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SUBJECTIVE PROBABILITY, OBJECTIVE PROBABILITY, AND COHERENCE

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According to Bayesianism, subjective probabilities are to be interpreted as rational degrees of belief. Degrees of belief are usually interpreted behavioristically in terms of what an agent's betting behavior would be in certain circumstances.[1] However, not all our betting behavior is rational, and most proponents of subjective probability stipulate some requirement that degrees of belief and subjective probabilities must satisfy in order to be considered rational. Almost all subjectivists require as a minimum that subjective probabilities satisfy the probability calculus. A probability assignment that satisfies the axioms of the probability calculus is called *coherent*, and one that does not satisfy the calculus is called *incoherent*. Although some subjectivists attempt to justify other restrictions upon subjective probabilities, many subjectivists think that coherence is the only requirement. The requirement of coherence has been defended by appealing to Dutch book theorems and axioms of preference, but many philosophers find these arguments unconvincing.[2] Hence many feel that the requirement of coherence is unjustified.

One attempt to justify the requirement of coherence, which finds its roots in Leibniz and Carnap, does not follow the Bayesians in interpreting subjective probability as degrees of belief. According to Leibniz, the probabilities in our mind should be proportional to what we believe the propensities in objects are.[3] A modern proponent of this position is Keith Lehrer. Lehrer defines the subjective probability of a proposition not as a degree of belief in a proposition, but as a subjective estimate of the objective probability of the proposition. Lehrer claims that this explains some phenomena that interpreting probability as degrees of belief seemed unable to explain. For example, according to Lehrer some people seem to assign a high subjective probability to a proposition, even though they are not very convinced that the proposition is true. This situation might happen in cases in which our belief in some

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proposition is based on some authority or testimony of others. We might accept the other's testimony and accordingly assign a high subjective probability to the proposition, even though we do not believe very strongly that the proposition is true.

Lehrer argues that an additional benefit of equating subjective probabilities with subjective estimates of objective probabilities is that we then have a reason to claim that subjective probabilities should satisfy the probability calculus. Lehrer says:

> Hence if subjective probabilities are interpreted as estimates of objective probabilities, limits of relative frequencies, then either subjective probabilities must satisfy the calculus or they will be demonstrably incorrect.
>
> The justification for requiring, as a condition of ideal rationality, that the subjective probabilities of a man conform to the calculus of probability is the same as the justification for requiring that a man be logically consistent. The violation of either requirement brings with it the certainty of error. If a man is logically inconsistent in what he affirms or believes, then he cannot possibly be correct in what he believes or affirms. Similarly, if the subjective estimates of a man concerning objective probabilities are incoherent, then he cannot possibly be correct in what he estimates. . . . Therefore, subjective probabilities must conform to the calculus of probability, not simply because dutch book can be made against one otherwise, but because subjective probabilities are estimates of ratios which, as a sheer matter of mathematics, conform to that calculus.[4]

Lehrer is assuming that objective probabilities are to be interpreted as the limits of relative frequencies. His basic argument does not depend upon that interpretation of probability, but rather it depends upon objective probabilities satisfying the probability calculus. Let us assume that objective probabilities, however they are interpreted, satisfy the probability calculus.[5] If subjective probabilities are estimates of objective probabilities, then incoherent subjective probabilities are guaranteed to be wrong. Since that is undesirable or irrational, subjective probabilities must also conform to the axioms of probability.

We might express Lehrer's claim that our subjective probabilities are estimates of objective probabilities by the following principle:

(1) If a person's subjective probability of S is r, then that person believes that the objective probability of S is r.

Let us now attempt to clarify why the above principle implies that subjective probabilities must conform to the probability calculus. Lehrer says that subjective probabilities must be coherent in order to avoid the certainty of error. It would seem that Lehrer means to hold something like the following:

(2) It is irrational to hold beliefs which cannot possibly be correct.

Given principles (1), (2), and that objective probabilities satisfy the probability calculus, one can conclude that subjective probabilities must also satisfy the probability calculus in order to be rational. If correct, Lehrer would have shown that if we interpret subjective probability as an estimate of objective probability instead of as degrees of belief then we can give a justification for requiring subjective probability assignments to be coherent.

Although principle (2) may initially appear correct, closer inspection reveals that some modifications are needed. Suppose that a student is taking a set theory class and has just recently studied the axiom of choice and Zorn's lemma, but has not yet studied the proof that they are equivalent. Suppose that this student believes the axiom of choice is correct, but believes that Zorn's lemma is false. This student's beliefs cannot possibly be correct, and thus according to principle (2) the student is irrational. However, it seems that if the student did not know that the axiom of choice and Zorn's lemma were equivalent then it would not necessarily be irrational to affirm one and deny the other, even though there was no possibility that the student's beliefs could be correct. Perhaps a normally reliable friend told the student that Zorn's lemma was false. If principle (2) were correct a person would have to be logically omniscient in order to be rational. One would have to know the logical consequences of each of his or her beliefs in order to avoid the possibility of having beliefs which are guaranteed to be false.

One way to avoid the above problem would be to require that an ideally rational person's set of beliefs be deductively closed. This would have the result that the student in the above example would have to know that the axiom of choice was equivalent to Zorn's lemma in order to be rational. Although this requirement will solve the above problem, most people would think that it is an unreasonable requirement. We commonly think that a person can be rational without knowing all of the consequences of his or her beliefs and that rationality does not require deductive closure. Thus this solution of the problem is inadequate.

Another solution would be to modify principle (2) and require that a rational person not hold any beliefs which he or she knows cannot possibly be true. Thus in the above example, since the student did not know that the axiom of choice was equivalent to Zorn's lemma, the student would not be required to give up one of his or her beliefs in order to be rational. But if the student learns of their equivalence in a future class, then he or she would be required to modify his or her beliefs in order to remain rational. We could express this modification of (2) as:

(2′) It is irrational to have beliefs which one knows cannot possibly be correct.

This modification avoids the above problems by not requiring that one's beliefs be deductively closed.

Although replacing principle (2) by (2′) avoids the above problems about rationality, Lehrer's argument that subjective probabilities must be coherent in order to be rational is no longer valid. First of all, a person might not know that objective probabilities satisfy the probability calculus. If the person lacked that belief, then it might be rational for him or her to have an incoherent subjective probability assignment. Let us avoid this problem by assuming that everyone knows that objective probability satisfies the probability calculus. One can thus conclude from principles (1) and (2′) that it is irrational to adopt a probability assignment which one knows is incoherent. From this one cannot conclude that the subjective probabilities of a rational person will be coherent, unless one also assumes that rational people are logically omniscient and know all theorems of the probability calculus. Thus Lehrer's argument that subjective probabilities must satisfy the probability calculus is seen to fail. Principles (1) and (2′) do not give us a reason to think that subjective probabilities satisfy the probability calculus; what they imply is that a rational person will not knowingly have incoherent subjective probability assignments.

Although the failure of Lehrer's argument for the requirement of coherence may initially be troubling, I think that upon closer inspection it will be seen to be a desirable result. In spite of the almost universal acceptance of coherence as a necessary condition for rationality, there are very good reasons for a rational person to have incoherent subjective probabilities. The requirement of coherence is very strict in that it does not tolerate error or doubt with regard to necessary truths. An axiom of the probability calculus requires that any necessary proposition receive a probability of 1. Thus in a coherent probability assignment, all necessarily true propositions must receive a probability of 1; there is no room for doubt. However, it seems very implausible to require a rational person to assign all necessarily true propositions a probability of 1. There are many propositions which are necessarily true, but we are incapable of knowing that they are necessarily true. Either Goldbach's conjecture, which is that every even number greater than two is the sum of two primes, or its denial would be an example; many other philosophical and mathematical truths are also necessarily true, even though they are not known. It would be unreasonable to assign these propositions a subjective probability of 1, even though that is a necessary condition for a coherent probability assignment.

Almost all people who extoll the virtues of subjective probability equate subjective probability with betting quotients. One result of this is that if one assigns a proposition S a subjective probability of 1, then he should be willing to enter a bet on S in which he will either gain nothing or lose everything. But surely it would be irrational to enter such a bet on Goldbach's conjecture. It is reasonable to express our ignorance about the truth of Goldbach's conjecture by assigning it a subjective probability between 0 and 1. Thus it would seem that rationality requires that our subjective probabilities not be coherent. Contrary to many claims, coherence cannot be a necessary condition of rationality, because incoherence is required of rational subjective probabilities.

The reason that coherence cannot be a necessary condition for rationality is because coherence imposes restraints upon subjective probabilities with regard to necessarily true propositions, necessarily false propositions, logically equivalent propositions, and entailment relations among propositions. However, rationality does not require that a rational person know which propositions are necessarily true, necessarily false, logically equivalent, or entailed by other propositions. Because our concept of rationality tolerates ignorance on those matters and coherence does not, coherence cannot be required for rationality. In order to be coherent a person would have to be logically omniscient, which is clearly not required by our concept of rationality. So Lehrer's attempt to justify the coherence of subjective probabilities fails, and there are good reasons to think that rational subjective probabilities must be incoherent.

Although principles (1) and (2′) do not justify coherence as a requirement of subjective probabilities, they do have other interesting results. If subjective probabilities are estimates of objective probabilities and if it is irrational to believe some proposition which one knows cannot possibly be true, then the following principles would appear to be correct:

(3) If one knows that the objective probability of a proposition S being true is not equal to r, then it is irrational for one's subjective probability of S to be r.
(4) If one knows that the objective probability of a proposition S being true is equal to r, then it is irrational for one's subjective probability of S to be other than r.

Principles (3) and (4) follow from principles (1) and (2′), and place strict requirements upon subjective probability assignments.

In spite of the attractiveness of principles (1) and (2′), there are serious problems that they face. More specifically, principle (3) leads to unintuitive results. Suppose that someone is going to assign a subjective probability to Goldbach's conjecture. This person knows that Goldbach's conjecture is either necessarily true or necessarily false, although no human knows which of these is the case. The following is an axiom of the probability calculus:

(5) If S is necessarily true, then $P(S) = 1$.

One can also derive the following theorem:

(6) If S is necessarily false, then $P(S) = 0$.

Since this person knows that Goldbach's conjecture is either necessarily true or necessarily false, he will know that the objective probability of it is either 1 or 0. Since he knows that the objective probability is either 1 or 0, he knows that it cannot be any number that is between 0 and 1.

Principle (3) would then require that his subjective probability of Goldbach's conjecture be either 0 or 1; any other number would be certain to be incorrect. This result could be generalized and applied to many mathematical and philosophical propositions. There are many mathematical and philosophical propositions which we know are either necessarily true or necessarily false, and yet we do not know which of these is the case. Principle (3) requires that our subjective probabilities of those propositions be either 0 or 1: intermediate values are not allowed.

The above result seems very unintuitive. We would normally think that a person who had a subjective probability of Goldbach's conjecture being true of either 0 or 1 was being unreasonable. Lehrer equates subjective probabilities with betting quotients. From this it follows that if one's subjective probability of a proposition is either 0 or 1, then he should be willing to enter a bet in which he would gain nothing or lose everything. However, most of us would consider it unreasonable given the evidence that we have to enter a bet on Goldbach's conjecture in which we would either gain nothing or lose everything. Thus principle (3) should be rejected.

The above counterexample does not essentially depend upon the objective probability of Goldbach's conjecture being either 0 or 1; we can generalize the counterexample to cases in which one knows that the objective probability of a proposition is either x or y, but neither x nor y is equal to 0 or 1. For example, suppose that one knows that the objective probability of a coin landing heads is either .2 or .8. However, if that is all one knew about the coin, it would be reasonable to have a subjective probability of heads appearing equal to .5, even though one knows that the objective probability is not .5. Without further information about how likely the coin is to have a certain bias, it would seem unreasonable to have a subjective probability of either .2 or .8 that the coin would land heads. Thus principle (3) is seen to be incorrect.

One might attempt to solve the above problem by looking at how likely it is that the coin has an objective probability of .8 to come up heads. Suppose that we knew that whether or not the coin had an objective probability of .8 to come up heads or an objective probability of .2 to come up heads was the result of flipping a fair coin. In this case we would say that the coin has a probability of .5 of having an objective probability of .8 to come up heads. One could then use the standard multiplication and addition axioms to conclude that there is a probability of .5 that the coin will come up heads. This would give a reasonable betting quotient, without appearing to break principles (1) and (2′).

Although the above may appear plausible, it will not solve all of the problems that the above counterexamples present to principle (3). In many cases, such as the original example, we do not know how likely the coin is to have a certain objective probability to come up heads. In other cases attempting to use second order objective probabilities will be impossible. Reconsider the case of Goldbach's conjecture. Since Goldbach's conjecture is either necessarily true or necessarily false, its objective probability is either 0 or 1. However, what is necessarily true is necessarily necessarily true, and thus it is necessary that Goldbach's conjecture is necessarily true or necessarily false. Thus it would be necessarily true or necessarily false that its objective probability is 0. The same would hold for the objective probability of Goldbach's conjecture being 1. From this one can infer that the objective probability that the objective probability of Goldbach's conjecture being 0 is 0 or 1, and the objective probability that the objective probability of Goldbach's conjecture being 1 is 0 or 1. But this provides no second order probabilities of the various objective probabilities that Goldbach's conjecture might have, and thus the original problem still stands.

Another problem is that in some situations, such as the above, principle (3) may be inconsistent with principle (4), which has independent support. One result of principle (4) is the following. If a person knows that the objective probability of a proposition S is either x or y, then that person's subjective probability of S cannot be x, unless that person's subjective probability of the objective probability of S being x is 1. Similarly, the person's subjective probability of S cannot be y, unless that person's subjective probability of the objective probability of S being y is 1.[6] In other words, in order to assign a subjective probability of x to some proposition S, one must be absolutely certain that the objective probability of S is x. But in many situations in which we know that the objective probability of some proposition S is either x or y, we do not know for certain whether the objective probability of S is x or whether the objective probability of S is y. In those cases principle (4) will conflict with principle (3), because principle (3) will require that the subjective probability of S be either x or y, whereas principle (4) will require that it be neither x nor y. Since principle (4) has independent support, principle (3) must be rejected.[7]

The above problems have arisen because we have accepted principle (3), which appears false. Since principle (3) is derived from principles (1) and (2′), we must reject either principle (1) or principle (2′). Principle (2′), or something very close to it, seems to be correct; it does seem irrational to hold beliefs that one knows are false.[8] If one held a belief that he knew was false, he would be in the position of both believing some proposition and its negation, which seems irrational. Thus, I think that we must reject the claim that subjective probabilities are estimates of objective probabilities. Given our incomplete knowledge of objective probabilities, it is impossible for subjective probabilities to be an estimate of them without breaking some criterion of rationality or breaking the requirement that the probability of a proposition be a single value. There is a connection between subjective probability and objective probability, as is illustrated by principle (4), but it does not appear that one is an estimate of the other.

In this paper I have argued that Lehrer's attempt to justify the requirement of coherence for subjective probability does not succeed. I also argued that, contrary to common opinion, coherence should not be required for a rational subjective probability assignment. It was then shown that interpreting subjective probability as estimates of objective probability had unintuitive results, and that other commonly accepted beliefs must be given up if we want to accept that interpretation of subjective probability.

NOTES

[1] There are several problems that arise from interpreting degrees of belief behaviouristically, although a discussion of them is beyond the scope of this paper. See Henry Kyburg, "Subjective Probability: Criticisms, Reflections, and Problems," *Journal of Philosophical Logic*, 7, 1978, pp. 157-180.

[2] See R. Kennedy and C. Chihara, "The Dutch Book Argument: Its Logical Flaws, Its Subjective Sources," *Philosophical Studies*, 36, 1979, pp. 19-33; Clark Glymour, *Theory and Evidence*, 1980, Princeton University Press, Princeton; and Ellery Eells, *Rational Decision and Causality*, 1982, Cambridge University Press, Cambridge.

[3] For an interesting discussion of Leibniz's views on probability see Ian Hacking, *The Emergence of Probability*, 1975, Cambridge University Press, Cambridge. For Carnap's views, see *The Logical Foundations of Probability*, 1962, University of Chicago Press, Chicago.

[4] Keith Lehrer, "Evidence, Meaning and Conceptual Change: A Subjective Approach," in Pearce and Maynard (eds.), *Conceptual Change*, 1973, D. Reidel Publishing Company, Dordrecht-Holland, pp. 98-99.

[5] Some philosophers claim that objective probabilities do not satisfy the probability calculus. See Bas C. van Fraassen, *The Scientific Image*, 1980, Clarendon Press, Oxford, and Paul Humphreys, "Why Propensities Cannot Be Probabilities," *Philosophical Review*, 94, 1985, pp. 557-570.

[6] Let X stand for the objective probability of S is x, and let Y stand for the objective probability of S is y. We are interested in the value of $P(S/X \vee Y)$. Assume that $P(X \vee Y)=1$, in other words, the only two possibilities are that the objective probability of S is x or the objective probability of S is y. We then have $P(S/X \vee Y)=P[S\&(X \vee Y)]\div P(X \vee Y)=P[S\&(X \vee Y)]=P[S\&X) \vee (S\&Y)]=P[S\&X]+P[S\&Y]=P(X)P(S/X)+P(Y)P(S/Y)=$(by principle (4)) $P(X)*x+P(Y)*y=P(X)*x+[1-P(X)]*y=[P(X)*(x-y)]+y$. Thus $P(S/X \vee Y)=[P(X)*(x-y)]+y$. Suppose $P(S/X \vee Y)=x$. Then $[P(X)*(x-y)]+y=x$, and hence $P(X)=1$. Similar reasoning would show that if $P(S/X \vee Y)=y$, then $P(Y)=1$. Thus principle (4) implies that if we know that the objective probability of S is either x or y, then our subjective probability of S cannot be x unless we are certain that the objective probability of S is y.

[7] See David Lewis, "A Subjectivist's Guide to Objective Chance," in Richard Jeffrey (ed.), *Studies in Inductive Logic and Probability*, Volume 2, 1980, University of California Press, Berkeley, pp. 263-293; and Brian Skyrms, *Causal Necessity*, 1980, Yale University Press, New Haven.

[8] The primary motivation behind principle (2) is that it is irrational to believe some proposition which is not even possibly true. This requirement is similar to a requirement of rationality used in Dutch book theorems, which is that it is irrational to be willing to be in such a way that you are guaranteed to lose, no matter what happens; both conceptions claim it is irrational to be in a position in which one's goals cannot possibly be realized. The motivation behind principle (2′) appears similar to the motivation which one might give for a modified Dutch book argument, which would be that it is irrational to be in such a situation in which it is possible to realize one's goals and yet act in such a way that one knows that it is not possible that one's goals be achieved.