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Mathematics for Economics

Note 11: Quasiconcave Programming

Note 11 is based on de la Fuente (2000, Ch. 7), Madden (1986, Ch. 20), Simon and Blume (1994, Ch. 21) and Dr. Walker's Lecture Notes for ECON 501B.

Quasiconcave Programming with Differentiability

As you will see during your first year of the PhD program, concave and quasiconcave functions arise naturally in many problems of interest. In addition, they provide much more structure in the analysis of the optimization problems that lie at the heart of economic theory: as we showed in Notes 7, 8 and 9 the first-order necessary conditions that characterize the solution of the general differentiable optimization problem are also sufficient conditions when we add some concavity assumptions. This section generalizes our previous results by allowing weaker restrictions.

The analysis that follows also introduces an alternative representation of the Kuhn-Tucker problem that is of fundamental relevance in ECON 501B.

Unconstrained Problem

Consider the problem

$$\max_{\mathbf{x}} \{f(\mathbf{x}) : \mathbf{x} \in U\} \tag{P.U}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and U is a convex set in $\mathbb{R}^n$ (not necessarily open!). The extensions of Theorems 3 and 6 in Note 6 require the notion of feasible direction.

Definition 1. (Feasible direction) Consider the problem (P.U). Take a point $\mathbf{x}$ in U and a direction vector $\mathbf{v}$ in $\mathbb{R}^n$. We say that $\mathbf{v}$ is a feasible direction from $\mathbf{x}$ if there exists some $\delta > 0$ such that

$$\mathbf{x} + t\mathbf{v} \in U \ \forall t \in (0, \delta)$$

that is, if any sufficiently small movement away from $\mathbf{x}$ in the direction of $\mathbf{v}$ leaves us inside the feasible set.

The next proposition extends Theorems 3 in Note 6.

Theorem 2. (**First-order necessary conditions for an unconstrained maximum**) Let f be a C^1 function, and let $\mathbf{x}^*$ be an optimal solution to problem (P.U). Then

$$Df(\mathbf{x}^*)\,\mathbf{v} \leq 0$$

for every directional vector $\mathbf{v} \in \mathbb{R}^n$ feasible from $\mathbf{x}^*$.

Proof. Let $\mathbf{x}^*$ be an optimal solution of (P.U), and $\mathbf{v}$ an arbitrary direction vector feasible from $\mathbf{x}^*$. Then there exists some $\delta > 0$ (which may depend on $\mathbf{v}$) such that $\mathbf{x}^* + t\mathbf{v} \in U$ for all $t \in (0, \delta)$. Because any feasible movement away from $\mathbf{x}^*$ reduces the value of the function, we have

$$f(\mathbf{x}^* + t\mathbf{v}) \leq f(\mathbf{x}^*)$$

for all t such that $\mathbf{x}^* + t\mathbf{v} \in U$. Rearranging terms and dividing by $t > 0$

$$\frac{f(\mathbf{x}^* + t\mathbf{v}) - f(\mathbf{x}^*)}{t} \leq 0 \tag{1}$$

and taking the limit to this expression as $t \to 0$

$$\lim_{t\to 0} \frac{f(\mathbf{x}^* + t\mathbf{v}) - f(\mathbf{x}^*)}{t} = Df(\mathbf{x}^*)\,\mathbf{v} \leq 0.$$

That is, the limit of the ratio of the left-hand side of (1) is the (one-sided) directional derivative of f in the direction of $\mathbf{v}$ evaluated at $\mathbf{x}^*$. Because f is C^1, the directional derivative exists and can be written as the scalar product of the derivative and the directional vector. ■

If U is an open set, then all its points are (by definition) interior. Therefore, given any $\mathbf{x}$ in U all directions are feasible from it. In this case $Df(\mathbf{x}^*)\,\mathbf{v} \leq 0$ will hold if only if $Df(\mathbf{x}^*) = \mathbf{0}$ and the statement in the last theorem reduces to the statement of Theorem 3 in Note 6. The same is true for an interior maximizer $\mathbf{x}^*$, that is, for an optimal solution to problem (P.U) that lies in the interior of the feasible set U.

We now provide sufficient conditions for a global maximum.

Theorem 3. (**Sufficient conditions for a global maximum**) Let f be a C^1 pseudoconcave function. If $\mathbf{x}^* \in U$ and for every directional vector $\mathbf{v} \in \mathbb{R}^n$ feasible from $\mathbf{x}^*$ we have $Df(\mathbf{x}^*)\,\mathbf{v} \leq 0$, then $\mathbf{x}^*$ is an optimal solution to problem (P.U).

Proof. We will prove the contrapositive statement. Assume f is a C^1 pseudoconcave function, and fix some $\mathbf{x}' \in U$. We will show that if $\mathbf{x}' \in U$ is not an optimal solution to (P.U), then it does not satisfy the first order condition $Df(\mathbf{x}')\,\mathbf{v} \leq 0$ for some feasible direction $\mathbf{v} \in \mathbb{R}^n$.

Suppose $\mathbf{x}' \in U$ is not optimal. Then there exists some point $\mathbf{x} \in U$ such that $f(\mathbf{x}) > f(\mathbf{x}')$. By the pseudoconcavity of f, $f(\mathbf{x}) > f(\mathbf{x}')$ implies

$$Df\left(\mathbf{x}'\right)\left(\mathbf{x} - \mathbf{x}'\right) > 0$$

where $(\mathbf{x} - \mathbf{x}')$ is a feasible direction vector from $\mathbf{x}'$, by the convexity of U. Then $\mathbf{x}'$ does not satisfy the first order condition. ■

The last two results improves the ones in Note 6 in two directions. First, they allow the analysis of problems where the maximum $\mathbf{x}^*$ lies on the boundaries of the constraint set. Second, the last statement substitutes concavity by pseudoconcavity, which is a weaker property. The next example sheds light on the first point.

Example 1. Let us consider the objective function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = -x^2$. This is a C^1 concave function, and it is then pseudoconcave. Let $U = \mathbb{R}$. In this case $f'(x^*) = 0$ at $x^* = 0$, and therefore 0 is the maximizer.

Let us assume now that $U = [1, 2]$. For all $x \in [1, 2]$, we have that $f'(x) = -2x < 0$. This means the function is always decreasing, and we cannot find a stationary point in the constraint set. Since the function is strictly decreasing on $[1, 2]$, then $x^* = 1$ is the global maximizer. Notice that, as Theorem 2 states, the point $x^* = 1$ satisfies $f'(x^*)\,v = -2x^*v \leq 0$ for every feasible direction v.

Let us assume now that $U = [-2, -1]$. For all $x \in [-2, -1]$, we have that $f'(x) = -2x > 0$. Since the function is strictly increasing on $[-2, -1]$, then $x^* = -1$ is the global maximizer. Notice that, as Theorem 2 states, the point $x^* = -1$ satisfies $f'(x^*)\,v = -2x^*v \leq 0$ for every feasible direction v. ▲

The Kuhn-Tucker Problem (with nonnegativity constraints)

In the original formulation, the Kuhn-Tucker problem entails nonnegativity constraints for the variables of choice, i.e. $x_i \geq 0$ for $i = 1, ..., n$. Although this problem can be solved with the tools developed in Note 8, for this particular formulation there is an alternative representation of the first-order conditions that facilitates its implementation.

We cover this special case as it appears quite often in economics. The reason is that the variables of choice are often bundles of goods or inputs and they all take values in the positive reals. (This sub-section is very useful for ECON 501B!)

Let $f : \mathbb{R}^n \to \mathbb{R}$, and consider the problem

$$\max_{\mathbf{x}} \left\{ f(\mathbf{x}) : \mathbf{x} \in U \right\} \tag{P.K-T$'$}$$

where

$$U = \left\{ \mathbf{x} \in \mathbb{R}^n_+ : g_1(\mathbf{x}) \leq b_1, ..., g_k(\mathbf{x}) \leq b_k \right\}.$$

Here $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, ..., k$, and we assume $k \leq n$. Notice that now $\mathbf{x} \in \mathbb{R}^n_+$.

Theorem 4. (Kuhn-Tucker conditions with nonnegativity constraints) Let $f, g_1, ..., g_k$ be C^1 functions. Suppose that $\mathbf{x}^*$ is a solution of (P.K-T$'$) and that the matrix $(\partial g_i / \partial x_j)$ has maximal rank at $\mathbf{x}^*$, where the i's vary over the indices of the g_i's constraints that are binding at $\mathbf{x}^*$ and the j's range over the indices j for which $x_j^* > 0$. Then, there exist multipliers $\lambda_1^*, ..., \lambda_k^*$ such that the conditions below hold.

	Marginal Conditions	Complementary Slackness Conditions
(K-T 1$'$)	$Df(\mathbf{x}^*) \leq \sum_{i=1}^k \lambda_i^* Dg_i(\mathbf{x}^*)$	$\left[Df(\mathbf{x}^*) - \sum_{i=1}^k \lambda_i^* Dg_i(\mathbf{x}^*) \right] \mathbf{x}^* = 0$
(K-T 2$'$)	$\lambda_1^* \geq 0, ..., \lambda_k^* \geq 0$	$\lambda_1^* [g_1(\mathbf{x}^*) - b_1] = 0, ..., \lambda_k^* [g_k(\mathbf{x}^*) - b_k] = 0$

Remark 1. An equivalent representation of (K-T 1$'$) is

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_j} \leq \sum\nolimits_{i=1}^k \lambda_i^* \frac{\partial g_i(\mathbf{x}^*)}{\partial x_j} \text{ with equality if } x_j^* > 0 \text{ for } j = 1, ..., n.$$

Here again, the K-T conditions are just necessary requirements for a local (and then a global!) maximum. Since we are interested in solutions to problem (P.K-T$'$), we need to add more structure to the problem. The next result modifies Theorem 5 in Note 8 in two directions (i) it requires the objective function to be pseudoconcave instead of concave; and (ii) it assumes the constraint functions are quasiconvex instead of convex. These extra requirements guarantee that the K-T necessary conditions are also sufficient conditions for a global maximizer.

Theorem 5. (Sufficient conditions for a global maximum) Let f be a C^1 pseudoconcave function, and the restrictions $-g_1, ..., -g_k$ be quasiconcave functions. Let $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ be a pair of vectors that satisfy the necessary conditions given in the Kuhn-Tucker theorem with nonnegativity constraints. Then $\mathbf{x}^*$ is a solution to problem (P.K-T$'$).

Remark 2. This result also holds for the K-T problem without the nonnegativity constraints.

Proof. (**Prove the statement following the proof of Theorem 5 in Note 8.**) ■

We end this section applying the results to a consumer problem with linear utility function.

Example 2. Let us consider one of the simplest problems in consumer theory. Assume a person has utility function $U(x_1, x_2) = ax_1 + bx_2$, faces prices p_1 and p_2 for these goods and his income is m. Suppose that $a > 0$, $b > 0$, $m > 0$ and $a/p_1 > b/p_2$. His problem is

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} \{U(x_1, x_2) : p_1x_1 + p_2x_2 \leq m\}.$$

Since the NDCQ is satisfied for every bundle in the feasible set (**Why?**), the K-T$'$ conditions are as below.

	Marginal Conditions	Complementary Slackness Conditions
(K-T 1$'$)	$a \leq \lambda^* p_1, b \leq \lambda^* p_2$	$x_1^*(a - \lambda^* p_1) = 0, x_2^*(b - \lambda^* p_2) = 0$
(K-T 2$'$)	$\lambda^* \geq 0$	$\lambda^*[p_1x_1^* + p_2x_2^* - m] = 0$

By the complementary slackness conditions in (K-T 1$'$), we have four possible cases to study (i) $x_1^* = x_2^* = 0$; (ii) $x_1^* > 0$ and $x_2^* > 0$; (iii) $x_1^* = 0$ and $x_2^* > 0$; and (iv) $x_1^* > 0$ and $x_2^* = 0$. Let us consider one by one.

[Case (i)] If $x_1^* = x_2^* = 0$, then, by (K-T 2$'$), $\lambda^* = 0$. By (K-T 1$'$) it follows that $a \leq \lambda^* p_1 = 0$. This contradicts the fact that $a > 0$.

[Case (ii)] Assume next that $x_1^* > 0$ and $x_2^* > 0$. Then, by (K-T 2$'$), we have that $a - \lambda^* p_1 = 0$ and $b - \lambda^* p_2 = 0$. But then $\lambda^* = a/p_1 = b/p_2$, which contradicts the fact that $a/p_1 > b/p_2$.

[Case (iii)] Let us try $x_1^* = 0$ and $x_2^* > 0$. Then, by (K-T 2$'$), $\lambda^* = b/p_2$. So, by (K-T 1$'$), $a \leq (b/p_2)\, p_1$ which is equivalent to $a/p_1 \leq b/p_2$. This is again a contradiction to $a/p_1 > b/p_2$.

[Case (iv)] The last possibility is $x_1^* > 0$ and $x_2^* = 0$. [**Check that** $(x_1^*, x_2^*, \lambda^*) = (m/p_1, 0, a/p_1)$ **satisfies all the K-T$'$ conditions.**]

Therefore, there is only one candidate for the maximum. The fact that $(x_1^*, x_2^*) = (m/p_1, 0)$ is the optimal bundle follows by Theorem 5. (**Why?**)

The answer is very intuitive. From an economical perspective, $a/p_1 > b/p_2$ means that the marginal utility per dollar spent is higher in good 1 than in good 2. Then the optimal behavior for the consumer consists on spending all his income in good 1. ▲

Concave Programming without Differentiability

Although the differentiability of the objective and constraint functions is a convenient assumption, the essence of many of the previous results goes through without it. To shed light on this fundamental point let us consider, for instance, the problem

$$\max_x \{f(x) : g(x) \leq b\} \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$. The Lagrangian function is $L(x, \lambda) = f(x) - \lambda [g(x) - b]$. In Note 8 we explained that if f and $-g$ are C^1 concave functions then x^* is a solution to (2) if and only if

(a) $\partial L(x^*, \lambda^*)/\partial x = f'(x^*) - \lambda^* g'(x^*) = 0$;

(b) $\lambda^* \geq 0$, $g(x^*) \leq b$; and

(c) $\lambda^* [g(x^*) - b] = 0$

for some positive λ.

Condition (a) admits another formulation. Given that f and $-g$ are C^1 concave functions, if $\lambda \geq 0$ then $f(x) - \lambda [g(x) - b]$ is a C^1 concave function of x for all $x \in \mathbb{R}$. For this kind of functions, the global maximizers are stationary points. Therefore, condition (a) is equivalent to the requirement

$$L(x^*, \lambda^*) = f(x^*) - \lambda^* [g(x^*) - b] \geq f(x) - \lambda^* [g(x) - b] = L(x, \lambda^*) \tag{3}$$

for all $x \in \mathbb{R}$. Notice that condition (3) does not entail differentiability! As a consequence, the necessary and sufficient conditions for x^* to be a solution to (2) can be rewritten as

(a) $L(x^*, \lambda^*) \geq L(x, \lambda^*)$ for all $x \in \mathbb{R}$;

(b) $\lambda^* \geq 0$, $g(x^*) \leq b$; and

(c) $\lambda^* [g(x^*) - b] = 0$.

To generalize our previous argument, let $f : \mathbb{R}^n \to \mathbb{R}$ and consider the problem

$$\max_{\mathbf{x}} \{f(\mathbf{x}) : \mathbf{x} \in U\} \tag{P.K-T}$$

where

$$U = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \leq b_1, ..., g_k(\mathbf{x}) \leq b_k\}.$$

Here $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, ..., k$. We assume $k \leq n$, that is, the number of constraints is at most equal to the number of decision variables.

Theorem 6. (K-T conditions without differentiability) Let $\mathbf{x}^*$ be a solution to the problem (P.K-T). Assume $f, -g_1, ..., -g_k$ are concave functions. Suppose further that there exists a point $\mathbf{x}' \in \mathbb{R}^n$ such that $g_1(\mathbf{x}') < b_1, ..., g_k(\mathbf{x}') < b_k$. Then there exists a vector of nonnegative multipliers $\boldsymbol{\lambda}^*$ such that

$$\begin{aligned} &f(\mathbf{x}^*) - \sum_{i=1}^{k} \lambda_i^* [g_i(\mathbf{x}^*) - b_i] \geq f(\mathbf{x}) - \sum_{i=1}^{k} \lambda_i^* [g_i(\mathbf{x}) - b_i] \text{ for all } \mathbf{x} \in \mathbb{R}_+ \\ &\lambda_i^* [g_i(\mathbf{x}^*) - b_i] = 0 \text{ for } i = 1, ..., m. \end{aligned}$$

Proof. See de la Fuente (2000), pp. 298-300. (Their set-up is slightly different.) ■

Theorem 7. (Sufficient conditions for a global maximum without differentiability) Consider the problem (P.K-T). Assume there exist vectors $\mathbf{x}^* \in \mathbb{R}^n$ and $\boldsymbol{\lambda}^* \in \mathbb{R}_+^k$ such that $\mathbf{x}^*$ is feasible, i.e. $\mathbf{x}^* \in U$, and

$$\begin{aligned} &f(\mathbf{x}^*) - \sum_{i=1}^{k} \lambda_i^* [g_i(\mathbf{x}^*) - b_i] \geq f(\mathbf{x}) - \sum_{i=1}^{k} \lambda_i^* [g_i(\mathbf{x}) - b_i] \text{ for all } \mathbf{x} \in \mathbb{R}_+ \\ &\lambda_i^* [g_i(\mathbf{x}^*) - b_i] = 0 \text{ for } i = 1, ..., m. \end{aligned}$$

Then $\mathbf{x}^*$ is an optimal solution to (P.K-T).

Proof. See de la Fuente (2000), p. 300. ■