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Mathematics for Economics

Note 9: Parametric Optimization - Envelope Theorem and Comparative Statics

Note 9 is based on de la Fuente (2000, Ch. 7), Madden (1986, Ch. 8) and Simon and Blume (1994, Ch. 19).

Parameterized Optimization in Economics

Most of the economic analysis considers whole families of optimization problems instead of just one in isolation. Let us consider a simple model of one-choice variable and a single parameter

$$\max_x \left\{\ln x : x^2 - \alpha \leq 0\right\} \tag{1}$$

where $\alpha \in \mathbb{R}_{++}$ and $x \in \mathbb{R}_{++}$. For any specification of α, this is a standard nonlinear programming problem. In particular, since the constraint set entails an inequality it is a Kuhn-Tucker problem. Problem (1) differs from the ones in Note 8 in that now both the maximizer x^* and the maximum value of the objective function $f(x^*) = \ln x^*$ will depend on the value of α. For this parametrized family of optimization problems two questions are of fundamental relevance: How does the maximum value of the objective function depend on α?; and How does the maximizer depend on α? We start answering them for problem (1), and then extend the idea to more general cases.

For any $\alpha \in \mathbb{R}_{++}$ and $x \in \mathbb{R}_{++}$ all the points in the feasible (or constraint) set satisfy the NDCQ. The Lagrangian function is

$$L(x, \lambda; \alpha) = \ln x - \lambda\left(x^2 - \alpha\right)$$

where we separate the variables of choice x and λ from the parameter α by a semi-colon. Since f is C^1 and nonstationary (see Corollary 3 in Note 8) the necessary first order conditions are

$$\begin{aligned} \frac{\partial L(x, \lambda; \alpha)}{\partial x} &= \frac{1}{x} - 2\lambda x = 0 \\ x^2 - \alpha &= 0 \\ \lambda &> 0. \end{aligned}$$

Then $x^* = +\sqrt{\alpha}$, $\lambda^* = 1/2\alpha$ and $f(x^*) = (1/2)\ln\alpha$. As we anticipated, all these functions depend on α. Moreover, $\partial x^*(\alpha)/\partial\alpha = (1/2)\alpha^{-1/2} > 0$ and $\partial f[x^*(\alpha)]/\partial\alpha = (1/2)\alpha^{-1} > 0$ for $\alpha \in \mathbb{R}_{++}$. This is the kind of statements economists care about. We next provide a general framework to address these issues.

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ be the choice variables and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_l)$ be a vector of all parameters appearing in our model. We will allow both objective and constraint functions to depend on $\boldsymbol{\alpha}$ and will write the values of these functions as $f(\mathbf{x}; \boldsymbol{\alpha})$ and $U(\boldsymbol{\alpha})$ respectively. The parameters must belong to some set Ω that we name the admissible parameter set.

We consider the general family of problems

$$\max_{\mathbf{x}} \{f(\mathbf{x}; \boldsymbol{\alpha}) : \mathbf{x} \in U(\boldsymbol{\alpha})\} \tag{F.P}$$

where $\boldsymbol{\alpha} \in \Omega \subseteq \mathbb{R}^l$ and $\mathbf{x} \in X \subseteq \mathbb{R}^n$. We assume both Ω and X are open sets.

The set of maximizers is described by the decision rule

$$S(\boldsymbol{\alpha}) = \arg\max \{f(\mathbf{x}; \boldsymbol{\alpha}) : \mathbf{x} \in U(\boldsymbol{\alpha})\}.$$

That is, $S(\boldsymbol{\alpha})$ is the set of elements $\mathbf{x}^*$ that are the optimal solutions to problem (F.P) for a given $\boldsymbol{\alpha} \in \Omega$. If the solution to (F.P) is unique for each value of $\boldsymbol{\alpha}$, then the decision rule becomes a function, and we write $\mathbf{x}^* = \mathbf{x}^*(\boldsymbol{\alpha})$.

The payoff accruing to an optimizing agent is given by the (maximum) value function, $V : \Omega \to \mathbb{R}$, defined by

$$V(\boldsymbol{\alpha}) = \max_{\mathbf{x}} \{f(\mathbf{x}; \boldsymbol{\alpha}) : \mathbf{x} \in U(\boldsymbol{\alpha})\} = f(\mathbf{x}^*; \boldsymbol{\alpha}), \text{ where } \mathbf{x}^* \in S(\boldsymbol{\alpha}).$$

As we noticed before, in most economic applications we are interested in asserting how the maximum value function is affected by changes in the parameters, and in the comparative statics of the decision rule $S(\boldsymbol{\alpha})$. That is, we would like to know how the maximum payoff and the behavior of the agent varies in response to changes in his environment (e.g. the prices he faces, the income, etc.). The next section answers the first question, and the last one focuses on the second question.

The Envelope Theorems

This section considers three versions of problem (F.P), that differ with respect to the constraint set, and discusses the corresponding theorems that study the effects of changing $\boldsymbol{\alpha}$ on the maximum value function $V(\boldsymbol{\alpha})$. Such theorems are called Envelope Theorems.

We start with the Envelope Theorem for unconstrained optimization problems.

Theorem 1. (Envelope Theorem for unconstrained optimization) Consider problem (F.P) with $U(\boldsymbol{\alpha}) = \mathbb{R}^n$, and assume $f(\mathbf{x};\boldsymbol{\alpha})$ is a C^1 function. Let $\mathbf{x}^*(\boldsymbol{\alpha})$ be a solution to this problem. Suppose $\mathbf{x}^*(\boldsymbol{\alpha})$ is a C^1 function of $\boldsymbol{\alpha}$. Then

$$\frac{\partial V(\boldsymbol{\alpha})}{\partial \alpha_i} = \frac{\partial f}{\partial \alpha_i}\left[\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha}\right]$$

for $i = 1, ..., l$.

Proof. Via the Chain rule

$$\begin{aligned}\frac{\partial V(\boldsymbol{\alpha})}{\partial \alpha_i} &= \sum\nolimits_{j=1}^{n} \frac{\partial f}{\partial x_j}\left[\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha}\right]\frac{\partial x_j^*}{\partial \alpha_i}(\boldsymbol{\alpha}) + \frac{\partial f}{\partial \alpha_i}\left[\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha}\right] \\ &= \frac{\partial f}{\partial \alpha_i}\left[\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha}\right]\end{aligned}$$

since $\partial f/\partial x_j\left[\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha}\right] = 0$ for $j = 1, 2, ..., n$ by the usual first order conditions. ■

In problem (F.P) each parameter α_i affects the maximum value function in two ways. In a direct way, α_i affects $V(\boldsymbol{\alpha})$ because it is part of the objective function $f(\mathbf{x};\boldsymbol{\alpha})$. Indirectly, α_i affects $V(\boldsymbol{\alpha})$ through its effect on the variables of choice $\mathbf{x}^*(\boldsymbol{\alpha})$. The Envelop Theorem states that to assert $\partial V/\partial \alpha_i$ we only need to consider the direct effect of α_i on $f(\mathbf{x};\boldsymbol{\alpha})$, $\partial f/\partial \alpha_i$, and evaluate this partial derivative at the optimal $\mathbf{x}^*(\boldsymbol{\alpha})$!

The next example applies this theorem to a standard firm problem.

Example 1. A competitive firm sells its product at a unit price $p = \alpha\bar{p}$, where $\alpha > 0$ is a positive parameter that captures the strength of demand. Assume the cost of producing y units is $C(y)$, with $C(0) = 0$ $C'(y) > 0$ and $C''(y) > 0$. The firm profit function is

$$\pi(\alpha) = \max\nolimits_y \left\{\alpha\bar{p}y - C(y)\right\}.$$

The conditions on the cost function guarantee that there is a nonzero profit-maximizing output $y^*(\alpha)$ which depends smoothly on α. By the Envelope Theorem for unconstrained optimization problems we get

$$\frac{\partial \pi(\alpha)}{\partial \alpha} = \bar{p} y^*(\alpha) > 0.$$

As expected, the firm makes more profits when the demand is stronger. ▲

Let us extend the analysis to constrained optimization problems. We start with the Lagrange problem, and end the analysis with the problem of Kuhn-Tucker.

Theorem 2. (Envelope Theorem for the Lagrange problem) Consider problem (F.P) with $U(\boldsymbol{\alpha}) = \{\mathbf{x} \in \mathbb{R}^n : h_1(\mathbf{x}; \boldsymbol{\alpha}) = 0, ..., h_m(\mathbf{x}; \boldsymbol{\alpha}) = 0\}$, and assume $f(\mathbf{x}; \boldsymbol{\alpha})$ is a C^1 function. Let $\mathbf{x}^*(\boldsymbol{\alpha})$ be a solution to this problem. Suppose $\mathbf{x}^*(\boldsymbol{\alpha})$ and the Lagrange multipliers $\boldsymbol{\mu}^*(\boldsymbol{\alpha})$ are C^1 functions of $\boldsymbol{\alpha}$ and that the NDCQ holds. Then

$$\frac{\partial V(\boldsymbol{\alpha})}{\partial \alpha_i} = \frac{\partial L}{\partial \alpha_i}[\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\mu}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$$

for $i = 1, ..., l$, where L is the natural Lagrangian function for this problem.

Proof. Form the Lagrangian for the maximization problem

$$L(\mathbf{x}, \boldsymbol{\mu}; \boldsymbol{\alpha}) = f(\mathbf{x}; \boldsymbol{\alpha}) - \sum_{t=1}^{m} \mu_t h_t(\mathbf{x}; \boldsymbol{\alpha}). \tag{2}$$

From the first order conditions we know that

$$\frac{\partial f}{\partial x_j}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] - \sum_{t=1}^{m} \mu_t^*(\boldsymbol{\alpha}) \frac{\partial h_t}{\partial x_j}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] = 0 \text{ for } j = 1, ..., n \tag{3}$$

$$h_t[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] = 0 \text{ for } t = 1, ..., m. \tag{4}$$

The partial derivative of the Lagrangian with respect to α_i at $[\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\mu}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$ is

$$\frac{\partial L}{\partial \alpha_i}[\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\mu}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] = \frac{\partial f}{\partial \alpha_i}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] - \sum_{t=1}^{m} \mu_t^*(\boldsymbol{\alpha}) \frac{\partial h_t}{\partial \alpha_i}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]. \tag{5}$$

We need to show that the derivative of the maximum value function with respect to α_i is equal to the last expression.

Let us then differentiate $V(\boldsymbol{\alpha})$ with respect to α_i

$$\frac{\partial V(\boldsymbol{\alpha})}{\partial \alpha_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] \frac{\partial x_j^*}{\partial \alpha_i}(\boldsymbol{\alpha}) + \frac{\partial f}{\partial \alpha_i}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]. \tag{6}$$

Next go back to the first order conditions and rearrange terms

$$\frac{\partial f}{\partial x_j}\left[\mathbf{x}^*\left(\boldsymbol{\alpha}\right);\boldsymbol{\alpha}\right]=\sum\nolimits_{t=1}^{m}\mu_t^*\left(\boldsymbol{\alpha}\right)\frac{\partial h_t}{\partial x_j}\left[\mathbf{x}^*\left(\boldsymbol{\alpha}\right);\boldsymbol{\alpha}\right]\text{ for } j=1,...,n. \tag{7}$$

Substituting (7) in (6) and rearranging terms we get

$$\frac{\partial V\left(\boldsymbol{\alpha}\right)}{\partial \alpha_i}=\sum\nolimits_{t=1}^{m}\mu_t^*\left(\boldsymbol{\alpha}\right)\sum\nolimits_{j=1}^{n}\frac{\partial h_t}{\partial x_j}\left[\mathbf{x}^*\left(\boldsymbol{\alpha}\right);\boldsymbol{\alpha}\right]\frac{\partial x_j^*}{\partial \alpha_i}\left(\boldsymbol{\alpha}\right)+\frac{\partial f}{\partial \alpha_i}\left[\mathbf{x}^*\left(\boldsymbol{\alpha}\right);\boldsymbol{\alpha}\right]. \tag{8}$$

The last trick is to go back to the first order conditions and differentiate (4) with respect to α_i

$$\sum\nolimits_{j=1}^{n}\frac{\partial h_t}{\partial x_j}\left[\mathbf{x}^*\left(\boldsymbol{\alpha}\right);\boldsymbol{\alpha}\right]\frac{\partial x_j^*}{\partial \alpha_i}\left(\boldsymbol{\alpha}\right)+\frac{\partial h_t}{\partial \alpha_i}\left[\mathbf{x}^*\left(\boldsymbol{\alpha}\right);\boldsymbol{\alpha}\right]=0\text{ for } t=1,...,m. \tag{9}$$

Substituting (9) in (8) we get

$$\frac{\partial V\left(\boldsymbol{\alpha}\right)}{\partial \alpha_i}=-\sum\nolimits_{t=1}^{m}\mu_t^*\left(\boldsymbol{\alpha}\right)\frac{\partial h_t}{\partial \alpha_i}\left[\mathbf{x}^*\left(\boldsymbol{\alpha}\right);\boldsymbol{\alpha}\right]+\frac{\partial f}{\partial \alpha_i}\left[\mathbf{x}^*\left(\boldsymbol{\alpha}\right);\boldsymbol{\alpha}\right].$$

which is identical to expression (5). ■

The last theorem is extremely important in economics. It allows, for instance, to provide a nice interpretation of the Lagrange multipliers.

Example 2. Consider a simple version of the Lagrange problem

$$\max\nolimits_{\mathbf{x}}\left\{f\left(\mathbf{x}\right):\mathbf{x}\in U\right\} \tag{10}$$

where $U=\left\{\mathbf{x}\in\mathbb{R}^n:h\left(\mathbf{x}\right)=0\right\}$. Now suppose that we perturb the constraint by an amount $\alpha\in\Omega\subset\mathbb{R}$ (with $0\in\Omega$) as follows

$$\max\nolimits_{\mathbf{x}}\left\{f\left(\mathbf{x}\right):h\left(\mathbf{x}\right)=\alpha\right\} \tag{11}$$

Notice that (11) reduces to (10) when $\alpha=0$. Assume the conditions of the Envelope Theorem hold. The Lagrangian is given by

$$L\left(\mathbf{x};\alpha\right)=f\left(\mathbf{x}\right)-\mu\left[h\left(\mathbf{x}\right)-\alpha\right].$$

Therefore,

$$\frac{\partial V\left(\alpha\right)}{\partial \alpha}=\mu^*\left(\alpha\right).$$

Then $\mu^*(\alpha)$ measures the change in the optimal value function with respect to the parameter α. That is, how much the maximum value function increases if we relax the constraint in one single unit. ▲

Theorem 3. (Envelope Theorem for the Kuhn-Tucker problem) Consider problem (F.P) with $U(\boldsymbol{\alpha}) = \{\mathbf{x} \in X : g_1(\mathbf{x}; \boldsymbol{\alpha}) \leq 0, ..., g_k(\mathbf{x}; \boldsymbol{\alpha}) \leq 0\}$, and assume $f(\mathbf{x}; \boldsymbol{\alpha})$ is a C^1 function. Let $\mathbf{x}^*(\boldsymbol{\alpha})$ be a solution to this problem. Suppose $\mathbf{x}^*(\boldsymbol{\alpha})$ and the multipliers $\boldsymbol{\lambda}^*(\boldsymbol{\alpha})$ are C^1 functions of $\boldsymbol{\alpha}$ and that the NDCQ holds. Then

$$\frac{\partial V(\boldsymbol{\alpha})}{\partial \alpha_i} = \frac{\partial L}{\partial \alpha_i}\left[\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\lambda}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}\right]$$

for $i = 1, ..., l$, where L is the natural Lagrangian function for this problem.

Proof. (**Prove this result.**) ■

The next two examples end this section.

Example 3. Consider the problem

$$\max_{\mathbf{x} \in \mathbb{R}^2_{++}} \left\{ x_1^{1/2} x_2^{1/2} : \alpha_1 x_1 + \alpha_2 x_2 - \alpha_3 \leq 0 \right\}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^3_{++}$. (**Find the solution function, multiplier function and maximum value function. Differentiate these expressions with respect to the three parameters, and compare these results with the claim of the Envelope Theorem for the Kuhn-Tucker problem.**) ▲

Example 4. (Roy's Identity) Suppose the conditions of Theorem 3 hold in the problem

$$\max_{\mathbf{x}} \left\{ U(\mathbf{x}) : \sum_{i=1}^{n} p_i x_i \leq m \right\}$$

where $(\mathbf{p}, m) \in \mathbb{R}^{n+1}_{++}$ and $\mathbf{x} \in \mathbb{R}^n_{++}$. Let $U(\mathbf{x})$ be strictly increasing. [**Use the last Envelope Theorem to prove that** $x_i^*(\mathbf{p}, m) = -\frac{\partial V / \partial p_i}{\partial V / \partial m}$ **for** $i = 1, ..., n$.] ▲

Smooth Dependence of the Maximizers on $\boldsymbol{\alpha}$

All the results in the previous section relied on a basic hypothesis: the smooth dependence of the maximizers on the parameters of the problem. In this section we look at this hypothesis more carefully.

Let us consider first the unconstrained problem of Theorem 1

$$\max_{\mathbf{x}} \{f(\mathbf{x}; \boldsymbol{\alpha}) : \mathbf{x} \in \mathbb{R}^n\} \tag{12}$$

where $\boldsymbol{\alpha} \in \Omega \subseteq \mathbb{R}^l$ and Ω is an open set. Let f be a C^2 function. Since we are assuming that a maximizer $\mathbf{x}^*(\boldsymbol{\alpha})$ exists, then $\mathbf{x}^*(\boldsymbol{\alpha})$ is a solution to the typical first-order conditions

$$\begin{aligned} \frac{\partial f}{\partial x_1}(\mathbf{x}; \boldsymbol{\alpha}) &= 0 \\ \vdots &= \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}; \boldsymbol{\alpha}) &= 0. \end{aligned}$$

By the IFT, we can solve these n equations for the n unknowns $x_1, ..., x_n$ as C^1 functions of the exogenous parameters $\boldsymbol{\alpha}$, provided that the Jacobian of these functions with respect to the endogenous variables $x_1, ..., x_n$ is non-singular at $[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$. But the Jacobian of the first-order partial derivatives $\partial f / \partial x_i$ is simply the Hessian of f at $[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$

$$D^2 f[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}. \tag{13}$$

The Hessian matrix evaluated at $[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$ is generally non-singular. Moreover, since (12) is a maximization problem, its determinant has the same sign as $(-1)^n$ by the usual second order sufficient conditions.

This means that we can replace the hypothesis that $\mathbf{x}^*(\boldsymbol{\alpha})$ is a C^1 function of α_i in Theorem 1, by the hypothesis that $\mathbf{x}^*(\boldsymbol{\alpha})$ is a non-degenerate critical point of f in the sense that the Hessian matrix (13) of f is non-singular at $[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$. The next theorem captures this observation.

Theorem 4. (Smoothness of $x^*(\boldsymbol{\alpha})$ on $\boldsymbol{\alpha}$ in the unconstrained problem) Consider problem (F.P) with $U(\boldsymbol{\alpha}) = \mathbb{R}^n$. Let $\mathbf{x}^*(\boldsymbol{\alpha})$ be the solution of the parametrized maximization problem. Fix the value of the parameters at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^0$. If the Hessian matrix (13) is non-singular at the point $\left[\mathbf{x}^*\left(\boldsymbol{\alpha}^0\right); \boldsymbol{\alpha}^0\right]$, then $\mathbf{x}^*(\boldsymbol{\alpha})$ is a C^1 function of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^0$.

A similar analysis works for constrained problems. Consider the parametrized constrained maximization problem of Theorem 2

$$\max_{\mathbf{x}} \{f(\mathbf{x}; \boldsymbol{\alpha}) : \mathbf{x} \in U(\boldsymbol{\alpha})\}$$

where $U(\boldsymbol{\alpha}) = \{\mathbf{x} \in \mathbb{R}^n : h_1(\mathbf{x}; \boldsymbol{\alpha}) = 0, ..., h_m(\mathbf{x}; \boldsymbol{\alpha}) = 0\}$, $\boldsymbol{\alpha} \in \Omega \subseteq \mathbb{R}^l$ and we assume that Ω is an open set. Let f be a C^2 function. Assume the NDCQ holds at $[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$, that is

$$\text{rank}\begin{pmatrix} \frac{\partial h_1}{\partial x_1}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] & \cdots & \frac{\partial h_1}{\partial x_n}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] & \cdots & \frac{\partial h_m}{\partial x_n}[\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] \end{pmatrix} = m. \tag{14}$$

Let us write the corresponding Lagrangian function

$$L(\mathbf{x}, \boldsymbol{\mu}; \boldsymbol{\alpha}) = f(\mathbf{x}; \boldsymbol{\alpha}) - \mu_1 h_1(\mathbf{x}; \boldsymbol{\alpha}) - ... - \mu_m h_m(\mathbf{x}; \boldsymbol{\alpha}).$$

The constrained maximizer $\mathbf{x}^*(\boldsymbol{\alpha})$ must satisfy the first order conditions

$$\begin{aligned} \frac{\partial L}{\partial x_1}(\mathbf{x}, \boldsymbol{\mu}; \boldsymbol{\alpha}) = 0, ..., \frac{\partial L}{\partial x_n}(\mathbf{x}, \boldsymbol{\mu}; \boldsymbol{\alpha}) = 0 \\ \frac{\partial L}{\partial \mu_1}(\mathbf{x}, \boldsymbol{\mu}; \boldsymbol{\alpha}) = 0, ..., \frac{\partial L}{\partial \mu_m}(\mathbf{x}, \boldsymbol{\mu}; \boldsymbol{\alpha}) = 0. \end{aligned}$$

This is a system of $n+m$ equations in $n+m$ unknowns $x_1, ..., x_n, \mu_1, ..., \mu_m$. Once again, we call on the IFT to provide conditions that guarantee that $\mathbf{x}^*(\boldsymbol{\alpha})$ and $\boldsymbol{\mu}^*(\boldsymbol{\alpha})$ will depend smoothly on the exogenous parameter α_i; the Jacobian of these equations with respect to the endogenous variables must be an $(n+m) \times (n+m)$ non-singular matrix at $[\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\mu}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$. This Jacobian is simply the Hessian of the Lagrangian

$$D^2 L[\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\mu}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}] = \begin{pmatrix} \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial \mu_1 \partial x_1} & \cdots & \frac{\partial^2 L}{\partial \mu_m \partial x_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 L}{\partial x_1 \partial x_n} & \vdots & \frac{\partial^2 L}{\partial x_n^2} & \frac{\partial^2 L}{\partial \mu_1 \partial x_n} & \cdots & \frac{\partial^2 L}{\partial \mu_m \partial x_n} \\ \frac{\partial^2 L}{\partial x_1 \partial \mu_1} & \cdots & \frac{\partial^2 L}{\partial x_n \partial \mu_1} & \frac{\partial^2 L}{\partial \mu_1^2} & \cdots & \frac{\partial^2 L}{\partial \mu_m \partial \mu_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 L}{\partial x_1 \partial \mu_m} & \cdots & \frac{\partial^2 L}{\partial x_n \partial \mu_m} & \frac{\partial^2 L}{\partial \mu_1 \partial \mu_m} & \cdots & \frac{\partial^2 L}{\partial \mu_m^2} \end{pmatrix} \tag{15}$$

evaluated at $[\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\mu}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$.

This is the matrix that we used before to check the second order sufficient conditions for a constrained maximum. The Hessian usually has non-zero determinant at $[\mathbf{x}^*(\boldsymbol{\alpha}), \lambda^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$. In fact, for a non-degenerate constrained maximization problem, its determinant has the same sign as $(-1)^n$. As a consequence, we can replace the conditions of Theorem 2 that $\mathbf{x}^*(\boldsymbol{\alpha})$ and $\boldsymbol{\mu}^*(\boldsymbol{\alpha})$ are C^1 functions of the parameter α_i and the requirement of the NDCQ by the non-degenerate second order condition for constrained problems, namely that the Hessian of the Lagrangian has a nonzero determinant at $[\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\mu}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}]$. (It can be shown that the NDCQ is a necessary condition for the latter to hold.)

Theorem 5. (Smoothness of $x^*(\boldsymbol{\alpha})$ and $\boldsymbol{\mu}^*(\boldsymbol{\alpha})$ on $\boldsymbol{\alpha}$ in the Lagrange problem) Consider problem (F.P) with $U(\boldsymbol{\alpha}) = \{\mathbf{x} \in \mathbb{R}^n : h_1(\mathbf{x}; \boldsymbol{\alpha}) = 0, ..., h_m(\mathbf{x}; \boldsymbol{\alpha}) = 0\}$. Let $\mathbf{x}^*(\boldsymbol{\alpha})$ be the solution of the parametrized maximization problem, and let $\boldsymbol{\mu}^*(\boldsymbol{\alpha})$ be the corresponding Lagrange multipliers. Fix the value of the parameters at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^0$. If the Hessian matrix is non-singular at the point $\left[\mathbf{x}^*\left(\boldsymbol{\alpha}^0\right), \boldsymbol{\mu}^*\left(\boldsymbol{\alpha}^0\right); \boldsymbol{\alpha}^0\right]$, then

(a) $\mathbf{x}^*(\boldsymbol{\alpha})$ and $\boldsymbol{\mu}^*(\boldsymbol{\alpha})$ are C^1 functions of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^0$; and

(b) the NDCQ holds at $\left[\mathbf{x}^*\left(\boldsymbol{\alpha}^0\right), \boldsymbol{\mu}^*\left(\boldsymbol{\alpha}^0\right); \boldsymbol{\alpha}^0\right]$.

Remark 1. The extension of the previous result to the K-T problem follows immediately.

We next extend Example 4 in Note 3.

Example 5. (Comparative Statics) Let us consider a firm that produces a good y by using a single input x. The firm sells the output and acquires the input in competitive markets. The market price of y is p, and the cost of each unit of x is just w. Its technology is given by $f : \mathbb{R}_+ \to \mathbb{R}_+$, where f is a C^2 function with $f' > 0$ and $f'' < 0$. Its profits are given by

$$\widetilde{\pi}(x; p, w) = pf(x) - wx. \tag{16}$$

The firm selects the input level, x, in order to maximize profits. We would like to know how its choice of x is affected by a change in w.

Assuming an interior solution, the first-order condition of this optimization problem is

$$\frac{\partial \widetilde{\pi}}{\partial x}(x; p, w) = pf'(x) - w = 0 \tag{17}$$

for some $x = x^*$. We know that if such x^* exists, it is a global maximum and it is unique. (**Why?**)

Notice that here

$$\frac{\partial^2 \widetilde{\pi}}{\partial x^2}(x^*; p, w) = pf''(x^*) < 0. \tag{18}$$

So the conditions of Theorem 4 are satisfied. It follows that if there is an $x^* = x(p, w)$ that satisfies (16) it is differentiable. Moreover, by the IFT we get

$$\frac{\partial x}{\partial p}(p, w) = -\frac{f'(x^*)}{pf''(x^*)} > 0 \text{ and } \frac{\partial x}{\partial w}(p, w) = -\frac{-1}{pf''(x^*)} < 0. \tag{19}$$

We conclude that if the price of the output increases, then the firm will acquire more of the input; and the opposite holds if the price of the input increases. ▲