Natalia Lazzati

Mathematics for Economics

Note 12: Homogeneous and Homothetic Functions

Note 12 is based on Madden (1986, Ch. 9) and Simon and Blume (1994, Ch. 20).

Homogeneous Functions

Many of the functions appearing from solutions of parameterized families of problems in economics are homogeneous functions. For instance, the optimal consumer bundle is an homogenous function of degree zero in prices and income. From an economical perspective, this means that if we multiply prices and income by the same constant, then the optimal consumer bundle does not change. This section describes this kind of functions.

A function $f : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}^n$, is said to be homogeneous of degree r, with $r \in \mathbb{R}$, if it satisfies

$$f(t\mathbf{x}) = t^r f(\mathbf{x}) \ \ \forall t > 0, \forall \mathbf{x} \in D.$$

Homogeneity is (in general) easy to check. Let us consider, for instance, the function $f : \mathbb{R}^2_{++} \to \mathbb{R}$ where $f(\mathbf{x}) = x_1^2/x_2$. Notice that

$$f(t\mathbf{x}) = (tx_1)^2 / tx_2 = tx_1/x_2 = tf(\mathbf{x}).$$

So that this function is homogeneous of degree one. Functions that are homogeneous of degree one are often named linearly homogenous.

The domain of a homogeneous function must satisfy the next requirement

$$\text{if } \mathbf{x} \in D \text{ then } t\mathbf{x} \in D \ \ \forall t > 0.$$

A set $D \subset \mathbb{R}^n$ which satisfies this property is called a cone. (**Show that a cone need not be a convex set, and that convex sets need not be cones.**)

A function of one variable $f : \mathbb{R}_{++} \to \mathbb{R}$ is homogeneous of degree r if and only if $f(x) = x^r f(1)$. Since $f(1)$ is just a real number, then $f : \mathbb{R}_{++} \to \mathbb{R}$ is homogeneous of degree r if and only if

$$f(x) = kx^r.$$

(**Show the last result.**) It follows that homogeneous functions of one variable $f : \mathbb{R}_{++} \to \mathbb{R}$ are C^2 functions with first and second order derivatives

$$f'(x) = rkx^{r-1} \text{ and } f''(x) = (r-1)\, rkx^{r-2}.$$

Then f is concave if $(r-1)\, rk \leq 0$ and it is convex if $(r-1)\, rk \geq 0$. Two other properties emerge. The first one is named Euler's theorem: $xf'(x) = rf(x)\,,\ \forall x \in \mathbb{R}_{++}$. The second one is that $f' : \mathbb{R}_{++} \to \mathbb{R}$ is homogenous of degree $(r-1)$. (**Show the last two properties.**)

We next generalize the latter results to homogenous functions of many variables.

Theorem 1. (Euler's Theorem) Suppose $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is C^1 and homogeneous of degree r. Then

$$\sum\nolimits_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(\mathbf{x}) = rf(\mathbf{x}) \ \ \forall \mathbf{x} \in \mathbb{R}^n_{++},\ \forall t > 0.$$

Proof. Since f is homogeneous of degree r

$$f(t\mathbf{x}) = t^r f(\mathbf{x}) \ \ \forall \mathbf{x} \in \mathbb{R}^n_{++},\ \forall t > 0.$$

Differentiate both sides with respect to t

$$\sum\nolimits_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(t\mathbf{x}) = rt^{r-1} f(\mathbf{x})\,.$$

Setting $t = 1$ gives

$$\sum\nolimits_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(\mathbf{x}) = rf(\mathbf{x}) \ \ \forall \mathbf{x} \in \mathbb{R}^n_{++}$$

which is the claim. ■

Then, for homogeneous functions, the sum of its argument variables times its partial derivatives with respect to these variables equals the degree of homogeneity times the value of the function. The Euler's theorem is also a sufficient condition for homogeneity of C^1 functions. We next extend the second property we mentioned earlier.

Theorem 2. Suppose $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is C^1 and homogeneous of degree r. Then $f_i'(\mathbf{x}) : \mathbb{R}^n_{++} \to \mathbb{R}$ is homogeneous of degree $r-1$, $i = 1, ..., n$.

Proof. (**Show this result.**) ■

The last theorem has a useful geometric consequence, that we will illustrate for homogenous functions of two arguments ($n = 2$). The slope of the level set through $\mathbf{x}$ of a function of two variables at $\mathbf{x}$, assuming $f_1'(\mathbf{x}) \neq 0$, is

$$-\frac{\partial f/\partial x_2(\mathbf{x})}{\partial f/\partial x_1(\mathbf{x})}.$$

So the slope at $t\mathbf{x}$ is

$$-\frac{\partial f/\partial x_2(t\mathbf{x})}{\partial f/\partial x_1(t\mathbf{x})} = -\frac{t^{r-1}\partial f/\partial x_2(\mathbf{x})}{t^{r-1}\partial f/\partial x_1(\mathbf{x})} = -\frac{\partial f/\partial x_2(\mathbf{x})}{\partial f/\partial x_1(\mathbf{x})}.$$

That is, for homogeneous functions the slope of the level set is the same at $\mathbf{x}$ than at $t\mathbf{x}$. In other words the level sets are parallel displacements of each other.

Homothetic Functions

Homogeneity is a cardinal property, not an ordinal one. To convince yourself notice that $g(z) = z + 1$ is a monotone increasing transformation and $u(\mathbf{x}) = x_1 x_2$ is a homogeneous function. However, $g[u(\mathbf{x})] = x_1 x_2 + 1$ is not a homogeneous function.

Nevertheless, many of the important properties that make homogenous functions so useful, e.g. that they preserve the shape of the level curves, are in fact ordinal properties. Homothetic functions form a class of functions that has all the ordinal properties of the homogeneous ones.

Definition 5. (Homotheticity) A function $v : \mathbb{R}^n \to \mathbb{R}$ is homothetic if it is a monotone increasing transformation of a homogeneous function, that is, if there is a monotone increasing transformation $z \to g(z)$ of $\mathbb{R}$ and a homogeneous function $u : \mathbb{R}^n \to \mathbb{R}$ such that $v(\mathbf{x}) = g[u(\mathbf{x})]$ for $\mathbf{x}$ in the domain.

(**Show that homotheticity is an ordinal property.**) Showing a function is homothetic via Definition 5 is quite hard in some cases. The next two theorems may be helpful in those circumstances.

Theorem 6. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a strictly monotone increasing function. Then u is homothetic if and only if for all $\mathbf{x}$ and $\mathbf{x}'$ in $\mathbb{R}^n$

$$u(\mathbf{x}) \geq u(\mathbf{x}') \iff u(t\mathbf{x}) \geq u(t\mathbf{x}') \text{ for all } t > 0.$$

Proof. See Simon and Blume (1994, pp. 502-503) ■

An ordinal property of homogeneity is that the slope of the level sets is constant along rays from the origin. This property provides a calculus-based necessary condition for homotheticity.

Theorem 7. Let u be a C^1 function on $\mathbb{R}^n$. Then u is homothetic if and only if all the slopes of the tangent planes to the level sets of u are constant along rays from the origin, i.e.

$$\frac{\partial u/\partial x_i\,(t\mathbf{x})}{\partial u/\partial x_j\,(t\mathbf{x})} = \frac{\partial u/\partial x_i\,(\mathbf{x})}{\partial u/\partial x_j\,(\mathbf{x})}\ \forall t > 0 \text{ and } \forall i, j.$$

Proof. (**Show the only if part.**) ■