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RESEARCH ARTICLE

Comparison of equilibrium actions and payoffs across players in games of strategic complements

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Abstract This paper provides sufficient conditions for comparing the choices of different players in games of strategic complements. The main results require a weak ordering relation on the best responses of players in the game and their constraint sets. Under additional restrictions, we can also compare their relative payoffs. We offer three applications of our idea to industrial organization and new models of behavioral economics. Specifically, we study horizontal mergers in oligopolies, competition among firms with differentiated demands and costs of production, and a model of biased perceptions.

Keywords Quasisupermodular games · Asymmetric equilibria · Single-crossing property

JEL Classification C72 · D03

1 Introduction

In many situations of interest, players that interact in the same environment display different characteristics, for example, firms face different costs of production, workers have different abilities, and people perceive reality in dissimilar ways. This paper explains how differences across players affect their relative equilibrium actions and payoffs. We focus on games of strategic complements.

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We model asymmetries among agents via an ordering relation on the set of players that involves two conditions: The first one compares their possibilities to select higher actions, and the second one contrasts their benefits from doing so via a natural restriction on their best-reply correspondences. We say a player is higher than another one if she has both a higher constraint set and higher incentives to increase own action. Defined in this way, we show that higher players select higher equilibrium actions. To perform these comparisons we use a binary relation on sets that is weaker than the strong set order, frequently encountered in the literature of monotone comparative statics. We finally show that our requirements on best-reply correspondences hold if a single-crossing condition is satisfied across players' marginal returns. From a methodological perspective, the closest precedent is Amir (2008) who provides sufficient conditions to contrast Nash equilibria of different games. Though the question we answer is quite different in nature, the approach we use is similar.

We offer three applications of our idea to industrial organization and new models of behavioral economics, which have been the motivation for this paper. The first application studies incentives to form coalitions in supermodular games with positive or negative externalities and can be used to extend one of the results of Davidson and Deneckere (1985) to horizontal mergers in Bertrand competition. They show that merging firms set higher prices at equilibrium and therefore make less profits than firms outside the coalition. We show their result holds with much weaker assumptions and is thereby robust to various specifications of demand and cost functions. The second application compares the strategies of firms in the market as a consequence of differentiated demands and costs of production. Our last application relates to the model of biased perceptions in Heifetz et al. (2007). In their setting, players choose their actions in the underlying game based on their perceived payoffs, but receive rewards according to the true payoff functions (often called material profits). Our result explains the behavior of the different players as a function of their preferences and allows us to contrast their relative performance. We discuss the implications of the latter findings regarding the dynamic evolution of preferences.

The remainder of the paper is organized as follows. Section 2 defines the class of games we consider and presents the main results, Sect. 3 elaborates on the three applications, Sect. 4 concludes.

2 Main results

Consider a game in normal form $\Gamma \equiv (A_i, \Pi_i; i \in N)$, where $N = \{1, 2 \ldots, n\}$ is the set of players. Here A_i indicates the (pure) strategy space of player i, and $\Pi_i : A_i \times \mathbf{A}_{-i} \to \mathbb{R}$, with $\mathbf{A}_{-i} \equiv \Pi_{j \neq i} A_j$, is her payoff function. Given a strategy profile $\mathbf{a}_{-i} \in \mathbf{A}_{-i}$, we denote by $b_i(\mathbf{a}_{-i})$ the best-response correspondence of player i in N, that is,

$$b_i(\mathbf{a}_{-i}) = \left\{ a_i \in A_i : a_i \in \arg\max_{a_i' \in A_i} \Pi_i\left(a_i', \mathbf{a}_{-i}\right) \right\}. \tag{1}$$

We write $\mathbf{b}(\mathbf{a}) \equiv \left[b_i(\mathbf{a}_{-i}); i \in N\right]$ for the best-response correspondence of all players to a given profile of strategies $\mathbf{a} \in \mathbf{A} \equiv \Pi_i A_i$.

The action profile $\mathbf{a}^*$ is a pure strategy Nash equilibrium if

$$\mathbf{a}^* \in \mathbf{b}\left(\mathbf{a}^*\right) \equiv \left[b_1\left(\mathbf{a}^*_{-1}\right) \times \ldots \times b_n\left(\mathbf{a}^*_{-n}\right)\right]. \tag{2}$$

That is, if $\mathbf{a}^*$ is a fixed point of $\mathbf{b}\,(.)$.

All our results derive from a specific binary relation on $\mathbf{b}\,(.)$ and $\mathbf{A}$. We impose all required conditions on the best-response correspondences and strategy sets and suggest afterward assumptions on the payoffs of players under which these conditions hold. Throughout the analysis, we restrict attention to games of strategic complements defined as follows.

Definition 1 We say Γ is a game of strategic complements if it satisfies, for all $i \in N$,

(i) $(A_i, \geq)$ is a complete sublattice of the lattice A;
(ii) $b_i\left(\mathbf{a}_{-i}\right)$ has a largest and a smallest element, $\overline{b}_i\left(\mathbf{a}_{-i}\right)$ and $\underline{b}_i\left(\mathbf{a}_{-i}\right)$; and
(iii) $\overline{b}_i\,(.)$ and $\underline{b}_i\,(.)$ are increasing in $\mathbf{a}_{-i}$ on $\mathbf{A}_{-i}$.[1]

We write $\overline{\mathbf{b}}\,(.)$ and $\underline{\mathbf{b}}\,(.)$ for the largest and smallest (coordinatewise) elements of $\mathbf{b}\,(.)$. Conditions (i)-(iii) in Definition 1 guarantee (by Tarski's fixed point theorem) that $\overline{\mathbf{b}}\,(.)$ and $\underline{\mathbf{b}}\,(.)$ have a greatest and a least fixed point. It follows that any game of strategic complements has a maximal and a minimal equilibrium, $\overline{\mathbf{a}}^*$ and $\underline{\mathbf{a}}^*$, with $\overline{\mathbf{a}}^* = \sup\left\{\mathbf{a} \in \mathbf{A} : \overline{\mathbf{b}}\,(\mathbf{a}) \geq \mathbf{a}\right\}$ and $\underline{\mathbf{a}}^* = \inf\left\{\mathbf{a} \in \mathbf{A} : \underline{\mathbf{b}}\,(\mathbf{a}) \leq \mathbf{a}\right\}$.

The approach we follow to compare equilibrium choices of players requires a binary relation on both the strategy sets and the best-response correspondences. Let Z be a partially ordered set. Let $X \subseteq Z$ and $Y \subseteq Z$. We write $X \succeq_o Y$ if for every $x \in X$ and $y \in Y$ such that $y > x$, we have $x \in Y$ and $y \in X$. It is clear from this definition that if the supremum (infimum) of X and Y exists and is contained in X and Y, respectively, then $\max Y \not> \max X$ ($\min Y \not> \min X$).[2]

Remark We write $\succeq_o$ instead of $\geq_o$ as the former relation is neither transitive nor antisymmetric, that is, it is not an order. When the underlying set, Z, is a chain, then $\succeq_o$ is equivalent to the strong set order, often invoked in the literature of games of strategic complements.[3]

The analysis that follows requires the comparison of best-reply correspondences for two given players $i, j \in N$. To facilitate the analysis, we write $b_i\left(a_j, \mathbf{a}_{-i,j}\right)$ for the best-reply correspondence of player i, where the first argument in b_i is player j's action and $\mathbf{a}_{-i,j}$ is the vector of actions of other players in N. In a similar way, we write $b_j\left(a_i, \mathbf{a}_{-i,j}\right)$ for the best-reply correspondence of player j.

We now define a binary relation on players. We say $i \succeq j$ if $A_i \succeq_o A_j$ and, for all $a' \in A_i \cap A_j$ and all $\mathbf{a}_{-i,j} \in \mathbf{A}_{-i,j} \equiv \Pi_{k \neq i,j} A_k$, we have

$$b_i\left(a', \mathbf{a}_{-i,j}\right) \succeq_o b_j\left(a', \mathbf{a}_{-i,j}\right). \tag{3}$$

[1] All our topological statements will tacitly be with the interval topology on A.

[2] I would like to thank John Quah for proposing this relation.

[3] According to the strong set order, we write $X \geq_S Y$ if, for every $x \in X$ and $y \in Y$, we have that

$$x \vee y \in X \text{ and } x \wedge y \in Y.$$

The following lemma states that if $i \succeq j$ then the action of player j cannot be higher than the one of player i at any fixed point of the extremal elements of the best-reply functions. Thus, the next result applies but is not restricted to the extremal equilibria of the game, that is, $\overline{\mathbf{a}}^*$ and $\underline{\mathbf{a}}^*$.

Lemma 1 *Suppose $i \succeq j$ and let $\mathbf{a}^*$ be a fixed point of either $\overline{\mathbf{b}}(.)$ or $\underline{\mathbf{b}}(.)$. Then $a_j^* \not> a_i^*$.*

Proof Let $\mathbf{a}^*$ be a fixed point of $\overline{\mathbf{b}}(.)$ and suppose on the contrary that $a_j^* > a_i^*$. Since $A_i \succeq_o A_j$, $a_j^* \in A_i$ and $a_i^* \in A_j$. Then

$$a_i^* = \overline{b}_i\left(a_j^*, \mathbf{a}_{-i,j}^*\right) \geq \overline{b}_i\left(a_i^*, \mathbf{a}_{-i,j}^*\right)$$

where the inequality follows by condition (iii) in Definition 1. But this means that

$$a_j^* = \overline{b}_j\left(a_i^*, \mathbf{a}_{-i,j}^*\right) > a_i^* \geq \overline{b}_i\left(a_i^*, \mathbf{a}_{-i,j}^*\right)$$

which contradicts (3) (with $a' = a_i^*$ and $\mathbf{a}_{-i,j} = \mathbf{a}_{-i,j}^*$).

The proof for the other case is similar, so we omit it. □

We now introduce a stronger version of $\succeq_o$. We write $X \gg_o Y$ if it does not exist $x \in X$ and $y \in Y$ such that $y > x$. This definition guarantees that if the infimum of X and the supremum of Y exist and are in X and Y, respectively, then $\max Y \not> \min X$.[4]

We say $i \succ j$ if $A_i \succeq_o A_j$ and, for all $a \in A_i \cap A_j$ and all $\mathbf{a}_{-i,j} \in \mathbf{A}_{-i,j}$, we have

$$b_i\left(a', \mathbf{a}_{-i,j}\right) \gg_o b_j\left(a', \mathbf{a}_{-i,j}\right). \tag{4}$$

Remark The strict relation $\succ$ relies on a stronger version of $\succeq_o$, $\gg_o$, for the best-reply correspondences but on the same condition for the constraint sets.

The next result extends the previous one to any Nash equilibrium.

Lemma 2 *Suppose that either $i \succ j$ or ($i \succeq j$ and their best-responses are single-valued) and let $\mathbf{a}^*$ be any fixed point of $\mathbf{b}(.)$. Then $a_j^* \not> a_i^*$.*

Proof Let $i \succ j$ and suppose on the contrary that $a_j^* > a_i^*$. Since $A_i \succeq_o A_j$, $a_j^* \in A_i$ and $a_i^* \in A_j$. Since $a_i^* \in b_i\left(a_j^*, \mathbf{a}_{-i,j}^*\right)$ we get

$$a_i^* \geq \underline{b}_i\left(a_j^*, \mathbf{a}_{-i,j}^*\right) \geq \underline{b}_i\left(a_i^*, \mathbf{a}_{-i,j}^*\right)$$

where the second inequality follows by condition (iii) in Definition 1. But this means that

$$\overline{b}_j\left(a_i^*, \mathbf{a}_{-i,j}^*\right) \geq a_j^* > a_i^* \geq \underline{b}_i\left(a_i^*, \mathbf{a}_{-i,j}^*\right)$$

[4] A stronger version of this relation has been used by Antoniadou (2007) to study comparative statics in the consumer problem.

which contradicts (4) (with $a' = a_i^*$ and $\mathbf{a}_{-i,j} = \mathbf{a}^*_{-i,j}$).

Let $i \succeq j$ and assume their best-responses are single-valued. Suppose on the contrary that $a_j^* > a_i^*$. Since $A_i \succeq_o A_j$, $a_j^* \in A_i$ and $a_i^* \in A_j$. Then

$$a_i^* = b_i\left(a_j^*, \mathbf{a}^*_{-i,j}\right) \geq b_i\left(a_i^*, \mathbf{a}^*_{-i,j}\right)$$

where the inequality follows by condition (iii) in Definition 1. But this means that

$$b_j\left(a_i^*, \mathbf{a}^*_{-i,j}\right) = a_j^* > a_i^* \geq b_i\left(a_i^*, \mathbf{a}^*_{-i,j}\right)$$

which contradicts (4) (with $a' = a_i^*$ and $\mathbf{a}_{-i,j} = \mathbf{a}^*_{-i,j}$). □

We finally provide a positive statement that builds directly on the last two lemmas. The new result requires the strategy spaces of players to be totally ordered.

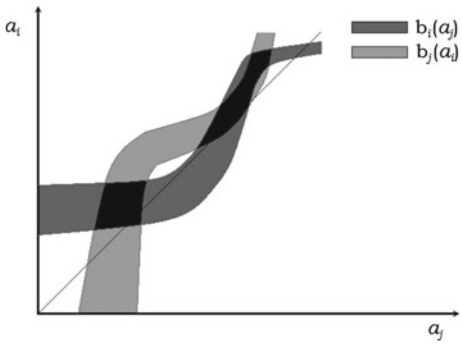
Proposition 3 *Assume the strategy space A is a chain. If $i \succeq j$, then $a_i^* \geq a_j^*$ at any fixed point $\mathbf{a}^*$ of either $\overline{\mathbf{b}}\,(.)$ or $\underline{\mathbf{b}}\,(.)$. If either $i \succ j$ or ($i \succeq j$ and their best-responses are single-valued), then $a_i^* \geq a_j^*$ at any pure strategy Nash equilibrium.*

Proof If A is a chain, then A_i is a chain for all $i \in N$. In this case, $a_j^* \not> a_i^*$ holds if and only if $a_i^* \geq a_j^*$ and the result follows by Lemmas 1 and 2. □

Let $b_i\,(\mathbf{a}_{-i})$ denote the best-response correspondence of player i to a profile of strategies $\mathbf{a}_{-i}$ of the other players. Figure 1 captures the insights behind Proposition 3 for a two-player game and one-dimensional action spaces. The black area contains all the intersections of the best-response correspondences of players i and j, that is, the pure strategy Nash equilibria. If condition (3) holds, then the reaction correspondence of player i is above the inverse image of the reaction correspondence of player j. As a consequence, the lowest and the highest intersections occur above the 45° line, and the extremal equilibrium actions of player i are higher than the ones of player j. Since a portion of the black area lies below the 45° line, the same comparison goes in the opposite direction at some intermediate equilibria.

As Fig. 2 illustrates, our result holds at any equilibrium when either players are strictly ordered (left panel)—and then the minimal selection of the best-response

Fig. 1 Comparing equilibrium actions when $i \succeq j$



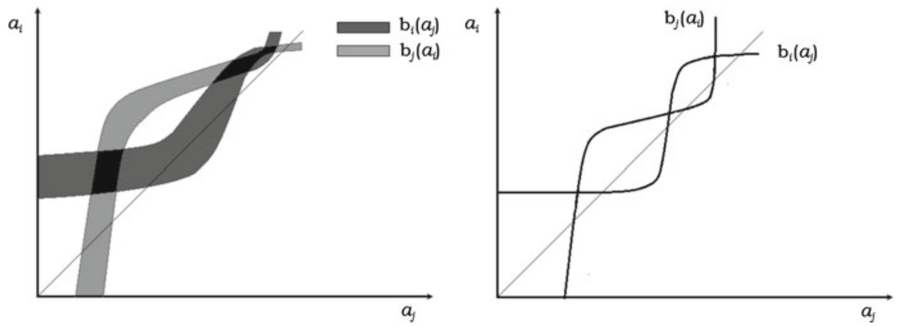


Fig. 2 Comparing equilibrium actions when (i) $i \succ j$ (*left panel*); and (ii) $i \succeq j$ and their best-responses are single-valued (*right panel*)

correspondence of player i is above the inverse image of any point in the reaction correspondence of player j—or players' best-responses are single-valued (right panel).[5]

We end this section by showing that the assumptions we impose on the best-reply correspondences are satisfied under specific restrictions on the payoff functions. Milgrom and Shannon (1994) show that conditions (ii) and (iii) in Definition 1 hold if, in addition to condition (i), Π_i is upper semicontinuous and quasisupermodular in a_i for any $\mathbf{a}_{-i}$ and has the single-crossing property in $(a_i; \mathbf{a}_{-i})$. Since any supermodular function is also quasisupermodular, and any function that has increasing differences in $(a_i; \mathbf{a}_{-i})$ also satisfies the single-crossing property, then ordinally supermodular games encompass all supermodular and log-supermodular games further studied by Milgrom and Roberts (1990), Vives (1990), and Topkis (1998). Quah and Strulovici (2009) identify a new order on functions, the interval dominance order, that generalizes the single-crossing properties. All of these papers provide alternative justifications for conditions (ii) and (iii) in Definition 1, based on various restrictions on $(A_i, \Pi_i; i \in N)$. We next propose sufficient conditions for expressions (3) and (4).

Throughout, we write $\Pi_i\left(a_i, a_j, \mathbf{a}_{-i,j}\right)$ for the payoff of player i. The first argument of Π_i is player i's choice, the second argument is the choice of player j, and the last one is the vector of actions of other players in N. In a similar way, we write $\Pi_j\left(a_j, a_i, \mathbf{a}_{-i,j}\right)$ for the payoff of player j.

Proposition 4 *Assume $A_i \succeq_o A_j$ and that, for all $a, a', a'' \in A_i \cap A_j$ with $a > a''$,*

$$\Pi_j\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_j\left(a'', a', \mathbf{a}_{-i,j}\right) \geq (>)\, 0 \implies \Pi_i\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_i\left(a'', a', \mathbf{a}_{-i,j}\right) \geq (>)\, 0. \quad (5)$$

Then $i \succeq j$. In addition, we obtain $i \succ j$ if (5) is replaced by

$$\Pi_j\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_j\left(a'', a', \mathbf{a}_{-i,j}\right) \geq 0 \implies \Pi_i\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_i\left(a'', a', \mathbf{a}_{-i,j}\right) > 0. \quad (6)$$

[5] Notice that the intermediate equilibrium of the right panel goes in the expected direction even when this equilibrium is unstable under a simple myopic best-response dynamics, that is, counterintuitive results are unrelated to the instability of equilibria. Our outcome is therefore quite different from the comparative statics analysis in supermodular games—see, for example, Echenique (2002).

Proof ($i \succeq j$) Assume on the contrary that there exist a, a' and a'' with $a > a''$ and $a' \in A_i \cap A_j$ such that $a'' \in b_i\left(a', \mathbf{a}_{-i,j}\right)$, $a \in b_j\left(a', \mathbf{a}_{-i,j}\right)$, and either $a \notin b_i\left(a', \mathbf{a}_{-i,j}\right)$ or $a'' \notin b_j\left(a', \mathbf{a}_{-i,j}\right)$. Since $A_i \succeq_o A_j$, then $a'' \in A_j$ and $a \in A_i$. If $a \notin b_i\left(a', \mathbf{a}_{-i,j}\right)$, then

$$\Pi_i\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_i\left(a'', a', \mathbf{a}_{-i,j}\right) < 0.$$

By the contrapositive of (5)

$$\Pi_j\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_j\left(a'', a', \mathbf{a}_{-i,j}\right) < 0$$

contradicting the fact that $a \in b_j\left(a', \mathbf{a}_{-i,j}\right)$. If $a'' \notin b_j\left(a', \mathbf{a}_{-i,j}\right)$, then

$$\Pi_j\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_j\left(a'', a', \mathbf{a}_{-i,j}\right) > 0 \Longrightarrow \Pi_i\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_i\left(a'', a', \mathbf{a}_{-i,j}\right) > 0$$

contradicting the fact that $a'' \in b_i\left(a', \mathbf{a}_{-i,j}\right)$. Then the initial claim is false and $i \succeq j$.

($i \succ j$) Assume on the contrary that there exist some a, a' and a'' with $a > a''$ and $a' \in A_i \cap A_j$ such that $a'' \in b_i\left(a', \mathbf{a}_{-i,j}\right)$ and $a \in b_j\left(a', \mathbf{a}_{-i,j}\right)$. Since $A_i \succeq_o A_j$, then $a'' \in A_j$ and $a \in A_i$. Since $a'' \in b_i\left(a', \mathbf{a}_{-i,j}\right)$, then

$$\Pi_i\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_i\left(a'', a', \mathbf{a}_{-i,j}\right) \leq 0.$$

By the contrapositive of (6)

$$\Pi_j\left(a, a', \mathbf{a}_{-i,j}\right) - \Pi_j\left(a'', a', \mathbf{a}_{-i,j}\right) < 0$$

which contradicts the fact that $a \in b_j\left(a', \mathbf{a}_{-i,j}\right)$. Then the initial claim is false and $i \succ j$. □

Condition (5) can be thought of as a single-crossing property in players' marginal returns to increasing own action. Given the strategy profile of the others, it states that if player j finds it profitable to increase own action so does player i. If payoffs are smooth, a stronger version of this condition is satisfied if $\partial\Pi_i\left(a, a', \mathbf{a}_{-i,j}\right)/\partial a_i \geq \partial\Pi_j\left(a, a', \mathbf{a}_{-i,j}\right)/\partial a_j$, for all $a, a' \in A_i \cap A_j$ and all $\mathbf{a}_{-i,j} \in \mathbf{A}_{-i,j}$. Condition (6) is just a strict version of the latter.

3 Applications

This section offers three applications of Propositions 3 and 4. The first one studies incentives to form coalitions in supermodular games with externalities and can be used to extend one of the results of Davidson and Deneckere (1985) to horizontal mergers in Bertrand competition. The second example refers to oligopolistic competition when firms face different demands and production costs. The last one pertains to the area of behavioral economics, and it is based on Heifetz et al. (2007).

3.1 Coalitions in supermodular games with externalities

Consider an n-player symmetric game.[6] Each player i chooses an action a_i from a compact set $A \subset \mathbb{R}$, which is identical for all the players. Her payoff is given by

$$\pi_i\left(a_i, \mathbf{a}_{-i}\right) : A \times \mathbf{A}_{-i} \rightarrow \mathbb{R} \tag{7}$$

where $\mathbf{a}_{-i} \in \mathbf{A}_{-i} \equiv A^{n-1}$. We assume $\pi_i\left(a_i, \mathbf{a}_{-i}\right)$ is continuous in $\mathbf{a}$ on $\mathbf{A} \equiv A^{\mathrm{n}}$ and has increasing differences in (a_k, a_l) for all $k \neq l, k, l \in N$. This game has (strict) positive externalities if $\pi_i\left(a_i, \mathbf{a}_{-i}\right)$ is (strictly) increasing in $\mathbf{a}_{-i}$. If $\pi_i\left(a_i, \mathbf{a}_{-i}\right)$ is (strictly) decreasing in $\mathbf{a}_{-i}$, we say the (strict) externalities are negative.

Suppose a group G_1 of the players decide to form a coalition to set their strategies together in order to maximize joint profits, for example, they decide to form a cartel. Let G_2 represent the other players. If the decision of joint maximization is common knowledge, then the coalition game can be studied as a new game where players perceive the following payoffs:

$$\begin{aligned} \Pi_i\left(a_i, \mathbf{a}_{-i}\right) &= \sum_{k \in G_1} \pi_k\left(a_k, \mathbf{a}_{-k}\right) && \text{if } i \in G_1 \\ \Pi_j\left(a_j, \mathbf{a}_{-j}\right) &= \pi_j\left(a_j, \mathbf{a}_{-j}\right) && \text{if } j \in G_2. \end{aligned} \tag{8}$$

Amir (2008) uses a similar approach to contrast Nash equilibrium payoffs with Pareto optimal outcomes in supermodular games with externalities.

Since increasing differences are preserved by addition, the new game $(A, \Pi_i; i \in N)$ also satisfies Definition 1. In addition, we observe that, with $a > a'$, $\mathbf{a}_{-k} = \left(a, \mathbf{a}_{-i,k}\right)$ and $\mathbf{a}'_{-k} = \left(a', \mathbf{a}_{-i,k}\right)$,

$$\begin{aligned} &\Pi_i\left(a, \mathbf{a}_{-i}\right) - \Pi_i\left(a', \mathbf{a}_{-i}\right) \\ &\quad = \pi_i\left(a, \mathbf{a}_{-i}\right) - \pi_i\left(a', \mathbf{a}_{-i}\right) + \sum_{k \in G_1, k \neq i} \left[\pi_k\left(a_k, \mathbf{a}_{-k}\right) - \pi_k\left(a_k, \mathbf{a}'_{-k}\right)\right] \end{aligned} \tag{9}$$

$$\Pi_j\left(a, \mathbf{a}_{-j}\right) - \Pi_j\left(a', \mathbf{a}_{-j}\right) = \pi_j\left(a, \mathbf{a}_{-j}\right) - \pi_j\left(a', \mathbf{a}_{-j}\right) \tag{10}$$

for players in G_1 and G_2, respectively. If the game has positive externalities, then $\pi_k\left(a_k, \mathbf{a}_{-k}\right) \geq \pi_k\left(a_k, \mathbf{a}'_{-k}\right)$ for all $k \in G_1, k \neq i$—when the externalities are strict, the weak inequality is strict. It follows that, for all $\left(a, a_j, \mathbf{a}_{-i,j}\right) = \left(a, a_i, \mathbf{a}_{-i,j}\right)$ and $\left(a', a_j, \mathbf{a}_{-i,j}\right) = \left(a', a_i, \mathbf{a}_{-i,j}\right)$, (9) is higher than (10), that is, the marginal returns of increasing own action are higher for the players in the coalition as compared to the others. Denoting by i a coalition member and by j a player outside the coalition, the last observation implies (by Proposition 4) that $i \succeq j$ $(i \succ j)$ when the externalities

[6] Formally, we say a game is symmetric if

$$\pi_i\left(a_1, \ldots, a_n\right) = \pi_{\sigma(i)}\left(a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}\right)$$

for all $i \in N$ and all $\mathbf{a} \in \mathbf{A}$, where $\sigma(i)$ is any permutation of N.

are positive (strictly positive). Since the opposite is true when the externalities are negative, the same holds if we revert the names of the players. The next proposition compares the equilibrium strategies of players inside and outside the coalition and contrasts their relative performance.

Proposition 5 *In the coalition game with positive (negative) externalities, members of the coalition select, at the extremal equilibria, higher (lower) actions than the others. Moreover, the profits of any coalition member are lower than the profits of any player outside the coalition irrespective of the nature of the externalities.*

If the externalities are strict, then these results hold at any pure strategy equilibrium.

The first sentence follows directly from Propositions 3 and 4. To show that a member outside the coalition gets a higher payoff when the externalities are positive, we should start by observing she shares $(n-2)$ adversaries with the coalition members and faces the same actions with respect to them. The extra competitor is a coalition member who selects a strategy higher than hers, and the opposite is true for players in G_1. Since payoffs have positive externalities in other players' actions, then the reward of a player outside the agreement is always higher than that of a coalition member. The result follows because non-coalition players establish strategies at their individual payoff-maximizing level while the others do not. The fact that a non-coalition member gets a higher payoff when the externalities are negative follows by a similar argument.

This result can be used to generalize one of the outcomes of Davidson and Deneckere (1985). They study incentives to merge when firms that produce symmetric differentiated products engage in price competition and show that mergers are beneficial to all firms but non-coalition members take a free ride and earn larger profits than the coalition members. Their model assumes constant marginal costs, continuously differentiable demand functions that increase in other firms' prices and decrease in own prices, and strictly concave profits that lead to single-valued best-responses. Proposition 5 shows their statement can be attained under much weaker conditions.

3.2 Oligopolistic competition

This application compares the equilibrium strategies of firms and their corresponding profits as a consequence of differentiated demands and costs of production.

Consider a set $N = \{1, 2 \ldots, n\}$ of firms that compete in prices. The payoff function of firm i is

$$\Pi_i (p_i, \mathbf{p}_{-i}) = (p_i - c_i) D_i (p_i, \mathbf{p}_{-i}) \tag{11}$$

where $\mathbf{p}_{-i}$ denotes the vector of other firms' prices and $D_i (p_i, \mathbf{p}_{-i})$ represents the demand function of firm i. Without loss of generality, we restrict p_i to some interval $[c_i, \overline{p}]$ for all $i \in N$. We assume the demand function $D_i (p_i, \mathbf{p}_{-i})$ has increasing differences in (p_i, p_j) for any firm j different from i, is decreasing and upper semi-continuous in p_i, and is increasing in p_j, that is, products are gross substitutes. All these conditions guarantee $([c_i, \overline{p}], \Pi_i; i \in N)$ is a supermodular game that satisfies

Definition 1.[7] In addition, let us denote the elasticity of demand of firm i as follows

$$\varepsilon_i(p_i, \mathbf{p}_{-i}) \equiv -p_i \partial \ln D_i(p_i, \mathbf{p}_{-i}) / \partial p_i.$$

Let us assume firm i has both higher per-unit costs and lower elasticity of demand than firm j. Then, for all $\mathbf{p}_{-i,j}$ and all $p, p' \in [c_i, \overline{p}] \cap \left[c_j, \overline{p}\right] = [c_i, \overline{p}]$,

$$1/(p - c_i) - \varepsilon_i\left(p, p', \mathbf{p}_{-i,j}\right)/p \geq 1/\left(p - c_j\right) - \varepsilon_j\left(p, p', \mathbf{p}_{-i,j}\right)/p \tag{12}$$

which is the same as to say $\partial \ln \Pi_i\left(p, p', \mathbf{p}_{-i,j}\right)/\partial p_i \geq \partial \ln \Pi_j\left(p, p', \mathbf{p}_{-i,j}\right)/\partial p_j$. In addition, the constraint set $[c, \overline{p}]$ is ascending in c so that $[c_i, \overline{p}] \succeq_o \left[c_j, \overline{p}\right]$. It follows, by Propositions 4, that $i \succeq j$. Therefore, by Proposition 3, the firm with higher per-unit costs and lower elasticity of demand will set a higher price (at the extremal equilibria). The next proposition captures this statement and compares firms' relative profits under the additional restriction of symmetric demands.

Proposition 6 *If firm i has both higher per-unit costs and lower elasticity of demand than firm j, then the price of firm i is higher than the price of j at the extremal equilibria. Moreover, if demands are symmetric, the profits of the higher cost firm are lower.*

The fact that firms with higher per-unit costs and symmetric demands make less profits follows from two sources. First, per-unit costs have a direct negative effect on firms' own payoffs. Second, low-cost firms set lower prices than high-cost firms, which, since the goods are gross substitutes, benefits the former and makes the latter ones worse-off.

3.3 Biased perceptions

Consider a game with symmetric payoffs where each player i chooses an action a_i from a compact chain A. Her payoff is given by

$$\pi_i(a_i, \mathbf{a}_{-i}, \omega) : A \times \mathbf{A}_{-i} \times \Omega \to \mathbb{R} \tag{13}$$

where $\mathbf{a}_{-i} \in \mathbf{A}_{-i} \equiv A^{n-1}$ denotes the vector of other agents' actions and $\omega \in \Omega \subseteq \mathbb{R}$ is a payoff-relevant parameter. Assume $\pi_i(a_i, \mathbf{a}_{-i}, \omega)$ is upper semicontinuous in a_i for $\mathbf{a}_{-i}$ fixed, is quasisupermodular in a_i, and has the single-crossing property in $(a_i; \mathbf{a}_{-i}, \omega)$.

Although all individuals share the same payoffs, they differ in the way they perceive the payoff-relevant parameter. Specifically, player i believes the value of ω is given by $\omega_i = \omega + \tau_i$ where $\tau_i \in \left[\underline{\tau}, \overline{\tau}\right] \subseteq \mathbb{R}$ and $\underline{\tau} < 0 < \overline{\tau}$. Optimistic players overestimate

[7] This game is also a game with strategic complementarities if $\ln\left[D_i\left(p_i, \mathbf{p}_{-i}\right)\right]$ has increasing differences in $\left(p_i, p_j\right)$ for any firm j different from i. When $D_i\left(p_i, \mathbf{p}_{-i}\right)$ is differentiable in p_i, the last condition is equivalent to the property that each firm's price elasticity of demand is a decreasing function of the prices of the other firms' products—see Topkis (1998).

ω, that is, $\tau_i > 0$; realistic players assess ω correctly, that is, $\tau_i = 0$; and pessimistic players underestimate the parameter, that is, $\tau_i < 0$.

Substituting ω by τ_i in (13), player i's perceived payoffs are given by

$$\Pi_i\left(a_i, \mathbf{a}_{-i}, \tau_i\right) = \pi_i\left(a_i, \mathbf{a}_{-i}, \omega + \tau_i\right). \tag{14}$$

These conditions describe a game of strategic complements that satisfies Definition 1. In addition, since π_i has the single-crossing property in $(a_i; \omega)$, if $\tau_i > \tau_j$, for all $a > a'$,

$$\Pi_j\left(a, a'', \mathbf{a}_{-i,j}, \tau_j\right) - \Pi_j\left(a', a'', \mathbf{a}_{-i,j}, \tau_j\right) \geq (>)\, 0 \Longrightarrow \Pi_i\left(a, a'', \mathbf{a}_{-i,j}, \tau_i\right) \\ -\Pi_i\left(a', a'', \mathbf{a}_{-i,j}, \tau_i\right) \geq (>)\, 0.$$

In words, if a player of a given type prefers a higher action, so does a player with a higher type. Thus, we can use Propositions 3 and 4 to compare the behavior of optimistic, pessimistic, and realistic players at the extremal equilibria. Moreover, if players' actions have either positive or negative externalities on other players' payoffs, we can also rank their true rewards $\left[\pi_i\left(a_i, \mathbf{a}_{-i}, \omega\right); i \in N\right]$.

Proposition 7 *In a game with biased perceptions, at the extremal equilibria, optimistic players select higher actions than realistic players, and realistic players choose higher actions than the pessimistic ones.*

If there are positive externalities, realistic players outperform optimistic players; if the externalities are negative, realistic players outperform the pessimistic ones.

The first statement follows directly from Propositions 3 and 4. To see that a realistic player outperforms any optimistic one if the externalities are positive, notice she shares $(n-2)$ competitors with the second one and therefore faces the same actions with respect to them. Her extra rival is an optimistic player that selects a strategy higher than hers, and the opposite is true for the optimistic player. Since players' strategies have positive externalities on other players' payoffs, then the realistic player's true payoff is higher than that of the optimistic one. The result follows because optimistic players do not set strategies at their individual payoff-maximizing level, while realistic players do. A similar argument applies to the case of negative externalities.

This result has implications for the dynamic evolution of preferences. Recent studies suggest that individuals within a generation behave rationally with respect to the inherited preferences but that the distribution of preferences across the population changes from one generation to the next under the pressure of differential material rewards—see, for example, Kockesen et al. (2000). According to our analysis, if the game has positive externalities, then optimistic players have less chances to survive in the long run as compared to realistic players. If the externalities are negative, the chances to survive are lower for the pessimistic players as compared to the realistic ones.[8]

[8] Our conclusion differs from the analysis of Heifetz et al. (2007) as they consider a strategic evolution of preferences where players select own types as a best-response to the types selected by the others.

4 Concluding remarks

This paper compares the equilibrium choices of players in games of strategic complements. We introduce differences across agents via an ordering relation on the set of players, which is based on two conditions: The first one compares their possibilities to select higher actions, and the second one contrasts their benefits from doing so via a restriction on their best-reply correspondences. We show this binary relation on the set of players has monotone implications on their equilibrium actions and (under additional conditions) allows us to compare their corresponding payoffs. We then provide sufficient conditions on the payoff functions of players under which the previous assumptions hold. The latter restrictions are based on the single-crossing property.

We offer three applications of our result and believe the idea might result useful in other models of behavioral economics.

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