A Finite Volume Code for 1D Gas Dynamics

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1 Introduction

A finite volume code is constructed to solve conservative systems, such as Euler’s Equations and the ideal magnetohydrodynamic (MHD) equations. Polynomial based reconstruction methods are implemented to obtain accurate spatial reconstruction, and both Runge-Kutta methods and characteristic tracing are implemented for temporal advancement.

We consider conservative systems,

$$U_t + (F(U))_x = 0, \quad (1)$$

where $U$ are the conservative variables and $F(U)$ is the analytical flux function. For the full Euler’s Equations, we have

$$U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \text{and} \quad F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{bmatrix}. \quad (2)$$

A simple ideal gas law equation of state is taken with a ratio of specific heat $\gamma = 1.4$, where the internal energy is defined as

$$e = \frac{p}{(\gamma - 1)\rho} \quad (3)$$

and the total energy per unit mass is a sum of the kinetic energy and internal energy,

$$E = \rho\left(\frac{u^2}{2} + e\right) \quad (4)$$

For the ideal MHD equations, we take the following conservative variables and flux function

$$U = \begin{bmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho u_z \\ B_y \\ B_z \\ E \end{bmatrix}, \quad \text{and} \quad F(U) = \begin{bmatrix} \rho u_x \\ \rho u_x^2 + p^* - B_x^2 \\ \rho u_x u_y - B_x B_y \\ \rho u_x u_z - B_x B_z \\ B_y u_x - B_z u_y \\ B_z u_x - B_x u_z \\ (E + p^*)u_x - B_x(B_x u_x + B_y u_y + B_z u_z) \end{bmatrix}. \quad (5)$$
In the MHD system the total pressure is defined as

\[ p^* = p + \frac{1}{2}(B_x^2 + B_y^2 + B_z^2). \] (6)

2 Results

In this section the results from four experiments that demonstrate the effectiveness of different slope limiters and approximate Riemann Solvers are presented. The five experiments are the Sod shock tube, the rarefaction problem, the blast problem, the Shu-Osher problem, and the Brio-Wu shock tube.

The Sod shock tube demonstrates the codes ability to capture shocks, rarefactions, and contact discontinuities within a single experiment. As illustrated in Figure 1, increasing the order of spatial reconstruction increases a methods ability to maintain sharp profiles. Since lower order methods have larger truncation error, their solutions are more diffusive and produce smoother profiles.

For the rarefaction problem, we study several approximate Riemann solvers abilities to capture smooth flow. The HLL and HLLC solvers both accurately capture the dynamics of the experiment, but the Roe solver does not converge to a solution.

The Blast2 problem involves strong shocks and narrow features. According to the results depicted in Figure 3, the MC slope limiter produces the sharpest profile, followed by the van Leer and minmod slope limiters.

The Shu-Osher problem simulates a right-moving shock front with artificial density fluctuations. As the order of spatial reconstruction and grid resolution is increased, the method captures smaller scale oscillations. Figure 4 compares first order Godunov against the piecewise linear method for varying grid resolution. Figure 5 shows the effect of varying the CFL number. There is little discrepancy as the CFL number changes, but sharper profiles are produced for larger values. The solution did not converge for a CFL number greater than one.

The Brio-Wu shock tube is an extension of the Sod shock tube to MHD. Unlike the hydrodynamics experiments, the Brio-Wu shock tube problem produces compound structures (i.e. a shock wave connected to a rarefaction), in addition shocks, rarefactions, and contact discontinuities. In Figure 6, the density and Mach number are plotted for the piecewise linear method and piecewise parabolic method. Mach number is defined as

\[ M = \frac{\text{flow speed}}{\text{speed of sound}} = \frac{\sqrt{u_x^2 + u_y^2 + u_z^2}}{\sqrt{\frac{\rho}{\gamma} p}}. \] (7)

3 Conclusion

The code constructed accurately models gas dynamics and MHD settings. MPI functions are included to allow parallel computations. The code can easily be extended to model other conservative systems once the appropriate eigensystems and physics are implemented. This requires defining the flux function and conversion between primitive and conservative variables for the new system.
Figure 1: The Sod shock tube problem on $N_x = 128$ using HLLC with the minmod slope limiter. The primitive variables are plotted at $t = 0.2$ for first order Godunov (FOG), the second order piecewise linear method (PLM), and the third order piecewise parabolic method (PPM).
Figure 2: The rarefaction problem on $N_x = 128$ using PLM with minmod. The primitive variables are plotted at $t = 0.15$ for HLL and HLLC. The solutions overlap and the test problem did not converge for the Roe solver.
Figure 3: The Blast2 problem on $N_x = 128$ using PLM and FOG with HLLC. The primitive variables are plotted at $t = 0.038$ for each minmod, van Leer’s, and MC slope limiters.
Figure 4: The Shu-Osher problem using PLM and FOG with MC and Roe. The primitive variables are plotted at $t = 1.8$ for grid resolution $N_x = 32, 64, 128,$ and $256$. 
Figure 5: The Shu-Osher problem using PLM with minmod and HLLC on $N_x = 128$. The primitive variables are plotted at $t = 1.8$ for each CFL number - CFL = 0.2, 0.4, 0.6, 0.8, 1.0. The results for CFL = 1.4 is not plotted because the solution did not converge.
Figure 6: The Brio-Wu shock tube problem using PLM and PPM with HLLC and minmod on $N_x = 400$. Density and Mach number are plotted.