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Journal of Economic Theory 163 (2016) 684–698

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Endogenous information acquisition in Bayesian games with strategic complementarities ☆

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Received 5 June 2015; final version received 1 March 2016; accepted 7 March 2016

Available online 10 March 2016

Abstract

This paper studies covert (or hidden) information acquisition in common value Bayesian games of strategic complementarities. Using the supermodular stochastic order to arrange the structures of information increasingly in terms of preferences, we provide novel, easily interpretable and verifiable, though restrictive conditions under which the value of information is increasing and convex, and study the implications in terms of the equilibrium configuration. Increasing marginal returns to information leads to extreme behavior in that agents opt either for the highest or the lowest quality signal, so that the final analysis of this complex game simplifies greatly into that of a two-action game. This result can rationalize the complete information game as an endogenous outcome. Finally, we also establish that higher-quality information leads players to select more dispersed actions in the Bayesian game.

JEL classification: C44; C72; D83

Keywords: Information acquisition; Supermodular Bayesian games; Stochastic supermodular order; Costly rationality

☆ The authors thank Adib Bagh, Federico Echenique, Keisuke Hirano, Johannes Hoerner, Wei He, Sjaak Hurkens, Juan J. Ganuza, Amilcar Menichini, Jean-Francois Mertens, Alessandro Pavan, Edward Schlee, Tim Van Zandt, Xavier Vives, and John Wooders for their valuable suggestions.

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http://dx.doi.org/10.1016/j.jet.2016.03.005
0022-0531/

1. Introduction

The emergence of information economics, grounded on solid theoretical foundations, owes much to the theory of Bayesian games (Harsanyi, 1967, and Mertens and Zamir, 1985). The basic formulation of this class of games, adopted by most economic applications, posits an exogenously given structure of information. An unknown payoff-relevant parameter is part of the structure of the game and players receive some exogenous partial information (or message) that guides their decisions in the game. This message is usually costless and its reliability (quality) is fully out of the control of the players. Yet, in many of the natural economic settings, it would be more appropriate to postulate that players have the option of acquiring information on the unknown parameter, beyond what is readily available, provided they pay a corresponding cost.

We endogenize information acquisition in the class of common value Bayesian supermodular games of Van Zandt and Vives (2007), or VZ-V. These are games in which actions are strategic complements; for each player, own actions and the unknown are complements; and interim beliefs increase in messages. A typical example would be price-setting firms that compete in an industry with demand known up to a parameter whose prior distribution is common knowledge. We add an initial stage where players can purchase covert information as a noisy signal.

We show that, for the class of common value Bayesian supermodular games, the informativeness of the signal relies on its association with the state of the world, a feature nicely captured by the supermodular stochastic order. This result extends Athey and Levin (2001), who establish a positive value for information in monotone decision problems, to settings with strategic interactions (see Neyman, 1991, for a related analysis). To do so, we adapt some results from VZ-V to take care of the effects generated by extra information on rivals' behavior for one player. We also investigate how information affects agents' strategies in the action phase of the game. We find that more informative signals lead players to take more extreme actions. The intuition is easy to grasp: With signals of higher quality, players place more faith in the messages they receive and are, thus, encouraged to make more extreme decisions in the Bayesian game.[1]

The central part of the paper deals with the second order properties of the value of covert information as a fruitful approach to identify a subclass of models that can possess only extreme pure strategy Bayesian information equilibria. We propose a novel assumption of convexity of the information structure in the supermodular order. The interpretation of this notion of convexity is quite natural for the setting at hand: Higher quality raises the informativeness of the signal with increasing returns in signal quality. For this subclass, we show that for each player, all interior choices of signal quality are strictly dominated, so that only the extremal signal qualities can emerge at equilibrium. In this way, we reduce the complex existence issue for the full game to one involving a simple two-action (matrix) game. Although the level of generality does not allow us to settle in a systematic way the existence issue for this matrix game, this part should be easy to accomplish for a particular application, where the extremal information games are well structured and thus easier to compare in terms of payoffs.

The second order implications of information acquisition have been extensively investigated in the single-player case. A key general conclusion is that the value of information tends to be convex, mainly near zero: Radner and Stiglitz (1984) and Chade and Schlee (2002); see also Dimitrova and Schlee (2003). Our analysis represents an extension to a multi-agent setting, using the supermodular order to derive *conditions for global convexity of expected payoffs*. Intuitively,

1 Roux and Sobel (2015) have independently derived a similar result to explain polarization of group decisions.

the fact that a player's signal quality increases in the supermodular order at an increasing rate combines well with the complementarity structure of a Bayesian supermodular game to yield the strong conclusion that each player will always go for extreme information.[2] In other words, the fit between the two structures is such that the game at hand inherits the properties of a decision problem with the same convexity assumption (for which our main result clearly holds), even though all the underlying complex strategic interactions are fully taken into account.

The complexity of this class of games with general payoffs is such that many previous authors ended up solving convenient specifications with closed form solutions.[3] Given the strength of our conclusion, it will come as no surprise that the stochastic convexity assumption is indeed restrictive (more on this point later). The tacit claim here is not that we have identified the universal structure for a general analysis of these games, but rather one tractable structure, which nicely complements the existing natural setting for Bayesian games of VZ-V. One would then hope that the present paper might constitute one more step in an ongoing process, which may require progress on several fronts, including existence in other classes of Bayesian games.

This framework might be appropriate as a theory of costly rationality in strategic settings. Some well-known violations of the hyper-rationality paradigm, such as those commonly observed in laboratory experiments, might be better explained by a recognition that, in many environments, good decisions require the acquisition of costly information, rather than the common postulate that economic agents are inherently irrational or boundedly rational (Radner, 2000).

2. The analytical framework

We consider a general model of hidden (or covert) information acquisition for a broad class of common value Bayesian games of strategic complementarities studied by VZ-V. This class of games has existing applications in various models of oligopolistic competition, arms races, bank runs, and R&D. Adding endogenous information would be of interest for these applications.

2.1. Payoffs and information structures: maintained assumptions

The set of players is $N = \{1, \ldots, n\}$. Agent i chooses an action a_i from a compact metric lattice A_i.[4] His payoff is given by $u_i(a_i, \mathbf{a}_{-i}, \omega) : A_i \times \mathbf{A}_{-i} \times \Omega \to \mathbb{R}$ where $\mathbf{a}_{-i} \in \mathbf{A}_{-i} \equiv \Pi_{j \neq i, j \in N} A_j$ denotes the vector of other agents' actions, and $\omega \in \Omega \subseteq \mathbb{R}$ is a realization of an exogenous payoff relevant random variable (the state of the world). Agents choose their actions before the state of the world, $\widetilde{\omega}$, is realized. The prior distribution of $\widetilde{\omega}$, $H(\cdot)$, is common knowledge. We assume the basic Assumptions 1–3 below hold throughout the paper.

Assumption 1. For all i, $u_i(\mathbf{a}, \omega)$ is uniformly bounded, measurable in ω, and continuous in $\mathbf{a}$.

[2] With the (a priori plausible) dual assumption that the signal quality increases at a decreasing rate, the sufficient conditions to obtain concave expected payoffs in own information seem to be more restrictive (see on-line appendix). This suggests an alternative to the VZ-V class of games might be needed for such an approach.

[3] In the on-line appendix, we also provide one natural example as a simple illustration of the present approach.

[4] Although players' actions may be multi-dimensional, we do not use bold letters for $a_i \in A_i$, but only for joint action profiles $\mathbf{a} \in \mathbf{A} \equiv \Pi_{i \in N} A_i$, and for the action profiles of all players other than i, $\mathbf{a}_{-i} \in \mathbf{A}_{-i}$. As for notation, we use $\geq$ for partial orders in general. We use "greater than" and "increasing" in a weak sense. We distinguish random variables from their realizations by using wiggles, e.g., ω denotes a realization of the random variable $\widetilde{\omega}$.

Complementarities arise through two different channels. First, the incremental returns of agent i in own action increase in others' actions. Second, the marginal profitability of an increase in a player's action increases in the state of the world. To fix ideas, the reader might keep in mind a Bertrand oligopoly with differentiated substitute products and demand uncertainty. The next assumption defines our setup as a supermodular game parameterized by ω.[5]

Assumption 2. u_i is supermodular in a_i and has strictly increasing differences in $(a_i; \mathbf{a}_{-i}, \omega)$.

Player i acquires information about the state of the world by purchasing a noisy signal. Let $S_i \subseteq \mathbb{R}$ be the set of all possible signal realizations. As in Ganuza and Penalva (2010), player i chooses from a family of joint distributions $\{F(s_i, \omega; \alpha_i)\}$ indexed by his expenditure $\alpha_i \in [\underline{\alpha}, \overline{\alpha}]$, where $\overline{\alpha}$ is his budget and $\underline{\alpha} \geq 0$.[6] Each α_i generates a statistical experiment, and increasing α_i raises the informativeness (or the quality) of the signal in a way to be formalized below.

We assume that signals are independent conditional on ω. Let $\mathbf{s} \in \mathbf{S} \equiv \Pi_{i \in N} S_i$ and $\boldsymbol{\alpha} \in [\underline{\alpha}, \overline{\alpha}]^n$ denote the realization of a vector of signals and a quality profile, respectively. Each $\boldsymbol{\alpha}$ induces a joint cumulative distribution function (cdf) $F(\mathbf{s}, \omega; \boldsymbol{\alpha}) : \mathbf{S} \times \Omega \to [0, 1]$.

Assumption 3. The conditional cdf's of $F(\mathbf{s}, \omega; \boldsymbol{\alpha})$ satisfy (i) $F(\mathbf{s} | \omega; \boldsymbol{\alpha})$ is first order stochastically increasing in ω and (ii) $F(\mathbf{s}_{-i}, \omega | s_i; \boldsymbol{\alpha})$ is first order stochastically increasing in s_i.

Thinking of Nature as an additional player of type $\omega \in \Omega$, these conditions mean that players' interim beliefs increase in messages in the sense of first order stochastic dominance.

2.2. The game and the equilibrium concept

The timing of the game is as follows. First, player i chooses $\alpha_i \in [\underline{\alpha}, \overline{\alpha}]$ independently of the others. Then, upon observing the realization of his own signal s_i (but neither others' information qualities α's nor others' signals), he chooses an action $a_i \in A_i$ in the game that follows. We thus focus on covert (as opposed to overt) information acquisition.[7] We denote this game by Γ.

A pure strategy for player i in Γ consists of a pair (α_i, σ_i), where $\alpha_i \in [\underline{\alpha}, \overline{\alpha}]$ and $\sigma_i : S_i \to A_i$ is a Borel measurable function from messages into actions. Let Σ_i denote the set of all possible σ_i. Given a strategy profile $(\boldsymbol{\alpha}, \boldsymbol{\sigma})$, with $\boldsymbol{\alpha} \in [\underline{\alpha}, \overline{\alpha}]^n$ and $\boldsymbol{\sigma} \in \boldsymbol{\Sigma} \equiv \Pi_{i \in N} \Sigma_i$, let $(\boldsymbol{\alpha}_{-i}, \boldsymbol{\sigma}_{-i})$ denote the profile where players other than i follow their corresponding strategies at $(\boldsymbol{\alpha}, \boldsymbol{\sigma})$.

The formal definition of the equilibrium notion under consideration is now given.

[5] See, e.g., Milgrom and Roberts (1990), Topkis (1998), and Vives (1990).

[6] Consistency requires $\mathrm{E}[F(s_i, \omega; \alpha_i) | \omega; \alpha_i] = H(\omega)$ for all $\alpha_i \in [\underline{\alpha}, \overline{\alpha}]$.

[7] This timing is the same as the one adopted by Persico (2000) for first and second price auctions, and by Hauk and Hurkens (2001) and Vives (1988, 2008) for Cournot games. This timing assumption is suitable in information acquisition settings characterized by non-verifiability of signal qualities, absence of communication channels and informational spillovers between players, and short time periods between the two phases of the game. Naturally, one can also identify other economic settings for which a two-period timing (overt information acquisition) would be more appropriate. The present approach to equilibrium existence does not appear to extend to this more complex setting. Other than studies with specific functional forms (quadratic payoffs and Gaussian noise), we know of no general treatment of this case. Thus this is an important open question for future work on this topic.

Definition 1. A pure strategy profile $(\boldsymbol{\alpha}^*, \boldsymbol{\sigma}^*) \in [\underline{\alpha}, \overline{\alpha}]^n \times \boldsymbol{\Sigma}$ is a Bayesian Nash equilibrium with endogenous information of Γ if, for each $i \in N$,

$$(\alpha_i^*, \sigma_i^*) \in \arg\max_{\alpha_i \in [\underline{\alpha},\overline{\alpha}], \sigma_i \in \Sigma_i} \left\{ \int_{\mathbf{S},\Omega} u_i\left(\sigma_i(s_i), \boldsymbol{\sigma}_{-i}^*(\mathbf{s}_{-i}), \omega\right) dF\left(\mathbf{s}, \omega; \alpha_i, \boldsymbol{\alpha}_{-i}^*\right) - \alpha_i \right\}.$$

Although Γ is a one-shot game, we distinguish between stage I (information acquisition) and stage II (action choice). Following Hauk and Hurkens (2001), we first consider the action stage game assuming an exogenous profile of information $\boldsymbol{\alpha}$ (denote this game by $\Gamma_{II}(\boldsymbol{\alpha})$). We then study incentives for unilateral deviations from $\boldsymbol{\alpha}$. An equilibrium of Γ happens when no such incentives are present at an equilibrium profile of the exogenous information game $\Gamma_{II}(\boldsymbol{\alpha})$.

3. The information acquisition game

3.1. Monotone equilibria for $\Gamma_{II}(\boldsymbol{\alpha})$ and incentives to deviate from $\boldsymbol{\alpha}$

Let $\boldsymbol{\alpha} \in [\underline{\alpha}, \overline{\alpha}]^n$ denote any exogenous information profile. We start with the analysis of the fictitious exogenous information game $\Gamma_{II}(\boldsymbol{\alpha})$, described above. The set of strategies Σ_i is a lattice when ordered with the pointwise (partial) order, i.e., $\sigma_i \geq \sigma_i'$ if $\sigma_i(s_i) \geq \sigma_i'(s_i)$ for every $s_i \in S_i$. The first existence result is the main conclusion of VZ-V (and Van Zandt, 2010).

Lemma 2. *For any given $\boldsymbol{\alpha} \in [\underline{\alpha}, \overline{\alpha}]^n$, $\Gamma_{II}(\boldsymbol{\alpha})$ has a greatest and a least Bayesian Nash equilibrium, both of which are monotone increasing and Borel measurable in own message.*

Henceforth, for a given $\boldsymbol{\alpha}$, we select the maximal equilibrium $\overline{\boldsymbol{\sigma}}$ for the game $\Gamma_{II}(\boldsymbol{\alpha})$. The monotonicity of $\overline{\boldsymbol{\sigma}}$ is a critical feature of our approach to order the information structures. Our analysis remains valid for any other monotone selection, e.g., the minimal equilibrium of $\Gamma_{II}(\boldsymbol{\alpha})$.

The next step in the analysis is to investigate conditions under which no player i has incentives to deviate from a given $\boldsymbol{\alpha}$. We initially focus on the returns to information acquisition. Two key aspects deserve attention. First, since we model hidden information acquisition, player i's deviation from α_i to α_i' will not be observable to others, so that there will be no strategic effect on the other players. Second, since player i knows the quality of his own signal, if he deviates at stage I to a given α_i', he will need to adjust his strategy $\overline{\sigma}_i(\cdot)$ accordingly at stage II to continue to best respond to the fixed strategy $\overline{\boldsymbol{\sigma}}_{-i}(\cdot)$. Formally, if player i unilaterally deviates from α_i to α_i' in stage I, he would switch from the strategy $\overline{\sigma}_i(\cdot)$ to a selection of

$$\varphi_i\left(s_i; \alpha_i'\right) = \arg\max_{a_i \in A_i} \int_{\mathbf{S}_{-i},\Omega} u_i\left(a_i, \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right) dF\left(\mathbf{s}_{-i}, \omega \,\middle|\, s_i; \alpha_i', \boldsymbol{\alpha}_{-i}\right). \quad (1)$$

The next lemma uses intermediate results in VZ-V and Van Zandt (2010).

Lemma 3. *The maximal and the minimal selections of $\varphi_i\left(s_i; \alpha_i'\right)$, $\overline{\varphi}_i\left(s_i; \alpha_i'\right)$ and $\underline{\varphi}_i\left(s_i; \alpha_i'\right)$, exist, are increasing, and Borel measurable in s_i.*

Using Lemma 3, the maximum expected payoff that player i can get (net of information costs) if he unilaterally deviates from α_i to α_i' in stage I is given by

$$U_i\left(\alpha_i', \boldsymbol{\alpha}\right) = \int_{S_i \times \Omega} \int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i; \alpha_i'\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right) dF\left(\mathbf{s}_{-i} \,|\, \omega; \boldsymbol{\alpha}_{-i}\right) dF\left(s_i, \omega; \alpha_i'\right). \quad (2)$$

3.2. Comparing information structures

Building on our previous results, this section characterizes the properties of $U_i\left(\alpha_i', \boldsymbol{\alpha}\right)$ as a function of α_i'. It also sheds light on the effect of α_i' on players' behavior at stage II.

3.2.1. First-order effects of the quality of information

To ascertain the impact of changing α_i' on $U_i\left(\alpha_i', \boldsymbol{\alpha}\right)$ we impose more structure on the family $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$. We assume $F(s_i, \omega; \alpha_i)$ increases in α_i in the supermodular (spm) stochastic order. Though close in spirit to Blackwell's (1953) order, the spm order is more appropriate here in view of the complementarity structure on payoffs.[8] As this order is central to our analysis, we state a formal definition and a convenient characterization (Tchen, 1980; Epstein and Tanny, 1980; Meyer and Strulovici, 2012).

Definition 4. Let $F(s_i, \omega; \alpha_i)$ and $F\left(s_i, \omega; \alpha_i'\right)$ be cdf's with the same marginals and $\alpha_i > \alpha_i'$. We say that $F(s_i, \omega; \alpha_i)$ is larger than $F\left(s_i, \omega; \alpha_i'\right)$ in the spm order if

$$F(s_i, \omega; \alpha_i) \geq F\left(s_i, \omega; \alpha_i'\right) \ \forall (s_i, \omega) \in S_i \times \Omega. \tag{3}$$

That is, if $F(s_i, \omega; \alpha_i)$ is increasing in α_i on $\left[\underline{\alpha}, \overline{\alpha}\right]$.

The spm order applies only to families $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$ that share the same marginals, as we assume.[9] Müller and Stoyan (2002) is a thorough survey. A useful characterization follows.

Lemma 5. *Inequality (3) holds if and only if*

$$\int\limits_{S_i \times \Omega} h(s_i, \omega)\, dF(s_i, \omega; \alpha_i) \geq \int\limits_{S_i \times \Omega} h(s_i, \omega)\, dF\left(s_i, \omega; \alpha_i'\right) \tag{4}$$

for all supermodular functions $h(s_i, \omega)$ for which these two expectations exist.

The next example offers a general way of constructing such a class of cdf's via mixing.

Example 6. Let $M(s_i, \omega)$ and $N(s_i, \omega)$ be two cdf's with the same marginals, and $M(s_i, \omega) \geq N(s_i, \omega)$, i.e. M is larger than N in the spm order. Assume $k : \left[\underline{\alpha}, \overline{\alpha}\right] \to [0, 1]$ is a smooth function, with $k'(\cdot) \geq 0$. Then $F(s_i, \omega; \alpha_i) = k(\alpha_i) M(s_i, \omega) + [1 - k(\alpha_i)] N(s_i, \omega)$ is a cdf with the same marginals as M and N, and $F(s_i, \omega; \alpha_i)$ increases in α_i in the spm order since

$$\frac{\partial F(s_i, \omega; \alpha_i)}{\partial \alpha_i} = k'(\alpha_i)\left[M(s_i, \omega) - N(s_i, \omega)\right] \geq 0.$$

The next result uses this order to convey a precise meaning to the quality of information.

[8] Due to its ordinal nature, Lehmann's (1988) order is not suitable, since our payoffs involve summations. In fact, the latter point is another key reason the VZ-V setting is appropriate here.

[9] We now argue that this assumption is actually w.l.o.g. Assume $\widetilde{s}_i$ has a continuous cdf. Convert player i's message s_i to $z_i = F(s_i; \alpha_i)$, which has the same informational content. Since signals enter payoffs in the game Γ only through strategies, this so-called "probability integral transformation" does not affect the game in any way, and makes $\widetilde{z}_i$ uniformly distributed on [0, 1]. Hence $\widetilde{z}_i$ is independent of α_i. Let's define $G(z_i, \omega; \alpha_i) = F\left(F^{-1}(z_i; \alpha_i), \omega; \alpha_i\right)$, where $F^{-1}(z_i; \alpha_i)$ is the right-continuous inverse of the marginal distribution $F(s_i; \alpha_i)$. It is easily verified that the new family $\{G(z_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$ share the same marginals.

Proposition 7. *If $F(s_i, \omega; \alpha_i)$ increases in α_i in the spm order, $U_i(\alpha_i', \boldsymbol{\alpha})$ increases in α_i'.*

This result has a simple economic interpretation. The strategic complementarities in the payoffs and the monotone restrictions of the conditional cdf's (A2 and A3) together with the fact that other players follow increasing strategies, lead player i to prefer high (low) actions when he predicts high (low) values of ω. When information structures are spm ordered, a higher α_i' increases the association between $\widetilde{s}_i$ and $\widetilde{\omega}$. This allows player i a better match between his actions, the state of the world and other players' choices, thereby increasing his expected payoff.

Athey and Levin (2001) establish a similar result for single-agent monotone decision problems, where the signal helps the agent to predict the state of the world. In our game, the signal plays a dual role: It helps each player to predict the state of the world as well as the equilibrium actions of the other players. The proof of Proposition 7 extends Athey and Levin (2001) by showing that, under A2–A3, the association between player i's signal and the state of the world is preserved after integrating over his rivals' equilibrium strategies. This step uses some results from VZ-V, e.g., the monotonicity of equilibrium strategies in signal realizations.

With the goal of providing a critique of the oft-made claim that more information to a player can hurt him in a game, Neyman (1991) argues that a Bayesian game is the appropriate setting to show that a player's welfare always increases with more unilateral information, of which his rivals are not aware. Though related to Proposition 7, Neyman's setting is different in that he considers the states-partition version of (finite) Bayesian games, with more information to a player being in the inclusion sense, i.e., a finer partition of the states of nature for that player.

We finally show that player i's strategy becomes more spread-out as he acquires more accurate signals. The intuition is simple: With higher-quality signals, he places more faith in his messages, and this encourages him to make more extreme decisions in the game that follows.

Proposition 8. *Assume the signal space S_i is either a compact real interval or a finite set, and $F(s_i, \omega; \alpha_i)$ increases in α_i in the spm order. For each $\alpha_i > \alpha_i'$, let*

$$s_i^*(\alpha_i, \alpha_i') = \inf_\omega \sup_t \left(\left\{t : F(\omega \mid t'; \alpha_i) - F(\omega \mid t'; \alpha_i') \geq 0 \text{ for all } t' \leq t\right\}\right)$$
$$s_i^{**}(\alpha_i, \alpha_i') = \sup_\omega \inf_t \left(\left\{t : F(\omega \mid t'; \alpha_i) - F(\omega \mid t'; \alpha_i') \leq 0 \text{ for all } t' \geq t\right\}\right).$$

Then, we have that

$$\overline{\varphi}_i(s_i; \alpha_i) \leq \overline{\varphi}_i(s_i; \alpha_i') \ \forall s_i < s_i^*(\alpha_i, \alpha_i') \quad \text{and} \quad \overline{\varphi}_i(s_i; \alpha_i) \geq \overline{\varphi}_i(s_i; \alpha_i') \ \forall s_i > s_i^{**}(\alpha_i, \alpha_i').$$

Levin (2001, pp. 665–666) provides examples of a spm-ordered family of cdf's that rotate around the unconditional mean of the signal as α_i varies. This also happens in our Motivating Example in the on-line appendix. In such settings, players' strategies rotate clockwise around the unconditional mean of s_i as the information becomes more precise, i.e., in terms of our Proposition 8, $E(s_i) = s_i^*(\alpha_i, \alpha_i') = s_i^{**}(\alpha_i, \alpha_i')$ for all $\alpha_i > \alpha_i'$. In such cases, our characterization of the effects of more accurate signals on players' behavior is fully informative.[10]

[10] A more complex case would have $\min(S_i) < s_i^*(\alpha_i, \alpha_i') \leq s_i^{**}(\alpha_i, \alpha_i') < \max(S_i)$. This happens, e.g., when S_i is finite (as in Roux and Sobel, 2015). However, in general one cannot rule out that $s_i^* = \min(S_i)$ and/or that $s_i^{**} = \max(S_i)$, which might render this result vacuous. This is due to the fact that $F(\omega \mid t; \alpha_i)$ may be badly behaved (highly oscillatory) for t near the end points of S_i, due to the fact that conditional cdf's are only measurable w.r.t. parameters. A sufficient condition to rule out $s_i^* = \min(S_i)$ and $s_i^{**} = \max(S_i)$ is to assume that $F(\omega \mid t; \alpha_i)$ is real-analytic in t, i.e., that it admits a power series expansion that converges to it (see, e.g., Krantz and Parks, 2002). These points can be seen from the proof of Proposition 8.

3.2.2. Second-order effects of information quality

We now study the second order effects of α_i' on $U_i\left(\alpha_i', \boldsymbol{\alpha}\right)$. While some of our previous results have related antecedents in the literature on information and games/decisions, the upcoming ones have no analogs in this literature. Besides their independent interest, the properties derived here are key to the study of equilibrium for the game Γ, and form the central part of the paper.[11]

We introduce a natural and useful notion of convexity in the spm order.

Definition 9. Assume the family $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$ shares the same marginals. We say that $F(s_i, \omega; \alpha_i)$ is convex in α_i in the spm order if, $\forall \alpha_i, \alpha_i' \in [\underline{\alpha}, \overline{\alpha}]$, $\forall \lambda \in [0, 1]$,

$$F\left(s_i, \omega; \lambda\alpha_i + (1-\lambda)\alpha_i'\right) \leq \lambda F(s_i, \omega; \alpha_i) + (1-\lambda) F\left(s_i, \omega; \alpha_i'\right) \ \forall (s_i, \omega) \in S_i \times \Omega. \quad (5)$$

That is, if $F(s_i, \omega; \alpha_i)$ is convex in α_i on $[\underline{\alpha}, \overline{\alpha}]$.

We now offer a characterization of this concept via expectations of supermodular functions.

Lemma 10. *$F(s_i, \omega; \alpha_i)$ is convex in α_i in the spm order iff, $\forall \alpha_i, \alpha_i' \in [\underline{\alpha}, \overline{\alpha}]$, $\forall \lambda \in [0, 1]$,*

$$\int_{S_i \times \Omega} h(s_i, \omega)\, dF\left(s_i, \omega; \alpha_i''\right) \leq \lambda \int_{S_i \times \Omega} h(s_i, \omega)\, dF(s_i, \omega; \alpha_i) + (1-\lambda) \int_{S_i \times \Omega} h(s_i, \omega)\, dF\left(s_i, \omega; \alpha_i'\right) \quad (6)$$

(here $\alpha_i'' = \lambda\alpha_i + (1-\lambda)\alpha_i'$) for all supermodular functions h for which the expectations exist.

The spm order ranks informativeness in the sense that a higher α_i leads to higher chances of observing high (low) realizations of the signal when the state of the world is high (low). This new notion of convexity means that increasing α_i raises informativeness with increasing returns.

We now provide some insights into the level of generality of the convexity assumption. In addition to having a natural economic interpretation in terms of increasing returns to information, convexity in the spm order enjoys several nice properties, such as preservation by useful operations such as convex combinations, pointwise maxima, and weak* limits. On the other hand, as is often the case with powerful assumptions (as this is shown to be below), this convexity property imposes substantial restrictions on the family of distributions. The first is that it restricts the informativeness of signals to increase without bound in the investment. Another point is that, since convexity in the spm order amounts to $F(s_i, \omega; \alpha_i)$ being convex in α_i, this distribution can possibly have atoms only along constant s_i or ω lines, so that the location of these atoms is always independent of the values of α_i.[12] This is clearly a potentially significant limitation in the present context since a higher quality of information necessarily leaves invariant those values of the signal and the fundamental (the atoms) that have a strictly positive probability. In this respect though, it is worth recalling that most studies in information economics assume non-atomic distributions (or density functions) at the outset.

[11] Related notions of stochastic convexity of integrals w.r.t. parameters of transition probabilities have been used in economic dynamics in Amir (1996a, 1996b and 1997) for first and second order stochastic dominance, respectively.

[12] In Example 11 below, $F(s_i, \omega; \alpha_i)$ inherits the atoms of the distributions M and N (as these are not assumed atomless), but the location of the atoms is invariant in α_i, as is easily seen from the definition of $F(s_i, \omega; \alpha_i)$.

We present a simple method to construct cdf families satisfying convexity in the spm order.

Example 11. Consider the family $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$ in Example 6. We show that if $M(s_i, \omega) \geq N(s_i, \omega)$, then $F(s_i, \omega; \alpha_i)$ is convex in α_i in the spm order if and only if $k''(\cdot) \geq 0$. Indeed,

$$\frac{\partial^2 F(s_i, \omega; \alpha_i)}{\partial \alpha_i^2} = k''(\alpha_i)\left[M(s_i, \omega) - N(s_i, \omega)\right] \geq 0.$$

This shows that the convex spm order can easily arise for an infinite family via simple mixing.

The usefulness of postulating information structures that are monotone and convex in the spm order for the VZ-V class of Bayesian games lies in the next key result for our analysis.

Theorem 12. *If $F(s_i, \omega; \alpha_i)$ is convex in α_i in the spm order, $U_i\left(\alpha_i', \boldsymbol{\alpha}\right)$ is convex in α_i'.*

The proof of Theorem 12 relies on two key steps. Lemma 10 states that if $F(s_i, \omega; \alpha_i)$ is convex in α_i in the spm order, the expected value of any supermodular function is convex in α_i as well. So the first part of the proof consists in showing that the problem of player i can be rewritten in terms of the expectation of a supermodular function in (s_i, ω). The second part relies on the fact that the pointwise maximum of any collection of convex functions is convex.

This theorem is consistent with well known results for decision problems showing that the value of information has a tendency towards a convex shape (Radner and Stiglitz, 1984, among other applications), and is convex near zero in general (Chade and Schlee, 2002).

Remark 13. All the notions and results in this section have dual counterparts in terms of concavity, except for Theorem 12, which seems to require stronger assumptions to yield concave payoffs according to the approach pursued here. Thus, while concave information structures are a priori perfectly plausible in some settings, they appear not to go well with the VZ-V complementarity structure to yield another tractable subclass of games. An alternative approach might be needed for the concave case with general payoffs, a topic of obvious interest for future work, as it would generalize the many economic applications known to date (in oligopoly and financial settings) where concave quadratic payoffs emerge under Gaussian information (Vives, 2008).

3.3. Bayesian Nash equilibrium with endogenous information

This section uses our previous results to characterize the equilibria of Γ. Considering the cost of information, a pure strategy profile $\left(\boldsymbol{\alpha}^*, \overline{\boldsymbol{\sigma}}^*\right)$ is a Bayesian Nash equilibrium of Γ if

$$\alpha_i^* \in \arg\max\left\{U_i\left(\alpha_i, \boldsymbol{\alpha}^*\right) - \alpha_i : \alpha_i \in \left[\underline{\alpha}, \overline{\alpha}\right]\right\} \text{ for all } i \in N, \tag{7}$$

that is, if no player has incentives to deviate from α^* at the highest equilibrium of $\Gamma_{II}(\alpha^*)$.

Proposition 7 and Theorem 12 state general conditions for $U_i\left(\alpha_i', \boldsymbol{\alpha}\right)$ to be increasing and convex in α_i'. Hence, the maximand in (7) is also convex in α_i', and its argmax is always given by an extremal element of the constraint set, as if the players were in a binary action game.

Lemma 14. *Assume $F(s_i, \omega; \alpha_i)$ is increasing and convex in α_i in the spm order for all $i \in N$. Then, the game with action set $[\underline{\alpha}, \overline{\alpha}]$ and payoff functions given by (7) is strategically equivalent (generically) to the game with action set $\{\underline{\alpha}, \overline{\alpha}\}$ and payoffs given by (7).*[13]

The power of the stochastic convexity assumption is captured by Lemma 14. The first implication is of a mathematical nature: The analysis of this complex class of games is drastically simplified. The resulting convexity of payoffs implies they are continuous in own action α_i on $(\underline{\alpha}, \overline{\alpha})$ and u.s.c. at $\underline{\alpha}$ and $\overline{\alpha}$, thereby dispensing with the need for dealing with measurability or other regularity conditions, and intricate fixed point considerations. The second is of a behavioral nature: Equilibrium entails the clear prediction of extreme behavior in information acquisition (i.e., for any initial $\boldsymbol{\alpha}$, every player has an incentive to deviate to $\underline{\alpha}$ or $\overline{\alpha}$).

We are ready to characterize the possible equilibria that can arise in the entire game Γ.

Proposition 15. *Assume the conditions of Lemma 14 hold. Then, we have that*

(i) any equilibrium profile $\boldsymbol{\alpha}^*$ that satisfies (7) is given by $\boldsymbol{\alpha}^* \in \{\underline{\alpha}, \overline{\alpha}\}^n$;
(ii) $\overline{\boldsymbol{\alpha}}$ is an equilibrium profile if $U_i(\overline{\alpha}, \overline{\boldsymbol{\alpha}}) - U_i(\underline{\alpha}, \overline{\boldsymbol{\alpha}}) \geq \overline{\alpha} - \underline{\alpha}$, $\forall i \in N$; and
(iii) $\underline{\boldsymbol{\alpha}}$ is an equilibrium profile if $U_i(\underline{\alpha}, \underline{\boldsymbol{\alpha}}) - U_i(\overline{\alpha}, \underline{\boldsymbol{\alpha}}) \geq \underline{\alpha} - \overline{\alpha}$, $\forall i \in N$.

If $\overline{\alpha}$ is the full information signal, one that reveals the state of the world with certainty, then Proposition 15 (ii) states sufficient conditions for the complete information game to emerge endogenously. Much applied work using game theory assumes the economic fundamentals are known with certainty. Our result identifies environments where this assumption can be fully justified. Conversely, in settings where the lowest signal is fully uninformative (i.e., the uniform distribution), and information is relatively costly, in equilibrium, the Bayesian game would involve level-1 play by all players (Stahl and Wilson, 1995). In the latter case, our approach would identify the least rational form of play commonly invoked in experimental economics as a fully rational outcome of an appropriately extended game with endogenous information.

Conditions (ii) and (iii) of Proposition 15 are not mutually exclusive: Both $\overline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\alpha}}$ are equilibria in settings where $U_i(\overline{\alpha}, \overline{\boldsymbol{\alpha}}) - U_i(\underline{\alpha}, \overline{\boldsymbol{\alpha}}) \geq \overline{\alpha} - \underline{\alpha} \geq U_i(\overline{\alpha}, \underline{\boldsymbol{\alpha}}) - U_i(\underline{\alpha}, \underline{\boldsymbol{\alpha}})$, for all i. This entails that $U_i(\alpha_i', \boldsymbol{\alpha})$ be supermodular in $(\alpha_i', \boldsymbol{\alpha})$ at the extreme information profiles. In addition, our result allows for the possibility of extreme asymmetric equilibria, where some players opt for the full information signal and others remain uninformed.

Elaborating on the previous remark, since it is well-known that every 2×2 game is either a supermodular game or a matching-pennies game (Echenique, 2004), the game at hand for the case of two players is also of one of these two types.[14] It follows that the game defined by (7) has a pure-strategy equilibrium if and only if the 2×2 game with action space $\{\underline{\alpha}, \overline{\alpha}\}$ is supermodular with respect to one of the four possible ways of ordering the two binary action sets in the spirit of Echenique (2004). In this sense, our approach to ranking information structures leads to existence of pure-strategy equilibrium in the two-stage game by exploiting strategic complementarities in both stages of the game, with the important caveat that these complementarities are required only for the two extreme information levels. Although complementarities do emerge for all levels of information acquisition in a variety of models with linear-quadratic payoffs and Gaussian

[13] "Generically" here is meant to rule out the uninteresting case where $U_i(\alpha_i', \boldsymbol{\alpha}) - \alpha_i'$ is a constant.

[14] A matching-pennies game is a 2×2 game with a unique Nash equilibrium, which is in mixed-strategies.

information (such as Vives, 1988, and Hellwig and Veldkamp, 2009), the scope for extending this powerful property to general formulations appears rather limited. Thus the present relaxation of this requirement, that it holds only for the two extreme levels of information acquisition, appears promising for economic applications that fit the VZ-V Bayesian structure and our assumption of increasing returns in information.

Arrow (1974) and Radner (2000) point out that the production of information often has increasing returns to scale, due to the presence of fixed costs. Adding fixed costs in information acquisition would bias results in our favor. The implied increasing returns (declining average cost of information) would favor full information, and in cases of high fixed costs may trigger the fully uninformative outcome. In addition, if players purchase the desired information from specialized suppliers, it is plausible to postulate a concave cost schedule as a reflection of pricing policies based on volume discounts (e.g., a piece-wise linear concave function). This would strongly reinforce the qualitative nature of our extreme information results.

4. Proofs

This section collects all the proofs of this paper, and some intermediate results. We begin with the usual characterization of (multi-dimensional) first order stochastic dominance.

Lemma 16. *Let $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{x}}'$ denote two random vectors with support $\mathbf{X} \subseteq \mathbb{R}^n$ and respective cdf's $F(\cdot)$ and $F'(\cdot)$. For any $n \geq 1$, $\widetilde{\mathbf{x}}$ first order stochastically dominates $\widetilde{\mathbf{x}}'$, denoted $F \succeq_1 F'$, if*

$$\int_{\mathbf{X}} h(\mathbf{x})\, dF(\mathbf{x}) \geq \int_{\mathbf{X}} h(\mathbf{x})\, dF'(\mathbf{x})$$

for all bounded increasing measurable functions $h : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Proof of Lemma 2. For any given $\boldsymbol{\alpha} \in \left[\underline{\alpha}, \overline{\alpha}\right]^n$, A1–A3 guarantee that $\Gamma_{II}(\boldsymbol{\alpha})$ is a supermodular Bayesian game as defined by VZ-V. Existence of extremal Bayesian Nash equilibria, in strategies increasing in own signal, follows from their main result on p. 344. Borel measurability of extremal strategies is shown in Van Zandt (2010). □

Proof of Lemma 3. It follows from Propositions 8 and 11 in VZ-V (and Van Zandt, 2010). □

Proof of Proposition 7. This proof consists of two steps. Step 1 shows that for any $\sigma_i \in \Sigma_i$ that is increasing, $\int_{\mathbf{S}_{-i}} u_i(\sigma_i(s_i), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega)\, dF(\mathbf{s}_{-i} | \omega; \boldsymbol{\alpha}_{-i})$ has increasing differences in (s_i, ω). Step 2 uses this result to show that $U_i\left(\alpha_i', \boldsymbol{\alpha}\right)$ increases in α_i'.

Step 1. Assume $\sigma_i \in \Sigma_i$ is increasing, $s_i \geq s_i'$ and $\omega \geq \omega'$, and consider the next inequalities

$$\begin{aligned}
&\int_{\mathbf{S}_{-i}} u_i(\sigma_i(s_i), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega)\, dF(\mathbf{s}_{-i} | \omega; \boldsymbol{\alpha}_{-i}) \\
&- \int_{\mathbf{S}_{-i}} u_i\left(\sigma_i\left(s_i'\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right) dF(\mathbf{s}_{-i} | \omega; \boldsymbol{\alpha}_{-i}) \\
= &\int_{\mathbf{S}_{-i}} \left[u_i(\sigma_i(s_i), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega) - u_i\left(\sigma_i\left(s_i'\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right)\right] dF(\mathbf{s}_{-i} | \omega; \boldsymbol{\alpha}_{-i}) \\
\geq &\int_{\mathbf{S}_{-i}} \left[u_i(\sigma_i(s_i), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega) - u_i\left(\sigma_i\left(s_i'\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right)\right] dF\left(\mathbf{s}_{-i} \middle| \omega'; \boldsymbol{\alpha}_{-i}\right)
\end{aligned}$$

$$\begin{aligned}&\geq \int_{\mathbf{S}_{-i}} \left[u_i\left(\sigma_i\left(s_i\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega'\right) - u_i\left(\sigma_i\left(s_i'\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega'\right)\right] dF\left(\mathbf{s}_{-i}\left|\omega'; \boldsymbol{\alpha}_{-i}\right.\right)\\ &= \int_{\mathbf{S}_{-i}} u_i\left(\sigma_i\left(s_i\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega'\right) dF\left(\mathbf{s}_{-i}\left|\omega'; \boldsymbol{\alpha}_{-i}\right.\right)\\ &\quad - \int_{\mathbf{S}_{-i}} u_i\left(\sigma_i\left(s_i'\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega'\right) dF\left(\mathbf{s}_{-i}\left|\omega'; \boldsymbol{\alpha}_{-i}\right.\right).\end{aligned}$$

First, since $\overline{\boldsymbol{\sigma}}_{-i}$ is increasing, $u_i\left(\sigma_i\left(s_i\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega\right) - u_i\left(\sigma_i\left(s_i'\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega\right)$ increases in $\mathbf{s}_{-i}$ by A2. In addition, by A3, $F\left(\mathbf{s}\left|\omega; \boldsymbol{\alpha}\right.\right)$ increases in ω in the sense of first order stochastic dominance. Then, by property MA of Theorem 3.3.10 in Müller and Stoyan (2002, p. 94), $F\left(\mathbf{s}_{-i}\left|\omega; \boldsymbol{\alpha}_{-i}\right.\right)$ increases in ω according to the same order. Thus, the first inequality follows by Lemma 16. By A2, $u_i\left(\sigma_i\left(s_i\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega\right) - u_i\left(\sigma_i\left(s_i'\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega\right)$ increases in ω, for all $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$; this justifies the second inequality. This completes Step 1.

Step 2. Let $\overline{\varphi}_i\left(s_i; \alpha_i'\right)$ be the maximal selection of $\varphi_i\left(s_i; \alpha_i'\right)$, as in (1), and $\alpha_i' > \alpha_i''$. Then,

$$\begin{aligned}U_i\left(\alpha_i', \boldsymbol{\alpha}\right) &= \int_{S_i \times \Omega} \int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i; \alpha_i'\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega\right) dF\left(\mathbf{s}_{-i}\left|\omega; \boldsymbol{\alpha}_{-i}\right.\right) dF\left(s_i, \omega; \alpha_i'\right)\\ &\geq \int_{S_i \times \Omega} \int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i; \alpha_i''\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega\right) dF\left(\mathbf{s}_{-i}\left|\omega; \boldsymbol{\alpha}_{-i}\right.\right) dF\left(s_i, \omega; \alpha_i'\right)\\ &\geq \int_{S_i \times \Omega} \int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i; \alpha_i''\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega\right) dF\left(\mathbf{s}_{-i}\left|\omega; \boldsymbol{\alpha}_{-i}\right.\right) dF\left(s_i, \omega; \alpha_i''\right)\\ &= U_i\left(\alpha_i'', \boldsymbol{\alpha}\right).\end{aligned}$$

The two equalities are true by definition and because players' signals are assumed independent given the state of the world, i.e. $F\left(\mathbf{s}_{-i}\left|s_i, \omega; \boldsymbol{\alpha}_{-i}\right.\right) = F\left(\mathbf{s}_{-i}\left|\omega; \boldsymbol{\alpha}_{-i}\right.\right)$. The first inequality follows by optimality. We know, by Lemma 3, that $\overline{\varphi}_i\left(s_i; \alpha_i''\right)$ increases in s_i. Then Step 1 guarantees the integral $\int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i; \alpha_i'\right), \overline{\boldsymbol{\sigma}}_{-i}\left(\mathbf{s}_{-i}\right), \omega\right) dF\left(\mathbf{s}_{-i}\left|\omega; \boldsymbol{\alpha}_{-i}\right.\right)$ is supermodular in (s_i, ω). Since $F\left(s_i, \omega; \alpha_i\right)$ increases in α_i in the spm order, the second inequality follows by Lemma 5. □

Proof of Proposition 8. The cdf $F\left(\omega, s_i; \alpha_i\right)$ and the survival function $\overline{F}\left(\omega, s_i; \alpha_i\right)$ write as

$$\begin{aligned}F\left(\omega, s_i; \alpha_i\right) &= \int_{\min(S_i)}^{s_i} F\left(\omega\left|t; \alpha_i\right.\right) dF(t) \text{ and}\\ \overline{F}\left(\omega, s_i; \alpha_i\right) &= 1 - F\left(s_i\right) - \int_{s_i}^{\max(S_i)} F\left(\omega\left|t; \alpha_i\right.\right) dF(t),\end{aligned}$$

where $F\left(s_i\right)$ stands for the marginal cdf of player i's message, and $F\left(\omega\left|t; \alpha_i\right.\right)$ is the conditional cdf of the fundamental given a message $s_i = t$. Let $\alpha_i, \alpha_i' \in \left[\underline{\alpha}, \overline{\alpha}\right]$ satisfy $\alpha_i > \alpha_i'$. Since $F\left(s_i, \omega; \alpha_i\right)$ increases in α_i in the spm order, we have that, for all $s_i \in S_i$ and $\omega \in \Omega$,

$$\int_{\min(S_i)}^{s_i} \left[F\left(\omega\left|t; \alpha_i\right.\right) - F\left(\omega\left|t; \alpha_i'\right.\right)\right] dF(t) = F\left(\omega, s_i; \alpha_i\right) - F\left(\omega, s_i; \alpha_i'\right) \geq 0, \tag{8}$$

$$\int_{s_i}^{\max(S_i)} \left[F\left(\omega\left|t; \alpha_i\right.\right) - F\left(\omega\left|t; \alpha_i'\right.\right)\right] dF(t) = \overline{F}\left(\omega, s_i; \alpha_i'\right) - \overline{F}\left(\omega, s_i; \alpha_i\right) \leq 0, \tag{9}$$

$$\int_{\min(S_i)}^{\max(S_i)} \left[F\left(\omega\left|t; \alpha_i\right.\right) - F\left(\omega\left|t; \alpha_i'\right.\right)\right] dF(t) = H(\omega) - H(\omega) = 0,$$

where (9) holds since the spm stochastic order implies that $\overline{F}(\omega, s_i; \alpha_i)$ is increasing in α_i for bivariate distributions (Müller and Stoyan, 2002).

Given (8), the set $T(\omega) \triangleq \{t : F(\omega | t'; \alpha_i) - F(\omega | t'; \alpha_i') \geq 0 \text{ for all } t' \leq t\}$ is non-empty. Since S_i is compact, $T(\omega)$ has a supremum in S_i for each ω. Then $s_i^*(\alpha_i, \alpha_i') = \inf_\omega \sup_t T(\omega)$ is in S_i. By construction, $F(\omega | s_i; \alpha_i') \succeq_1 F(\omega | s_i; \alpha_i)$ for all $s_i < s_i^*(\alpha_i, \alpha_i')$. Proceeding in a similar way, we get $F(\omega | s_i; \alpha_i) \succeq_1 F(\omega | s_i; \alpha_i')$ for all $s_i > s_i^{**}(\alpha_i, \alpha_i')$.

By A3, $F(\mathbf{s} | \omega; \boldsymbol{\alpha})$ first order stochastically increases in ω. By Müller and Stoyan (2002, p. 94), property MA of Theorem 3.3.10, $F(\mathbf{s}_{-i} | \omega; \boldsymbol{\alpha}_{-i})$ increases in ω according to the same order. In addition, by Lemma 2, $\overline{\boldsymbol{\sigma}}_{-i}$ is increasing, and, by A2, u_i is supermodular in a_i and has increasing differences in $(a_i; \mathbf{a}_{-i}, \omega)$. Therefore, if $a_i \geq a_i'$ and $w \geq w'$, by Lemma 16

$$\begin{aligned} I(\omega; \boldsymbol{\alpha}_{-i}) &\triangleq \int_{\mathbf{S}_{-i}} \left[u_i(a_i, \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega) - u_i(a_i', \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega)\right] dF(\mathbf{s}_{-i} | \omega; \boldsymbol{\alpha}_{-i}) \\ &\geq \int_{\mathbf{S}_{-i}} \left[u_i(a_i, \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega) - u_i(a_i', \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega)\right] dF(\mathbf{s}_{-i} | \omega'; \boldsymbol{\alpha}_{-i}) \\ &\geq \int_{\mathbf{S}_{-i}} \left[u_i(a_i, \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega') - u_i(a_i', \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega')\right] dF(\mathbf{s}_{-i} | \omega'; \boldsymbol{\alpha}_{-i}) \\ &= I(\omega'; \boldsymbol{\alpha}_{-i}) \end{aligned}$$

where the last inequality follows from A2. Hence $I(\omega; \boldsymbol{\alpha}_{-i})$ increases in ω, for all $a_i \geq a_i'$.

Applying Lemma 16 again we get that

$$\begin{aligned} &\int_\Omega \int_{\mathbf{S}_{-i}} u_i(a_i, \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega) \, dF(\mathbf{s}_{-i} | \omega; \boldsymbol{\alpha}_{-i}) \, dF(\omega | s_i; \alpha_i) \\ &- \int_\Omega \int_{\mathbf{S}_{-i}} u_i(a_i', \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega) \, dF(\mathbf{s}_{-i} | \omega; \boldsymbol{\alpha}_{-i}) \, dF(\omega | s_i; \alpha_i) \end{aligned}$$

increases (decreases) whenever $s_i < s_i^*(\alpha_i, \alpha_i')$ ($s_i > s_i^{**}(\alpha_i, \alpha_i')$) and player i chooses α_i' instead of α_i (with $\alpha_i' < \alpha_i$). Then the incremental returns with respect to own action of the maximand in (1) are higher (smaller) at α_i' than at α_i when the message is small (high). Since the latter is supermodular in a_i, Proposition 8 follows from Lemma 7 in VZ-V (p. 348). □

Proof of Lemma 10. Let $\alpha_i, \alpha_i' \in [\underline{\alpha}, \overline{\alpha}]$. As $\lambda F(s_i, \omega; \alpha_i) + (1-\lambda) F(s_i, \omega; \alpha_i')$ is a convex combination of two cdf's with the same marginals, it is a cdf with the same marginals.

Let $\alpha_i'' = \lambda \alpha_i + (1-\lambda)\alpha_i'$. Since $F(s_i, \omega; \alpha_i'')$ and $\lambda F(s_i, \omega; \alpha_i) + (1-\lambda) F(s_i, \omega; \alpha_i')$ have the same marginal distributions, we can try to compare them in terms of the supermodular stochastic order. Let $h(s_i, \omega)$ be a supermodular function with finite expectation with respect to both cdf's. By Lemma 5, the following two conditions are equivalent

(i) $F(s_i, \omega; \alpha_i'') \leq \lambda F(s_i, \omega; \alpha_i) + (1-\lambda) F(s_i, \omega; \alpha_i') \; \forall (s_i, \omega) \in S_i \times \Omega$
(ii) $\int_{S_i \times \Omega} h(s_i, \omega) \, dF(s_i, \omega; \alpha_i'') \leq \int_{S_i \times \Omega} h(s_i, \omega) \, d\left[\lambda F(s_i, \omega; \alpha_i) + (1-\lambda) F(s_i, \omega; \alpha_i')\right]$.

Since expectation is a linear operator, condition (ii) is in turn equivalent to

(iii) $\int_{S_i \times \Omega} h(s_i, \omega) dF(s_i, \omega; \alpha_i'') \leq \lambda \int_{S_i \times \Omega} h(s_i, \omega) dF(s_i, \omega; \alpha_i) + (1-\lambda) \int_{S_i \times \Omega} h(s_i, \omega) dF(s_i, \omega; \alpha_i')$.

Thus (i) is satisfied if and only if (iii) is fulfilled, which is exactly our claim. □

Proof of Theorem 12. Let $\lambda \in [0,1]$, α_i' and α_i'' denote two arbitrary elements of $[\underline{\alpha}, \overline{\alpha}]$, and $\alpha_i''' = \lambda\alpha_i' + (1-\lambda)\alpha_i''$. The following inequalities show our statement,

$$\begin{aligned}
&\lambda U_i\left(\alpha_i', \boldsymbol{\alpha}\right) + (1-\lambda) U_i\left(\alpha_i'', \boldsymbol{\alpha}\right) \\
&= \lambda \int\limits_{S_i\times\Omega} \int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i;\alpha_i'\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right) dF(\mathbf{s}_{-i}|\omega;\boldsymbol{\alpha}_{-i})\, dF\left(s_i,\omega;\alpha_i'\right) \\
&\quad + (1-\lambda) \int\limits_{S_i\times\Omega} \int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i;\alpha_i''\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right) dF(\mathbf{s}_{-i}|\omega;\boldsymbol{\alpha}_{-i})\, dF\left(s_i,\omega;\alpha_i''\right) \\
&\geq \lambda \int\limits_{S_i\times\Omega} \int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i;\alpha_i'''\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right) dF(\mathbf{s}_{-i}|\omega;\boldsymbol{\alpha}_{-i})\, dF\left(s_i,\omega;\alpha_i'\right) \\
&\quad + (1-\lambda) \int\limits_{S_i\times\Omega} \int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i;\alpha_i'''\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right) dF(\mathbf{s}_{-i}|\omega;\boldsymbol{\alpha}_{-i})\, dF\left(s_i,\omega;\alpha_i''\right) \\
&\geq \int\limits_{S_i\times\Omega} \int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i;\alpha_i'''\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right) dF(\mathbf{s}_{-i}|\omega;\boldsymbol{\alpha}_{-i})\, dF\left(s_i,\omega;\alpha_i'''\right) = U_i\left(\alpha_i''', \boldsymbol{\alpha}\right)
\end{aligned}$$

where the first inequality is true by optimality, and the second one follows from the following argument. Consider the function

$$\int_{\mathbf{S}_{-i}} u_i\left(\overline{\varphi}_i\left(s_i;\alpha_i'''\right), \overline{\boldsymbol{\sigma}}_{-i}(\mathbf{s}_{-i}), \omega\right) dF(\mathbf{s}_{-i}|\omega;\boldsymbol{\alpha}_{-i}). \tag{10}$$

Lemma 2 ensures $\overline{\varphi}_i\left(s_i;\alpha_i'''\right)$ is increasing in s_i, so we can use Step 1 in the proof of Proposition 7 to confirm that (10) is supermodular in (s_i, ω). Then the second inequality follows by Lemma 10, since $F(s_i, \omega; \alpha_i)$ is convex in α_i in the spm stochastic order. □

Proof of Lemma 14. By Proposition 7 and Theorem 12, our conditions imply $U_i\left(\alpha_i', \boldsymbol{\alpha}\right) - \alpha_i'$ is convex in α_i', $\forall \boldsymbol{\alpha} \in [\underline{\alpha}, \overline{\alpha}]^n$ and $\forall i \in N$, since the sum of convex functions is convex. It follows from this convexity property that $U_i\left(\alpha_i', \boldsymbol{\alpha}\right) - \alpha_i'$ is continuous in $\alpha_i' \in (\underline{\alpha}, \overline{\alpha})$ and upper semi-continuous at $\alpha_i' = \underline{\alpha}$ and $\alpha_i' = \overline{\alpha}$ ($\forall \boldsymbol{\alpha} \in [\underline{\alpha}, \overline{\alpha}]^n$). Thus, $U_i\left(\alpha_i', \boldsymbol{\alpha}\right) - \alpha_i'$ achieves its maximum in $\alpha_i' \in \{\underline{\alpha}, \overline{\alpha}\}$ for each $\boldsymbol{\alpha} \in [\underline{\alpha}, \overline{\alpha}]^n$. The same convexity property also implies that, irrespective of the initial $\boldsymbol{\alpha} \in [\underline{\alpha}, \overline{\alpha}]^n$, each player's best reply will always be in $\{\underline{\alpha}, \overline{\alpha}\}$, i.e., one of the two extreme qualities at stage I, for generic games. Hence, the game is strategically equivalent to the game with action set $\{\underline{\alpha}, \overline{\alpha}\}$ and payoffs in (7). □

Proof of Proposition 15. Part (i) is a direct consequence of Lemma 14 (via dominance arguments). Parts (ii)–(iii) then follow via direct comparison of the binary payoffs of players. □

Appendix A. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.jet.2016.03.005.

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