Online Appendix
Endogenous Information Acquisition in Bayesian Games with Strategic Complementarities

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This online appendix contains two items. First, in Appendix A we provide a simple model —with closed form solution— that offers some intuitive view of both our approach and the main results in the paper. Second, in Appendix B we elaborate on the possibility of decreasing marginal returns of information acquisition, i.e., concavity of the value of information.

1 Appendix A: Motivating example

We present a simple version of the beauty-contest in the spirit of our model.[1] Versions of this model were previously used by Hellwig and Veldkamp (2009) and Myatt and Wallace (2012). The setting is also similar to the oligopoly models of Hauk and Hurkens (2001) and Vives (1988, 2008).[2] Although the analysis can be easily extended to an arbitrary number of players, we restrict attention to only two, for expositional ease.

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[1] The paper endogenizes information in the class of games studied by VZ-V. Thus, all their applications and examples display payoffs that fulfill our assumptions. In addition, Examples 6 and 11 in the paper provide simple ways to construct CDFs that satisfy our restrictions regarding information structure. Further applications of our framework would then follow by combining the former VZ-V applications with our constructions.

[2] In oligopoly models, endogenous information acquisition has been also studied by Hwang (1993), Jansen (2008), Li, McKelvey and Page (1987) and Dimitrova and Schlee (2003). As we do in this example, these papers assume specific functional forms for the payoffs and often Gaussian signal distributions.

A player's payoff depends on the proximity of his action, $a_i \in \mathbb{R}$, to both the underlying state variable, ω, and the average group action, $\overline{a} = (a_1 + a_2)/2$,

$$u_i = \overline{u} - (1-\gamma)(a_i - \omega)^2 - \gamma (a_i - \overline{a})^2, \quad i = 1, 2 \tag{1}$$

where $\gamma \in (0,1)$, and ω is the realization of a random parameter drawn from a normal prior with mean μ and variance υ_ω^2. The important feature of this model is that the marginal returns of a player's own action, increase with the action of the other player and with the state ω.

Player i decides how much to spend on information by choosing α_i. In doing so, he determines his information structure from a family of cdf's $\{F(s_i, \omega; \alpha_i)\}$ indexed by $\alpha_i \in [0,1]$.[3] We assume the joint distribution of $(\widetilde{s}_i, \widetilde{\omega})$ is the bivariate normal

$$\mathcal{N}[A, B(\alpha_i)], \text{ with } A = (\mu, \mu) \text{ and } B(\alpha_i) = \begin{pmatrix} 1 & \alpha_i \upsilon_\omega \\ \alpha_i \upsilon_\omega & \upsilon_\omega^2 \end{pmatrix}. \tag{2}$$

The variance of $\widetilde{s}_i$ is normalized to 1 to highlight the fact that, in our setting, what really matters to player i is the correlation between own signal and the state of the world.[4]

We assume signals are independent conditional on ω. Given a profile $\boldsymbol{\alpha}$, our conditions imply that $\mathrm{E}(\omega \,| s_i; \alpha_i) = \mu + \alpha_i \upsilon_\omega (s_i - \mu)$ and $\mathrm{E}(s_{-i} \,| s_i; \boldsymbol{\alpha}) = \mu + \alpha_i \alpha_{-i} (s_i - \mu)$. The bivariate normal distribution has many additional nice features. The most relevant for us is that α_i orders the family of signals according to the supermodular stochastic order.

Recall the timing of the game is as follows: First, participants select how much to spend on information (i.e., α_i). After observing the realization of their own signals, but neither the experiment selected nor the message received by the other player, each of them chooses an action. As observed by Hauk and Hurkens (2001), in this simple setting, it is quite easy to solve for the Bayesian information Nash equilibrium. First, we solve for the (unique) Bayesian Nash equilibrium in the second stage. Then, substituting players' actions by the corresponding strategies in the utility functions, we consider the game where only information levels need to be chosen.

Assume for the moment that participants acquired a profile of information $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ at stage I, and focus on the game that follows. For each $\boldsymbol{\alpha} \in [0,1]^2$, it is well known that (for the present formulation) the equilibrium strategies in the second stage are affine with respect to

[3] We restrict α_i to the interval $[0,1]$ to simplify the exposition.

[4] For this example, since players' payoffs are both quadratic and supermodular, information could be modeled with a mean-variance order. We opted for correlation to better connect the example to our general framework. The reason is that bivariate normal distributions with the same marginals are ranked in the supermodular stochastic order according to their correlation coefficient.

messages. To get them, we will assume player $-i$ follows a strategy $\sigma_{-i}(s_{-i}) = a + b_{-i}(s_{-i} - \mu)$. Inserting this expression in (1) we get

$$u_i = \overline{u} - (1-\gamma)(a_i - \omega)^2 - \gamma \left(a_i/2 - (a + b_{-i}(s_{-i} - \mu))/2\right)^2.$$

If player i receives a message s_i, his interim payoff is

$$E(u_i | s_i; \boldsymbol{\alpha}) = M_i - (1 - (3/4)\gamma) a_i^2 + 2(1-\gamma) a_i (\mu + \alpha_i \upsilon_\omega (s_i - \mu)) + \gamma a_i (a + b_{-i}\alpha_i \alpha_{-i}(s_i - \mu))/2. \tag{3}$$

where the conditional expectation is calculated with respect to the Bayesian updated beliefs. In (3), $M_i = E\left(\overline{u} - (1-\gamma)\omega^2 - \gamma (a + b_{-i}(s_{-i} - \mu))^2/4 \,|s_i; \boldsymbol{\alpha}\right)$. We avoid the latter calculation as M_i does not play any fundamental role in the analysis that follows. Computing the first order condition, his best-response is given by

$$a_i = \frac{1}{(1 - (3/4)\gamma)} \left\{ (1-\gamma)\mu + \gamma a/4 + \left[(1-\gamma)\alpha_i \upsilon_\omega + \gamma b_{-i}\alpha_i \alpha_{-i}/4\right](s_i - \mu) \right\}. \tag{4}$$

Substituting i by $-i$, we can get the corresponding expression for the other player. Combining results, the equilibrium strategies (as a function of $\boldsymbol{\alpha}$) are

$$\sigma_i(s_i) = \mu + \upsilon_\omega (1-\gamma) \frac{1 - (3/4)\gamma + (1/4)\gamma \alpha_{-i}^2}{(1 - (3/4)\gamma)^2 - (\gamma \alpha_i \alpha_{-i}/4)^2} \alpha_i (s_i - \mu), \; i = 1, 2. \tag{5}$$

Hence the sensitivity of player i's strategy with respect to unexpected shocks $(s_i - \mu)$ increases in both α_i and α_{-i}.

To attain the profile of information at equilibrium, we need to go one step back and find the conditions under which no player i has an incentive to deviate from α_i. Two issues deserve attention. First, since we model covert information acquisition, if player i deviates at stage I from α_i to α_i', there is no strategic effect on the other player. Second, if player i selects α_i' instead of α_i, he will use this information to update his interim strategy in the second stage as follows

$$\varphi_i(s_i; \alpha_i') = \mu + \upsilon_\omega (1-\gamma) \frac{1 - (3/4)\gamma + (1/4)\gamma \alpha_{-i}^2}{(1 - (3/4)\gamma)^2 - (\gamma \alpha_i \alpha_{-i}/4)^2} \alpha_i' (s_i - \mu) \tag{6}$$

where (6) is obtained by substituting $\sigma_{-i}(s_{-i})$, as defined in (5), in (4). Notice that (6) depends on both α_i' and α_i; the reason is that player i can improve his strategy after deviating but cannot affect the other player's beliefs and these beliefs do affect i's optimal behavior.

Let's assume player $-i$ follows the equilibrium strategy at stage II corresponding to the profile $\boldsymbol{\alpha}$. We let $U_i(\alpha_i'; \boldsymbol{\alpha})$ denote player i's highest expected payoff if he deviates from α_i to

α_i'. Taking the unconditional expectation of u_i after substituting a_i by $\varphi_i(s_i;\alpha_i')$ and a_{-i} by $\sigma_{-i}(s_{-i})$ we get

$$U_i(\alpha_i';\boldsymbol{\alpha}) = E(M_i|\boldsymbol{\alpha}) + \left(1-\frac{3}{4}\gamma\right)\left\{\mu^2 + \left(\upsilon_\omega(1-\gamma)\frac{1-(3/4)\gamma+(1/4)\gamma\alpha_{-i}^2}{(1-(3/4)\gamma)^2-(\gamma\alpha_i\alpha_{-i}/4)^2}\right)^2\alpha_i'^2\right\}.$$

It follows that player i's highest expected payoff is increasing and convex in α_i'. In other words, the marginal returns to information are positive and increasing.

Taking into account the cost of information acquisition, player i will not have any incentives to deviate from α_i if $\alpha_i \in \arg\max_{\alpha_i'\in[0,1]}\{U_i(\alpha_i';\boldsymbol{\alpha})-\alpha_i'\}$. We conclude that a profile of information $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ constitutes an equilibrium if it satisfies, simultaneously, the system[5]

$$\begin{aligned}\alpha_1 &\in \arg\max_{\alpha_1'\in[0,1]}\left\{\left(1-\frac{3}{4}\gamma\right)\left(\upsilon_\omega(1-\gamma)\frac{1-(3/4)\gamma+(1/4)\gamma\alpha_2^2}{(1-(3/4)\gamma)^2-(\gamma\alpha_1\alpha_2/4)^2}\right)^2\alpha_1'^2-\alpha_1'\right\}\\ \alpha_2 &\in \arg\max_{\alpha_2'\in[0,1]}\left\{\left(1-\frac{3}{4}\gamma\right)\left(\upsilon_\omega(1-\gamma)\frac{1-(3/4)\gamma+(1/4)\gamma\alpha_1^2}{(1-(3/4)\gamma)^2-(\gamma\alpha_1\alpha_2/4)^2}\right)^2\alpha_2'^2-\alpha_2'\right\}.\end{aligned}$$

Since $U_i(\alpha_i';\boldsymbol{\alpha})-\alpha_i'$ is strictly convex in α_i' and the constraint set is $[0,1]$, its argmax is either 0, 1, or both, for $i=1,2$. Then any equilibrium candidate $\boldsymbol{\alpha}$ reflects extreme behavior regarding information acquisition: Players acquire the full information signal or no information at all as Proposition 15 —in the paper— states. Here, a maximal and a minimal information equilibrium always exist because the induced game at stage I is supermodular in the α's.

In this setting $\boldsymbol{\alpha}=(1,1)$ is an information equilibrium if

$$\left(1-\frac{3}{4}\gamma\right)\upsilon_\omega^2 \geq 1$$

and $\boldsymbol{\alpha}=(0,0)$ constitutes an equilibrium if

$$\upsilon_\omega^2(1-\gamma)^2\Big/\left(1-\frac{3}{4}\gamma\right)\leq 1.$$

Thus the complete information game emerges endogenously if either the prior is very uninformative (υ_ω large) or there are strong complementarities in the second stage (γ small). The opposite is true for the common uncertainty, no private information, game. In addition, if

$$\left(1-\frac{3}{4}\gamma\right)\upsilon_\omega^2 \geq 1 \geq \upsilon_\omega^2(1-\gamma)^2\Big/\left(1-\frac{3}{4}\gamma\right)$$

then the extreme information profiles are both equilibria.

Several insights can be gleaned from this example. First, expected payoffs from deviating increase in the correlation between the signal and the fundamental —see Footnote 4. Thus,

[5] We eliminated two constant terms in this system, as they don't affect the maximizers.

players care about information because it allows a better match between the actions selected and the underlying state variable. Given the complementarities in the model, this also allows a better match between the players' strategies. Although the value of information is always positive, it is also costly and players sometimes decide not to get information at all.

Second, the quality of the signal decreases player i's optimal strategy when he receives low messages, and increases it for high messages. Formally,

$$\frac{\partial \varphi_i (s_i; \alpha_i')}{\partial \alpha_i'} = \upsilon_\omega (1-\gamma) \frac{1-(3/4)\gamma + (1/4)\gamma\alpha_{-i}^2}{(1-(3/4)\gamma)^2 - (\gamma\alpha_i\alpha_{-i}/4)^2} (s_i - \mu)$$

which is positive if $s_i \geq \mu$ and negative otherwise. This result follows from Proposition 8 —in the paper— if we let $\mu = s_i^* (\alpha_i, \alpha_i') = s_i^{**} (\alpha_i, \alpha_i')$ for all $\alpha_i > \alpha_i'$.

Third, as stated in Lemma 14 —in the paper—, the convexity of the payoffs induces players to behave in an extreme fashion with respect to information acquisition: Either both pick the full information signal or remain fully uninformed. Thus, our result explains the emergency of the full information game as an endogenous outcome.[6]

2 Appendix B: Concavity of information quality

This appendix explores the scope for a player's maximal expected payoff to be concave, as opposed to convex, in own information level. The interest in such a property is obvious, since this result would in particular guarantee existence of pure strategy equilibrium via the standard topological approach.

For (one-player) decision problems under uncertainty, several studies have elaborated on the difficulties involved in getting the value of information to be globally concave, most notably Radner and Stiglitz (1984) and Chade and Schlee (2002). We show that the present approach also sheds light on this issue.

Before presenting our main result, we introduce the dual version of Definition 9 in the paper.

Definition 1 *Assume the family* $\{F(s_i, \omega; \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$ *shares the same marginals. We say* $F(s_i, \omega; \alpha_i)$ *is concave in* α_i *in the supermodular order if,* $\forall \alpha_i, \alpha_i' \in [\underline{\alpha}, \overline{\alpha}]$*,* $\forall \lambda \in [0,1]$

$$F\left(s_i, \omega; \lambda\alpha_i + (1-\lambda)\alpha_i'\right) \geq \lambda F(s_i, \omega; \alpha_i) + (1-\lambda) F\left(s_i, \omega; \alpha_i'\right) \quad \forall (s_i, \omega). \tag{7}$$

That is, if $F(s_i, \omega; \alpha_i)$ *is concave in* α_i *on* $[\underline{\alpha}, \overline{\alpha}]$.

[6]Note that when $\alpha_i = 1$ then the conditional variance of $\widetilde{\omega}$ given s_i vanishes and the signal reveals the value of the fundamental with certainty.

We next offer a characterization of this definition via expectations of supermodular functions.

Lemma 2 *$F(s_i,\omega;\alpha_i)$ is concave in α_i in the spm order iff, $\forall\alpha_i,\alpha_i'\in[\underline{\alpha},\overline{\alpha}]$, $\forall\lambda\in[0,1]$,*

$$\int_{S_i\times\Omega} h(s_i,\omega)\ dF\left(s_i,\omega;\alpha_i''\right)\geq\lambda\int_{S_i\times\Omega} h(s_i,\omega)\ dF(s_i,\omega;\alpha_i)+(1-\lambda)\int_{S_i\times\Omega} h(s_i,\omega)\ dF\left(s_i,\omega;\alpha_i'\right) \tag{8}$$

(here $\alpha_i''=\lambda\alpha_i+(1-\lambda)\alpha_i'$) for all supermodular functions h for which the expectations exist.

Proof: Let $\alpha_i,\alpha_i'\in[\underline{\alpha},\overline{\alpha}]$. As $\lambda F(s_i,\omega;\alpha_i)+(1-\lambda)F(s_i,\omega;\alpha_i')$ is a convex combination of two cdf's with the same marginals, it is a cdf with the same marginals.

Let $\alpha_i''=\lambda\alpha_i+(1-\lambda)\alpha_i'$. Since $F(s_i,\omega;\alpha_i'')$ and $\lambda F(s_i,\omega;\alpha_i)+(1-\lambda)F(s_i,\omega;\alpha_i')$ have the same marginal distributions, we can try to compare them in terms of the supermodular stochastic order. Let $h(s_i,\omega)$ be a supermodular function with finite expectation with respect to both cdf's. By Lemma 5 in the paper, the following two conditions are equivalent

(i) $F(s_i,\omega;\alpha_i'')\geq\lambda F(s_i,\omega;\alpha_i)+(1-\lambda)F(s_i,\omega;\alpha_i')\ \forall(s_i,\omega)\in S_i\times\Omega$

(ii) $\displaystyle\int_{S_i\times\Omega} h(s_i,\omega)\,dF(s_i,\omega;\alpha_i'')\geq\int_{S_i\times\Omega} h(s_i,\omega)\,d\left[\lambda F(s_i,\omega;\alpha_i)+(1-\lambda)F(s_i,\omega;\alpha_i')\right].$

Since expectation is a linear operator, condition (ii) is in turn equivalent to

(iii) $\displaystyle\int_{S_i\times\Omega} h(s_i,\omega)\,dF(s_i,\omega;\alpha_i'')\geq\lambda\int_{S_i\times\Omega} h(s_i,\omega)\,dF(s_i,\omega;\alpha_i)+(1-\lambda)\int_{S_i\times\Omega} h(s_i,\omega)\,dF(s_i,\omega;\alpha_i').$

Thus (i) is satisfied if and only if (iii) is fulfilled, which is exactly our claim. □

In terms of our set-up, the spm order ranks informativeness in the sense that a higher α_i leads to higher chances of observing high realizations of the signal when the state of the world is high, and low realizations when the state of the world is low. This new notion of concavity refers to how fast α_i raises informativeness. It means that α_i raises informativeness with decreasing returns.

Since Lemma 2 is simply the dual of Lemma 10 in the paper, the first part of our proof of convex information value works for concavity as well. However, the second part of the proof fails as the pointwise maximum of a collection of concave functions need not be concave. While a similar approach can characterize concavity of $U_i(\alpha_i',\boldsymbol{\alpha})$ in α_i', it entails a very restrictive joint concavity condition on own second-stage actions and signals, as captured by the next result.[7]

[7] In addition, the construction of information structures in Example 11 in the paper admits an obvious concave analog, obtained by simply reversing the relevant inequalities. In this construction, concavity of the information structure entails no more restrictiveness (in the mathematical sense) than its convexity.

Proposition 3 *Assume that*

$$\int_{S_i \times \Omega} \int_{\mathbf{S}_{-i}} u_i \left(a_i \left(s_i \right), \overline{\boldsymbol{\sigma}}_{-i} \left(\mathbf{s}_{-i} \right), \omega \right) dF \left(\mathbf{s}_{-i} \left| \omega ; \boldsymbol{\alpha}_{-i} \right. \right) dF \left(s_i, \omega ; \alpha_i' \right) \tag{9}$$

is jointly concave on $A_i \times [\underline{\alpha}, \overline{\alpha}]$*, i.e., in* $\left(a_i \left(s_i \right), \alpha_i' \right)$*. Then,* $U_i \left(\alpha_i', \boldsymbol{\alpha} \right)$ *is concave in* α_i'*.*

Proof: Let $\lambda \in [0, 1]$, α_i' and α_i'' denote two arbitrary elements of $[\underline{\alpha}, \overline{\alpha}]$, and $\alpha_i''' = \lambda \alpha_i' + (1 - \lambda) \alpha_i''$. The following inequalities prove the result,

$$\begin{aligned}
U_i \left(\alpha_i''', \boldsymbol{\alpha} \right) &= \int_{S_i \times \Omega} \int_{\mathbf{S}_{-i}} u_i \left(\overline{\varphi}_i \left(s_i ; \alpha_i''' \right), \overline{\boldsymbol{\sigma}}_{-i} \left(\mathbf{s}_{-i} \right), \omega \right) dF \left(\mathbf{s}_{-i} \left| \omega ; \boldsymbol{\alpha}_{-i} \right. \right) dF \left(s_i, \omega ; \alpha_i''' \right) \\
&\geq \int_{S_i \times \Omega} \int_{\mathbf{S}_{-i}} u_i \left(\lambda \overline{\varphi}_i \left(s_i ; \alpha_i' \right) + (1 - \lambda) \overline{\varphi}_i \left(s_i ; \alpha_i'' \right), \overline{\boldsymbol{\sigma}}_{-i} \left(\mathbf{s}_{-i} \right), \omega \right) dF \left(\mathbf{s}_{-i} \omega ; \boldsymbol{\alpha}_{-i} \right) dF \left(s_i, \omega ; \alpha_i''' \right) \\
&\geq \lambda \int_{S_i \times \Omega} \int_{\mathbf{S}_{-i}} u_i \left(\overline{\varphi}_i \left(s_i ; \alpha_i' \right), \overline{\boldsymbol{\sigma}}_{-i} \left(\mathbf{s}_{-i} \right), \omega \right) dF \left(\mathbf{s}_{-i} \left| \omega ; \boldsymbol{\alpha}_{-i} \right. \right) dF \left(s_i, \omega ; \alpha_i' \right) \\
&\quad + (1 - \lambda) \int_{S_i \times \Omega} \int_{\mathbf{S}_{-i}} u_i \left(\overline{\varphi}_i \left(s_i ; \alpha_i'' \right), \overline{\boldsymbol{\sigma}}_{-i} \left(\mathbf{s}_{-i} \right), \omega \right) dF \left(\mathbf{s}_{-i} \left| \omega ; \boldsymbol{\alpha}_{-i} \right. \right) dF \left(s_i, \omega ; \alpha_i'' \right) \\
&= \lambda U_i \left(\alpha_i', \boldsymbol{\alpha} \right) + (1 - \lambda) U_i \left(\alpha_i'', \boldsymbol{\alpha} \right)
\end{aligned}$$

where the first inequality follows from the optimality of $\overline{\varphi}_i$ and the second from the assumption of joint concavity. □

The joint concavity condition of Proposition 3 does not lend itself to a simple decomposition into separate components placed directly on the primitives of the game. Indeed, it can be seen by inspection that the integral (9) will be concave in $a_i(s_i)$ if we assume that u_i is concave in a_i. In addition, the integral (9) will also be concave in α_i' if $F(s_i, \omega; \alpha_i)$ is concave in α_i in the supermodular order (the proof of the latter fact is a direct dual of that for the convex case). However, although the stated conditions guarantee the concavity of (9) in each of the two arguments $a_i(s_i)$ and α_i', they are not sufficient for the needed joint concavity in $(a_i(s_i), \alpha_i')$. This is the reason for which we claim that the sufficient conditions for concavity *seem to be* more restrictive than their counterparts for convexity.

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