state Montague's theorem and discuss Skyrms' treatment of it in an attempt to elicit the philosophical significance of each.

Montague actually proves four theorems that purport to show that necessity cannot be construed as a metalinguistic predicate. Suppose our object language $T$ is an extension of Robinson's Arithmetic; then the metalanguage of $T$ is expressible in $T$ by the use of Gödel numbering. Furthermore, let us suppose necessity is a metalinguistic predicate. Corresponding to the concept of necessity will be a predicate, say $N$, which will range over the Gödel numbers of sentences of $T$. Let us denote the Gödel number of $P$ by $G(P)$. Then $N(G(P))$ is a sentence of $T$ that corresponds to the proposition that the sentence designated by $G(P)$ is necessary. Montague's most interesting theorem is that if for all $P, Q$ of $T$,

\begin{align*}
(1) & \vdash_T N(G(P)) \rightarrow P \\
(2) & \vdash_T N(G(N(G(P))) \rightarrow P)) \\
(3) & \vdash_T N(G(P \rightarrow Q)) \rightarrow (N(G(P)) \rightarrow N(G(Q))) \\
(4) & \vdash_T N(G(P)), \text{ if } P \text{ is a logical axiom,}
\end{align*}
then \( T \) is inconsistent. One should immediately notice that conditions (1)—(4) correspond to theorems in most systems of modal logic. Thus Montague claims to have shown that if we wish to insist that necessity is a metalinguistic predicate, almost all of modal logic must be sacrificed. It is this claim that Skyrms disputes.

Skyrms attacks Montague's conclusions by constructing a consistent object language and metalanguage in which necessity is treated as a metalinguistic predicate. Skyrms lets the base language, called '\( L_O \)', be any language as long as it contains the propositional calculus, contains only sentences of finite length, and in every model each sentence is either true or false. From this base language, Skyrms constructs a modal language, \( L_M \), which is our object language. \( L_M \) is constructed from \( L_O \) be letting \( L_M \) be the closure of \( L_O \) under truth functions and \( \Box \) as a sentential operator:

\[
L_M: \quad \text{The sentences of } L_M \text{ are the smallest set satisfying the following conditions:}
\]

1. If \( S \) is a sentence of \( L_O \), then \( S \) is a sentence of \( L_M \).
2. If \( S \) and \( T \) are sentences of \( L_M \), then \( (S \lor T), (S \land T), (S \supset T), \sim S, \) and \( \Box S \) are in \( L_M \).\(^3\)

This is the object language Skyrms is working with.

Skyrms then constructs a metalanguage, \( L_\omega \), by means of a hierarchy of languages. \( L_\omega \) will be the union of many metalanguages, the \( L_n \)'s, which are constructed from \( L_O \) in the following way.

\[
L_n: \quad \text{The sentences of } L_{n+1} \text{ are the smallest set satisfying the following conditions:}
\]

1. If \( S \) is a sentence of \( L_n \), then \( S \) and \( \star Q(S) \) are sentences of \( L_{n+1} \).
2. If \( S \) and \( T \) are sentences of \( L_{n+1} \), then \( (S \lor T), (S \land T), (S \supset T), \sim S, \) and \( \sim T \) are sentences of \( L_{n+1} \).\(^4\)

Again, the metalanguage \( L_\omega \) is the union of all the \( L_n \)'s, \( n \in \omega \). In \( L_\omega \), Skyrms wants us to interpret \( Q(S) \) to be a name of \( S \), and \( \star \) to be interpreted as the necessity predicate. Thus \( \star Q(S) \) says that the sentence designated by \( Q(S) \) is necessary. Skyrms then connects the object language and metalanguage with a mapping \( C \):
$C$ is a mapping from the sentences of $L_M$ to the sentences of $L_\omega$ such that:

(i) if $S$ is free of modalities, $C(S) = S$,
(ii) if $S$ is $\square R$, then $C(S) = *Q(C(R))$,
(iii) if $S$ is $(R \lor T), (R \land T), (R \supset T)$, or $\neg R$, then $C(S)$ is $(C(R) \lor C(T)), (C(R) \land C(T)), (C(R) \supset C(T))$, or $\neg C(R)$, respectively.\(^5\)

The function $C$ gives us the metalinguistic counterpart of a member of the object language. With this, Skyrms has given us an object language, a metalanguage, and the mapping between them.

Skyrms remarks that there are two concepts of necessity — validity (truth in all models) and provability — and he discusses them separately. To simplify our discussion, I will confine my attention to his section on validity. Basically a model is a function which assigns every sentence a value of 1 or 0, taken to represent truth and falsehood respectively. The models for the base language $L_0$ will determine the models for the $L_n$. Letting $f_0$ denote a model of $L_0$, we can define a corresponding model for $L_\omega$ as follows:

The model $f_{n+1}$ of $L_{n+1}$ induced by a model $f_0$ of $L_0$ is the smallest extension of the model $f_n$ of $L_n$ induced by $f_0$ such that:

(i) $f(Q(S)) = S$
(ii) $f(*X) = 1$ if $X$ is $Q(S)$ and $S$ is true in all models of $L_n$ and
    $f(*X) = 0$ otherwise.
(iii) The sentential connectives: $\lor, \land, \supset, \neg$, are interpreted as denoting truth functions in the usual way.

The model $f$ of $L_\omega$ induced by a model $f_0$ of $L_0$ is the union of the models $f_n$ induced by $f_0, (n \in \omega)$.\(^6\)

Thus each model of $L_0$ will generate a corresponding model of $L_\omega$. Let us define $V_\omega$ to be the set of all sentences true in all models of $L_\omega$. We will find it convenient to write $\vdash_{V_\omega} Q$ for 'Q is a theorem of $V_\omega$', or 'Q is in $V_\omega$'.

At this point let us consider how Skyrms handles Montague's theorems. The first thing to notice is that Skyrms does not really discuss Montague's theorems directly; he discusses theorems by Löb. To simplify matters, I will discuss Montague's third theorem:
Suppose $T$ is an extension of Robinson's Arithmetic and for all sentences $P, Q$ of $T$,

(i) $\vdash_T N(G(P)) \rightarrow P$,

(ii) $\vdash_T N(G(P))$, whenever $P$ is a sentence such that $\vdash_T \neg P$,

then $T$ is inconsistent.\(^7\)

Comparing this with our theory $V_\omega$, Skyrms establishes:

(i') $\vdash_{V_\omega} *Q(P) \rightarrow P$,

(ii') $\vdash_{V_\omega} *Q(P)$, whenever $P$ is a sentence such that $\vdash_{V_\omega} \neg P$.

Furthermore, we can let $V_\omega$ be an extension of Robinson's Arithmetic. The similarity between (i), (ii) and (i'), (ii') lead one to think that $V_\omega$ is inconsistent. But there is a slight difference in that Montague's theorem uses Gödel numbers as names, whereas Skyrms is using the function $Q$ as a naming function; different naming functions are employed. One would not expect this to make a difference, but the question must be investigated.

When we examine the connection between $V_\omega$ and Montague's theorem we see that conditions (i) and (ii) of Montague's theorem are not necessarily satisfied in $V_\omega$. If we let $V_\omega$ be rich enough, standard constructions will yield a provability predicate $S$, such that if $S$ is in $V_\omega$, then $S(G(S))$ is in $V_\omega$. Thus condition (ii) of Montague's theorem will be satisfied. However, condition (i) may still fail to be satisfied.\(^8\) Thus we have no reason to believe that $V_\omega$ is inconsistent.

Since there is no obvious connection between Gödel numbering and the $Q$ function, we might try to prove Montague's theorem using $Q$ as the naming function instead of Gödel numbering. To begin, let us prove a modified version of Tarski's diagonal lemma, which uses $Q$ as a naming function instead of Gödel numbering.

**DIAGONAL LEMMA.** If $M$ is a metalanguage, $O$ an object language, $O$ contains the predicate calculus and the function $Q$, and the metalinguistic function $\Delta(\psi(x)) = \psi(Q(\psi(x)))$ is weakly represented in $O$ by $\delta$\(^9\), then for any $\psi$ there is a $\theta$ such that $\vdash_o \psi(Q(\theta)) = \theta$.

**Proof:**

Let $\theta = \psi(\delta(Q(\psi(\delta(x))))).$ Then 

$\vdash_M \Delta(\psi(\delta(x))) = \psi(\delta(Q(\psi(\delta(x))))) = \theta$ 

since $\delta$ weakly represents $\Delta$ in $O$. 


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\[ \vdash_{O} \delta(Q(\psi(\delta(x)))) = Q(\theta) \]
since \( \theta \equiv \theta \) is a theorem,
\[ \vdash_{O} \psi(\delta(Q(\psi(\delta(x))))) \equiv \psi(\delta(Q(\psi(\delta(x))))) \]
by substitution,
\[ \vdash_{O} \psi(Q(\theta)) \equiv \psi(\delta(Q(\psi(\delta(x))))) , \text{ and} \]
\[ \vdash_{O} \psi(Q(\theta)) = \theta. \]

With this lemma we can prove a theorem very similar to Montague's third theorem. We will interpret \( \psi \) in the diagonal lemma as \( \sim^{*} \) (not necessary).

THEOREM. If \( M \) and \( O \) are the same as in the diagonal lemma, and for all formulas \( P \) in \( O \)

(i) \[ \vdash_{O} *Q(P) \rightarrow P. \]
(ii) \[ \vdash_{O} *Q(P), \text{ whenever } P \text{ is a sentence such that } \vdash_{O} P. \]

then \( O \) is inconsistent.

Proof:

(1) \[ \sim * Q(P) = P \]
diagonal lemma
(2) \[ *Q(P) \rightarrow \sim P \]
1
(3) \[ *Q(P) \rightarrow P \]
assumption (i)
(4) \[ \sim * Q(P) \]
2, 3
(5) \[ P \]
1, 4
(6) \[ *Q(P) \]
5, assumption (ii)
(7) \[ *Q(P) \& \sim *Q(P) \]
4, 6

Since Skyrms repeatedly claims his metalanguage is consistent, it will profit us to see if he has truly escaped the results of the previous theorem. The object langauge in question will be \( V_{\omega} \). First of all, \( V_{\omega} \) contains both the predicate calculus and a naming function \( Q \). We also have for all formulas \( P \) in \( V_{\omega} \):

(i) \[ \vdash_{V_{\omega}} *Q(P) \rightarrow P, \]
(ii) \[ \vdash_{V_{\omega}} *Q(P), \text{ whenever } P \text{ is such that } \vdash_{V_{\omega}} P. \]

Thus conditions (i) and (ii) of the above theorem are satisfied. The only requirement left is that the metalinguistic function \( \Delta(\psi(\chi)) = \psi(Q(\psi(\chi))) \) be weakly represented by some \( \delta \) in \( V_{\omega} \). Let us take a closer look at this requirement. \( \delta \) weakly represents \( \Delta \) in \( V_{\omega} \) if whenever \( \vdash_{M} R = \Delta(P) \) is true, \( \vdash_{V_{\omega}} Q(R) = \delta(Q(P)) \) is true. However it is impossible for \( \vdash_{V_{\omega}} Q(R) = \delta(Q(P)) \)
The formation rules of the language $L_\omega$ require that every occurrence of a 'Q’ be preceded by an asterisk; thus $Q(R) = \delta(Q(P))$ is not well formed in $L_\omega$. In our proof of the diagonal lemma, by assuming that $\delta$ weakly represented $\Delta$, we assumed that $\neg\sigma\delta(Q(\psi(\sigma(x)))) = \sigma(\sigma)$ was true. But when our object language is $L_\omega$, we see that the above sentence is not well formed because the occurrences of 'Q' are not preceded by asterisks; hence it cannot be a theorem of $V_\omega$. Thus we see that because of the way the naming function $Q$ is restricted in $L_\omega$, $\Delta$ is not weakly represented by $\delta$ in $V_\omega$. A function $\delta$ that would weakly represent $\Delta$ in $V_\omega$ is not a well formed formula in $V_\omega$.

Given that $\Delta$ is not weakly representable in Skyrm's metalanguage, it is not surprising that his metalinguistic conception of necessity is consistent. Montague's theorems showed that if Godel numbers are considered to be names of sentences and necessity is a predicate of names of sentences, then the language would be inconsistent if it were a sufficiently strong language. The preceding theorem showed that if instead of Godel numbers, we take the naming function to be a quotation function, the same results hold, given that the language is sufficiently strong. This could be generalized to say that for any naming function, if the language is sufficiently strong it is inconsistent to let necessity be a metalinguistic predicate. Skyrm's language is consistent only because it is weakened by restricting the naming function.

Skyrm has shown that in severely restricted languages necessity can be treated as a metalinguistic predicate. Montague's theorem and my extension of it have shown that if the restrictions are taken off of these languages, then necessity cannot consistently be treated as a metalinguistic predicate. Thus we must side with Montague and claim that necessity cannot be a metalinguistic predicate for a reasonably rich language. Let us take $L^*_\omega$ to be the metalanguage for $L_M$ that we get my loosening the various restrictions that Skyrm has placed on $Q$. All of the theorems of $L_\omega$ will be contained in $L^*_\omega$; $V_\omega$ is a subset of $V^*_\omega$. What my extension of Montague's theorem shows is that $V^*_\omega$ is inconsistent, and Skyrm has shown that $V_\omega$ is consistent. It appears that Skyrm's results do not really bear upon the question whether the metalinguistic conception of necessity is inconsistent for rich languages. The preceding extension of Montague's theorem appears to show that it must be.

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NOTES

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4 Montague, p. 159.


6 Skyrms, p. 369.

7 Skyrms, p. 370.


9 The metalinguistic function $\Delta$ is weakly represented by the function symbol $\delta$ in $O$ if and only if for all sentences $\varphi_1$ and $\varphi_2$, if $\Delta(\varphi_1) = \varphi_2$ is true in $M$, then $\delta(\varphi_1) = Q(\varphi_2)$ is true in $O$. 