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Mathematics for Economics

Note 5: Convex Sets and Concave Functions

Note 5 is based on Madden (1986, Ch. 1, 2, 4 and 7) and Simon and Blume (1994, Ch. 13 and 21).

Concave functions play a key role in optimization as they provide much more structure to the problem. For instance, in unconstrained optimization problems if the objective function is C^1 and concave, then the first order conditions fully characterize global maxima.

We start the analysis with the notion of convex sets, as they are the domain of concave functions. (Do not confuse convex sets with convex functions!) Then we define concave functions and characterize them in alternative ways. We end this note describing some useful properties of this kind of functions.

Convex Sets

Let us consider a set $S \subset \mathbb{R}$ and suppose $x, x' \in S$ with $x \leq x'$. The set $[x, x']$ is then a set of real numbers between (and including) x and x', or visually, the set of all points on the line joining x to x'. Now it may or may not be that $[x, x'] \subset S$. If it is the case that for all $x, x' \in S$ with $x \leq x'$ we have that $[x, x'] \subset S$, then we say that S is a convex set. The next definition formalizes this idea.

Definition 1. (Convex set) A set $S \subset \mathbb{R}$ is convex if, $\forall \lambda \in [0, 1]$ and $\forall x, x' \in S$,

$$\lambda x + (1 - \lambda)\, x' \in S.$$

If $x, x' \in \mathbb{R}$ and $\lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)\, x'$ is said to be a convex combination of x and x'. Definition 1 simply says that S is a convex set if any convex combination of every two elements of S is also in S.

Example 1. Consider the interval $[a, b] \subset \mathbb{R}$. We next show that this interval is a convex set. Let $x, x' \in [a, b]$ be two arbitrary elements. We need to prove that $\lambda x + (1 - \lambda)\, x' \in [a, b]$ for all $\lambda \in [0, 1]$.

Select an arbitrary λ in $[0,1]$. Since $x, x' \in [a,b]$, then $x, x' \leq b$. As $\lambda \in [0,1]$, it follows that $\lambda x + (1-\lambda) x' \leq b$. Using a similar argument $\lambda x + (1-\lambda) x' \geq a$. As λ, x and x' were arbitrarily chosen, then $\lambda x + (1-\lambda) x' \in [a,b] \ \forall x, x' \in [a,b]$ and $\forall \lambda \in [0,1]$. ▲

Let us move to higher dimensions. The Cartesian product of n sets A_1, A_2, ..., A_n, written $A_1 \times A_2 \times ... \times A_n$, is the set of all ordered tuples $(x_1, x_2, ..., x_n)$ such that $x_1 \in A_1$, $x_2 \in A_2$, ..., and $x_n \in A_n$. A convex set in $\mathbb{R}^n$ can be visualized in the same way as in $\mathbb{R}$; $S \subset \mathbb{R}^n$ is convex if and only if the line joining any two points in the set lies entirely within the set.

Definition 2. A set $S \subset \mathbb{R}^n$ is convex if, $\forall \lambda \in [0,1]$ and $\forall \mathbf{x}, \mathbf{x}' \in S$,

$$\lambda \mathbf{x} + (1-\lambda) \mathbf{x}' \in S.$$

(**Show that** $[a,b] \times [a,b] \subset \mathbb{R}^2$ **is a convex set.**)

The next theorem states that the intersection of two convex sets is a convex set.

Theorem 3. If S and T are two convex sets in $\mathbb{R}^n$ then $S \cap T$ is a convex set.

Proof. Let $\mathbf{x}, \mathbf{x}' \in S \cap T$. Then $\mathbf{x}, \mathbf{x}' \in S$ and $\mathbf{x}, \mathbf{x}' \in T$. Since S and T are convex sets it follows that $\mathbf{x}'' \in S$ and $\mathbf{x}'' \in T$ where

$$\mathbf{x}'' = \lambda \mathbf{x} + (1-\lambda) \mathbf{x}' \text{ and } \lambda \in [0,1].$$

Hence $\mathbf{x}'' \in S \cap T$. Since this is true for any $\mathbf{x}, \mathbf{x}' \in S \cap T$ and any $\lambda \in [0,1]$, $S \cap T$ is convex. ■

In fact the intersection of an infinite collection of convex sets is convex as well. (**Show that the union of two convex sets need not be convex. Hint: find a counter-example.)**

A very important family of subsets of $\mathbb{R}^n$ are the hyperplanes.

Definition 4. A set $H \subset \mathbb{R}^n$ is a hyperplane if it can be described as

$$H = \{\mathbf{x} \in \mathbb{R}^n : \textstyle\sum_{i=1}^n \alpha_i x_i = \beta\}$$

for some $\beta \in \mathbb{R}$ and some $(\alpha_1, ..., \alpha_n) \subset \mathbb{R}^n$ such that $\alpha_i \neq 0$ for some i.

In the case of $n = 1$, then H contains just the single point β/α. In the case of $n = 2$ and $\alpha_1, \alpha_2 \neq 0$, then the hyperplane is a straight line with equation $\alpha_1 x_1 + \alpha_2 x_2 = \beta$. In the usual representation (x_2 measured vertically and x_1 horizontally) if $\alpha_1 = 0$ then H corresponds to an horizontal line. If, alternatively, $\alpha_2 = 0$ then H corresponds to a vertical line.

Generally, a hyperplane in $\mathbb{R}^n$ "splits" $\mathbb{R}^n$ in two parts: those points above or below the hyperplane. Therefore, associated to H we can define

$$H^+ = \{\mathbf{x} \in \mathbb{R}^n : \textstyle\sum_{i=1}^n \alpha_i x_i \geq \beta\}$$

and

$$H^- = \{\mathbf{x} \in \mathbb{R}^n : \textstyle\sum_{i=1}^n \alpha_i x_i \leq \beta\}.$$

Notice that $H^+ \cup H^- = \mathbb{R}^n$ and $H^+ \cap H^- = H$.

Theorem 5. A hyperplane H and its associated half spaces H^+ and H^- are always convex sets.

Proof. (**Show this result.**) ■

The major importance of hyperplanes stems from the celebrated theorems about (general) convex sets. We describe one of them without proof.

Theorem 6. **(The Separating Hyperplane Theorem)** If S and T are two disjoint convex sets in $\mathbb{R}^n$ then there is a hyperplane $H \subset \mathbb{R}^n$ such that $S \subset H^+$ and $T \subset H^-$.

(**To convince yourself about the result draw two pictures, one for $H \subset \mathbb{R}$ and another one for $H \subset \mathbb{R}^2$.**)

Concave Functions

Concave Functions on $\mathbb{R}$

For functions of one variable there is a familiar and useful visual representation. The graph of f, which we denote G_f, is described as follows

$$G_f = \left\{(y, x) \in \mathbb{R}^2 : y = f(x), x \in \mathbb{R}\right\}.$$

Then G_f is the set of all ordered pairs $[f(x), x]$ for $x \in \mathbb{R}$. When $x \in \mathbb{R}$, then $G_f \subset \mathbb{R}^2$.

If we draw the graph of a function $f : \mathbb{R} \to \mathbb{R}$ we may or may not find that this graph possesses the following geometric feature

(GF) The straight line joining any two points on the graph lies entirely on or below the graph.

The concept of a concave function evolves from (GF). A function of one variable is concave if and only if its graph possesses the geometric feature (GF). We next formalize this idea.

Definition 7. (**Concavity in** $\mathbb{R}$) A function $f : \mathbb{R} \to \mathbb{R}$ is concave if

$$f\left[\lambda x + (1-\lambda) x'\right] \geq \lambda f(x) + (1-\lambda) f\left(x'\right)$$

$\forall \lambda \in [0,1]$ and $\forall x, x' \in \mathbb{R}$. For strict concavity replace $\geq$ by $>$ and $[0,1]$ by $(0,1)$ whenever $x \neq x'$.

Remark. In Definition 7 the domain of f is the whole real line, which is a convex set. As we shall emphasize at the end of this note, the domain of concave functions can be restricted to any other convex set $S \subset \mathbb{R}$.

(**Use Definition 7 to show that** $f(x) = -x^2$ **is a concave function on** $\mathbb{R}$.) By doing so you will realize that this demonstration is tortuous even when the function is very simple. Fortunately, at least for differentiable functions, there are easier ways to identify concavity.

If a function $f : \mathbb{R} \to \mathbb{R}$ is concave, then it is continuous. Nevertheless it may not be differentiable, e.g. $f(x) = -|x|$ is strictly concave on $\mathbb{R}$ but it is not differentiable at $x = 0$. Consider for the moment the graph of a concave function that is both continuous and differentiable. After a minute of reflection you will realize the graph has the next property

(New GF) At any point of the graph, the tangent line to the graph at that point lies entirely on or above the graph.

Using the tangent line formula, the last observation translate to the statement

$$f\left(x'\right) + \left(x - x'\right) f'\left(x'\right) \geq f(x)$$

$\forall x, x' \in \mathbb{R}$. Our argument states that if the function is differentiable and concave, then the last inequality must hold. The next theorem states that for C^1 functions the opposite is also true.

Theorem 8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1. Then f is concave if and only if

$$f\left(x'\right) + \left(x - x'\right) f'\left(x'\right) \geq f(x)$$

$\forall x, x' \in \mathbb{R}$. For strict concavity replace $\geq$ by $>$ whenever $x \neq x'$.

Proof. (If) Suppose $f(x') + (x - x') f'(x') \geq f(x)$ $\forall x, x' \in \mathbb{R}$. We have to show that this implies

$$f\left[\lambda x + (1 - \lambda) x'\right] \geq \lambda f(x) + (1 - \lambda) f\left(x'\right)$$

$\forall \lambda \in [0,1]$ and $\forall x, x' \in \mathbb{R}$. To this end choose any two $x, x' \in \mathbb{R}$ and any $\lambda \in [0,1]$. We write x'' for $\lambda x + (1 - \lambda) x'$. From the initial assumption it follows that

$$f\left(x''\right) + \left(x - x''\right) f'\left(x''\right) \geq f(x) \tag{1}$$

and

$$f\left(x''\right) + \left(x' - x''\right) f'\left(x''\right) \geq f\left(x'\right). \tag{2}$$

Multiplying (1) by λ and (2) by $(1 - \lambda)$ and adding gives, since $\lambda \in [0,1]$,

$$f\left(x''\right) + f'\left(x''\right)\left[\lambda x + (1 - \lambda) x' - x''\right] \geq \lambda f(x) + (1 - \lambda) f\left(x'\right).$$

As $x'' = \lambda x + (1 - \lambda) x'$ then

$$f\left[\lambda x + (1 - \lambda) x'\right] \geq \lambda f(x) + (1 - \lambda) f\left(x'\right).$$

Since this is true $\forall \lambda \in [0,1]$ and $\forall x, x' \in \mathbb{R}$ then f is concave. (Note that $f'(x'') < \infty$ because f is C^1.)

(Only if) Now suppose f is concave. We need to show that this implies

$$f\left(x'\right) + \left(x - x'\right) f'\left(x'\right) \geq f(x)$$

$\forall x, x' \in \mathbb{R}$. If $x = x'$ the implication follows immediately, so suppose this is not the case. By concavity

$$f\left[\lambda x + (1 - \lambda) x'\right] \geq \lambda f(x) + (1 - \lambda) f\left(x'\right)$$

$\forall \lambda \in [0,1]$ and $\forall x, x' \in \mathbb{R}$. Or rearranging terms

$$f\left[x' + \lambda\left(x - x'\right)\right] - f\left(x'\right) \geq \lambda\left[f(x) - f\left(x'\right)\right]$$

$\forall \lambda \in [0,1]$ and $\forall x, x' \in \mathbb{R}$. Assuming λ differs from 0 we can get

$$\left(x - x'\right) \frac{f\left[x' + \lambda\left(x - x'\right)\right] - f\left(x'\right)}{\lambda\left(x - x'\right)} \geq f\left(x\right) - f\left(x'\right)$$

$\forall \lambda \in (0,1]$ and $\forall x, x' \in \mathbb{R}$. Substituting $\lambda\left(x - x'\right)$ by $\triangle x$

$$\left(x - x'\right) \frac{f\left[x' + \triangle x\right] - f\left(x'\right)}{\triangle x} \geq f\left(x\right) - f\left(x'\right).$$

Taking take the limit when $\lambda \to 0$

$$\left(x - x'\right) f'\left(x'\right) \geq f\left(x\right) - f\left(x'\right)$$

or

$$f\left(x'\right) + \left(x - x'\right) f'\left(x'\right) \geq f\left(x\right)$$

$\forall x, x' \in \mathbb{R}$. This completes the proof. ■

(**Now use Theorem 8 to show that** $f\left(x\right) = -x^2$ **is a concave function on** $\mathbb{R}$.) The characterization of concavity takes an even simpler form for functions that are twice continuously differentiable. This characterization has again an obvious geometric meaning. The slope of the graph diminishes as x increases; that is $f''\left(x\right) \leq 0$ everywhere. The next theorem states that the "reverse" implication is also true.

Theorem 9. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^2. Then f is concave if and only if

$$f''\left(x\right) \leq 0$$

$\forall x \in \mathbb{R}$. Moreover, if $f''\left(x\right) < 0$ $\forall x \in \mathbb{R}$, then f is strictly concave.

Proof. (**Show this result.**) ■

(**Use Theorem 9 to show that** $f\left(x\right) = -x^2$ **is a concave function on** $\mathbb{R}$.) Note that $f''\left(x\right) < 0$ for all $x \in \mathbb{R}$ is sufficient (but not necessary) for f to be strictly concave. For instance, the function $f\left(x\right) = -x^4$ is strictly concave but $f''\left(0\right) = 0$.

Convex functions can be defined in a similar way. In particular $f : \mathbb{R} \to \mathbb{R}$ is a convex function if and only if $-f : \mathbb{R} \to \mathbb{R}$ is concave. So you do not need to remember more theorems!

There is an alternative characterization of concave and convex functions that does not require differentiability. It is in terms of the convexity of certain sets associated with the graphs of the corresponding functions. For $f : \mathbb{R} \to \mathbb{R}$ the hypograph of f, HG_f is

$$HG_f = \left\{ (y, x) \in \mathbb{R}^2 : y \le f(x) \text{ for some } x \in \mathbb{R} \right\}.$$

Similarly, the epigraph of f, EG_f is

$$EG_f = \left\{ (y, x) \in \mathbb{R}^2 : y \ge f(x) \text{ for some } x \in \mathbb{R} \right\}.$$

Visually the hypograph of f is the set of points in $\mathbb{R}^2$ which lie on or below the graph of f, while the epigraph is the set of points in $\mathbb{R}^2$ which lie on or above the graph of f. (**Illustrate these concepts for** $f(x) = -|x|$.)

The next theorem characterizes concave functions through their hypographs.

Theorem 10. A function $f : \mathbb{R} \to \mathbb{R}$ is concave if and only if its hypograph is a convex set.

Proof. (If) Suppose HG_f is a convex set and choose any $x, x' \in \mathbb{R}$; let $f(x) = y$ and $f(x') = y'$. Then $(x, y) \in HG_f$ and $(x', y') \in HG_f$. Since HG_f is a convex set, we have

$$\left[\lambda x + (1 - \lambda) x', \lambda y + (1 - \lambda) y' \right] \in HG_f$$

$\forall \lambda \in [0, 1]$. That is

$$f \left[\lambda x + (1 - \lambda) x' \right] \ge \lambda y + (1 - \lambda) y'$$

$\forall \lambda \in [0, 1]$. Since this is true for all $x, x' \in \mathbb{R}$ then f is concave.

(Only if) Suppose f is concave and let $(x, y) \in HG_f$ and $(x', y') \in HG_f$; that is

$$f(x) \ge y \text{ and } f\left(x'\right) \ge y'.$$

Multiplying the first inequality by λ, the second one by $(1 - \lambda)$, and adding

$$\lambda f(x) + (1 - \lambda) f\left(x'\right) \ge \lambda y + (1 - \lambda) y'$$

$\forall \lambda \in [0, 1]$. But since f is concave

$$f \left[\lambda x + (1 - \lambda) x' \right] \ge \lambda f(x) + (1 - \lambda) f\left(x'\right)$$

$\forall \lambda \in [0,1]$. Hence

$$f\left[\lambda x + (1-\lambda)x'\right] \geq \lambda y + (1-\lambda)y'$$

$\forall \lambda \in [0,1]$. Then $[\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y'] \in HG_f$, $\forall \lambda \in [0,1]$. Since this is true for all $(x,y) \in HG_f$ and $(x',y') \in HG_f$ it follows that HG_f is a convex set. ■

In a similar way, the convexity of a function can be characterized through its epigraph.

Theorem 11. A function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if its epigraph is a convex set.

The characterizations of concave and convex functions in Theorems 10 and 11, respectively, are related to the notion of level sets. Level sets are of fundamental relevance in economics, as stated in Chapter 13 of SB. The level set of f for the value y in the image, written $C_f(y)$, is the complete set of $x \in \mathbb{R}$ at which the value of f equals y

$$C_f(y) = \{x \in \mathbb{R} : f(x) = y\}.$$

Notice that $C_f(y)$ is a subset of $\mathbb{R}$ whereas G_f is a subset of $\mathbb{R}^2$. Two associated concepts are the upper and lower level sets

$$\begin{aligned} UC_f(y) &= \{x \in \mathbb{R} : f(x) \geq y\} \\ LC_f(y) &= \{x \in \mathbb{R} : f(x) \leq y\}. \end{aligned}$$

The next result relates the concavity of f with $UC_f(y)$, and it follows as a direct corollary of Theorem 10.

Corollary 12. If $f : \mathbb{R} \to \mathbb{R}$ is concave then $UC_f(y)$ is a convex set for all y in the range of f.

(**Check that the upper level sets of $f(x) = -|x|$ are convex sets**.) It is important to stress that the last theorem goes in only one way. To convince yourself consider $f(x) = x^3$. The range is $\mathbb{R}$ and $UC_f(y) = \left\{x \in \mathbb{R} \mid x^3 \geq y\right\} = \left[y^{1/3}, +\infty\right)$. So the upper level sets are intervals, and then convex sets, but the function is not concave. Hence the class of functions which has convex upper level sets includes, but is not limited to, the concave functions; they are known as quasi-concave functions and are extremely important in economics (and in your first year at the PhD!). We will study quasiconcave functions at the end of the course.

In a similar way, the convexity of f is related with $LC_f(y)$.

Corollary 13. If $f : \mathbb{R} \to \mathbb{R}$ is convex then $LC_f(y)$ is a convex set for all y in the range of f.

Concave Functions on $\mathbb{R}^n$

This section extends the results in the previous one to higher dimensions. Here $f : \mathbb{R}^n \to \mathbb{R}$. The graph of f, which we denote G_f, is now a subset of $\mathbb{R}^{n+1}$

$$G_f = \left\{ (y, \mathbf{x}) \in \mathbb{R}^{n+1} : y = f(\mathbf{x}), \mathbf{x} \in \mathbb{R} \right\}$$

where $\mathbf{x}^T = (x_1, ..., x_n)$. Some other sets associated with $f : \mathbb{R}^n \to \mathbb{R}$ are the following

1. the level set of f for the value y in the range of f

$$C_f(y) = \{ \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = y \};$$

2. the upper level set of f for the value y in the rang of f

$$UC_f(y) = \{ \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq y \}; \text{ and}$$

3. the lower level set of f for the value y in the rang of f

$$LC_f(y) = \{ \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq y \}.$$

When $n = 2$ the level sets can be visually represented, as they are subsets of $\mathbb{R}^2$. [**Draw the level sets, upper level sets and lower level sets of** $f(x_1, x_2) = x_1 x_2$ **and** $f(x_1, x_2) = x_1 + x_2$ **for** $y = 1/4$ **and** $(x_1, x_2) \in [0, 1]^2$.]

For higher dimensions the notion of concavity is as follows.

Definition 14. (**Concavity in** $\mathbb{R}^n$) A function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if and only if

$$f\left[\lambda \mathbf{x} + (1 - \lambda) \mathbf{x}'\right] \geq \lambda f(\mathbf{x}) + (1 - \lambda) f\left(\mathbf{x}'\right)$$

$\forall \lambda \in [0, 1]$ and $\forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$. For strict concavity replace $\geq$ by $>$ and $[0, 1]$ by $(0, 1)$ whenever $\mathbf{x} \neq \mathbf{x}'$.

Example 2. A very important utility function in economics receives the name of Leontief or "perfect complements" and in its simplest form is given by $U : \mathbb{R}^2_+ \rightarrow \mathbb{R}$ with

$$U(x_1, x_2) = \min\{x_1, x_2\}.$$

We want to show that this function is concave. Let $(x_1, x_2) \in \mathbb{R}^2_+$ and $(x_1', x_2') \in \mathbb{R}^2_+$ denote two arbitrary bundles. Let us define $(x_1'', x_2'') = \lambda(x_1, x_2) + (1-\lambda)(x_1', x_2') \in \mathbb{R}^2_+$. There are four cases to consider: $x_1 \geq x_2$ and $x_1' \geq x_2'$; $x_1 \geq x_2$ and $x_1' \leq x_2'$; $x_1 \leq x_2$ and $x_1' \geq x_2'$; and $x_1 \leq x_2$ and $x_1' \leq x_2'$. We next check that the required condition for concavity holds in the first case.

Select an arbitrary λ in $[0,1]$. Assume $x_1 \geq x_2$ and $x_1' \geq x_2'$. Then $U(x_1, x_2) = x_2$ and $U(x_1', x_2') = x_2'$. In addition,

$$x_1'' = \lambda x_1 + (1-\lambda)x_1' \geq \lambda x_2 + (1-\lambda)x_2' = x_2''.$$

Therefore $U(x_1'', x_2'') = x_2''$. As a consequence,

$$U\left(x_1'', x_2''\right) = x_2'' = \lambda x_2 + (1-\lambda)x_2' = \lambda U(x_1, x_2) + (1-\lambda)U\left(x_1', x_2'\right).$$

Since λ was arbitrarily selected, the claim follows. (**Check the other three cases.**) ▲

It is still true that concavity implies continuity, but it does not imply differentiability. However, if the function is continuously differentiable then there are other alternative characterizations of this property. We next relate concavity with first-order derivatives.

Theorem 15. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is C^1. Then f is concave if and only if

$$f\left(\mathbf{x}'\right) + Df\left(\mathbf{x}'\right)\left(\mathbf{x} - \mathbf{x}'\right) \geq f(\mathbf{x})$$

$\forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$. For strict concavity replace $\geq$ by $>$ whenever $\mathbf{x} \neq \mathbf{x}'$.

Proof. See Madden (1986, pp. 52-54) ■

Although the last theorem looks similar to the one-variable analysis, notice that here

$$Df\left(\mathbf{x}'\right)\left(\mathbf{x} - \mathbf{x}'\right) = \textstyle\sum_{i=1}^n \left[\partial f\left(\mathbf{x}'\right)/\partial x_i\right]\left(x_i - x_i'\right).$$

For $n = 2$ the theorem says that a C^1 function of two variables is concave if and only if the tangent plane to its graph $(\subset \mathbb{R}^3)$ at any point lies entirely on or above the graph.

Now, we relate the concavity of a function f with its second-order derivatives. If the function is C^2 then the next characterization holds.

Theorem 16. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^2. Then f is concave if and only if $D^2 f(\mathbf{x})$ is negative semi-definite for all $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$. Moreover, if $D^2 f(\mathbf{x})$ is negative definite for all $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$, then f is strict concave.

Proof. See Madden (1986, pp. 52-54) ■

Example 3. Another very important utility function in economics receives the name of Cobb-Douglas and in its simplest form is given by $U : \mathbb{R}^2_+ \to \mathbb{R}$ with

$$U(x_1, x_2) = x_1^a x_2^b$$

where $a, b > 0$ and $a + b = 1$. This utility function is C^2. (**Check concavity directly by using Definition 14, and then by using Theorems 15 and 16.**) ▲

Finally we mention the hypograph (epigraph) characterization of concave (convex) functions. The hypograph and epigraph of $f : \mathbb{R}^n \to \mathbb{R}$ are

$$HG_f = \left\{(y, \mathbf{x}) \in \mathbb{R}^{n+1} : y \leq f(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\right\}$$

and

$$EG_f = \left\{(y, \mathbf{x}) \in \mathbb{R}^{n+1} : y \geq f(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\right\}$$

respectively. Like G_f, these are subsets of $\mathbb{R}^{n+1}$. The next theorem states alternative definitions of concave and convex functions.

Theorem 17. A function $f : \mathbb{R}^n \to \mathbb{R}$ is concave (convex) if and only if HG_f (EG_f) is a convex set.

The next statement is a direct corollary of the previous one.

Corollary 18. If the function $f : \mathbb{R}^n \to \mathbb{R}$ is concave (convex) then $UC_f(y)$ $(LC_f(y))$ is a convex set for all y in the range of f.

(**Check that Corollary 18 holds using the utility function in Example 3 above.**)

Convex Domains

The previous sections developed all the results for functions defined on $\mathbb{R}$ and on $\mathbb{R}^n$, i.e. $f : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$. The analysis extends immediately (after some changes in notation) to open and convex domains on $\mathbb{R}$ and $\mathbb{R}^n$ respectively. Part of the analysis also extends to close and convex domains on $\mathbb{R}$ and $\mathbb{R}^n$ respectively. (Open domains are needed for all those results that involve taking derivatives.)

Properties of the Concave Functions

Beside the previous characterizations, concave functions have some additional nice properties. In some cases, these properties help to check whether a given function is, in fact, concave.

Theorem 19. Suppose $D \subset \mathbb{R}^n$ is a convex set, and suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are two functions such that $f(\mathbf{x}) = kg(\mathbf{x})$ where $k \in \mathbb{R}$. If g is concave then f is concave if $k > 0$ and convex if $k < 0$.

Proof. (Show this result.) ■

Moreover, the sum of concave functions is always concave.

Theorem 20. Suppose $D \subset \mathbb{R}^n$ is convex and suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are two concave functions. Then $h(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})$ is also concave.

Proof. (Show this result.) ■

Let us finally consider functions of functions, i.e. $f(\mathbf{x}) = h[g(\mathbf{x})]$. To be precise suppose $g : D \to \mathbb{R}$ where $D \subset \mathbb{R}^n$ has range $E \subset \mathbb{R}$, and suppose that $h : E \to \mathbb{R}$. Then f is defined by

$$f : D \to \mathbb{R} \text{ with values } f(\mathbf{x}) = h[g(\mathbf{x})].$$

The last result involves the notion of monotone increasing transformations (often called monotonic transformations). (Note 10 elaborates more on this concept.)

Definition 21. Let I be an interval of the real line. Then $h : I \to \mathbb{R}$ is a monotone increasing transformation of I if h is a strictly increasing function on I. Furthermore, if h is a monotone increasing transformation and g is a real-valued function of n variables, then we say that

$$h(g) : \mathbf{x} \to h[g(\mathbf{x})]$$

is a monotone increasing transformation of g.

The next theorem states that any monotone increasing transformation of a concave function is concave if the monotone transformation is itself a concave function.

Theorem 22. Suppose $D \subset \mathbb{R}^n$ is convex, suppose $g : D \to \mathbb{R}$ has range E, and $h : E \to \mathbb{R}$. Define $f : D \to \mathbb{R}$ by $f(\mathbf{x}) = h[g(\mathbf{x})]$. If h is monotone increasing and concave, and g is concave, then f is concave.

[**Use Theorem 22 and the main result of Example 2 to show that** $V : \mathbb{R}^2_+ \to \mathbb{R}$ **with** $V(x_1, x_2) = \ln(\min\{x_1, x_2\})$ **is a concave function.**]