10. Find the equation for the plane containing the parallel lines given by the vector equations:

\[ \mathbf{r}_1(s) = \langle 2, -3, -5 \rangle + s \langle 1, -1, 2 \rangle \quad (1) \]

\[ \mathbf{r}_2(t) = \langle 1, 2, 3 \rangle + t \langle -2, 2, -4 \rangle \quad (2) \]

**Solution**

To find the equation of a plane we need:

(a) A point \( P \) on the plane.

(b) A vector \( \mathbf{n} \) normal to the plane.

Choosing (for example) \( s = 0 \) in equation (1) gives us the point \( P(2, -3, -5) \) on the line \( \mathbf{r}_1(t) \). This takes care of (a).

Choosing \( t = 0 \) and \( t = 1/2 \) in equation (2) gives us the points \( Q(1, 2, 3) \) and \( R(0, 3, 1) \) on the line \( \mathbf{r}_2 \). Hence the vector

\[ \mathbf{n} = \mathbf{PQ} \times \mathbf{PR} \]

is normal to the plane. We find

\[ \mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \langle -1, 5, 8 \rangle \times \langle -2, 6, 6 \rangle \]

\[ = \left| \begin{array}{ccc} i & j & k \\ -1 & 5 & 8 \\ -2 & 6 & 6 \end{array} \right| = \langle -18, -10, 4 \rangle \]
This takes care of (b).

Using the point $P$ and the normal vector $\mathbf{n}$ we find that the equation of the plane is

$$-18(x - 2) - 10(y + 3) + 4(z + 5) = 0.$$ 

This simplifies (after dividing by -2) to:

$$9x + 5y - 2x = 13.$$

11. Find the equation for the line which is parallel to the line

$$L : \frac{x - 1}{2} = \frac{y}{3} = 2 - z$$

and contains the point $(3, -5, 7)$.

Solution

If we rewrite $L$ in its ‘standard form’:

$$\frac{x - 1}{2} = \frac{y}{3} = \frac{z - 2}{-1}$$

the denominators of these fractions tell us the line $L$ has direction $\mathbf{d} = (2, 3, -1)$. Since the line we are asked to find has the same direction $\mathbf{d}$ and passes through the point $(3, -5, 7)$, its equation (in symmetric form) is:

$$\frac{x - 3}{2} = \frac{y + 5}{3} = \frac{z - 7}{-1},$$

or, more simply:

$$\frac{x - 3}{2} = \frac{y + 5}{3} = 7 - z.$$

12. Find the length of the curve given by $\mathbf{r}(t) = \langle 1 + t^2, 2 - 3t^2, 5t^2 \rangle$ for $t \in [1, 3]$.

Solution

Recall that the arc length of a curve $\mathbf{r}(t)$ from $t = a$ to $t = b$ is given by

$$L = \int_a^b \| \mathbf{r}'(t) \| \, dt = \int_a^b \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt.$$ 

\footnote{Scale down}
Hence

\[
L = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt
\]

\[
= \int_1^3 \sqrt{(2t)^2 + (-6t)^2 + (10t)^2} \, dt
\]

\[
= \int_1^3 \sqrt{140t^2} \, dt
\]

\[
= 2\sqrt{35} \int_1^3 t \, dt
\]

\[
= 2\sqrt{35} \left[ \frac{t^2}{2} \right]_1^3
\]

\[
= \sqrt{35} (9 - 1) = 8\sqrt{35}.
\]

13. A particle has position vector \( \mathbf{r}(t) = \langle 4 \sin t, 9 \cos t \rangle \) for \( t \in [0, 2\pi) \).

(a) Find the velocity and the acceleration vectors.

Solution

The velocity vector is given by

\[
\mathbf{v}(t) = \mathbf{r}'(t) = \langle 4 \cos t, -9 \sin t \rangle
\]

and the acceleration is

\[
\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{v}'(t) = \langle -4 \sin t, -9 \cos t \rangle.
\]

(b) Find a non-parametric equation for the curve and sketch the path of the particle.

Solution

From the expression \( \mathbf{r}(t) \) we find

\[
x = 4 \sin t, \quad y = 9 \cos t.
\]

Since \( \sin^2 t + \cos^2 t = 1 \), we have

\[
\left( \frac{x}{4} \right)^2 + \left( \frac{y}{9} \right)^2 = 1.
\]

The expression above describes the path of the particle: it travels (clockwise) around an ellipse with \( x \)-radius 4 and \( y \)-radius 9.
(c) Draw the velocity and acceleration vectors on the curve at the point when $t = \pi/6$.

Solution
The velocity vector should be drawn tangent to the ellipse starting at the point $(2, 9\sqrt{3}/2)$ and pointing in a clockwise direction. The acceleration vector will be normal to the ellipse, starting at the point $(2, 9\sqrt{3}/2)$ and ending at the origin.

Here are the explicit computations to help you with your sketch:

\[
\mathbf{v} \left( \frac{\pi}{6} \right) = \langle 2, \sqrt{3} \rangle
\]

\[
\mathbf{a} \left( \frac{\pi}{6} \right) = \langle -2, -\frac{9\sqrt{3}}{2} \rangle.
\]

14. Find the parametric equations for the line tangent to the curve

\[
\mathbf{r}(t) = \langle 6 \sin t, 2 \cos t, 3 \sin t + \cos t \rangle
\]

when $t = 3\pi/4$.

Solution
The tangent line at $t = 3\pi/4$ passes through the point

\[
\mathbf{r}(3\pi/4) = \left( 3\sqrt{2}, -\sqrt{2}, \sqrt{2} \right),
\]

and has direction

\[
\mathbf{r}'(3\pi/4) = \langle -3\sqrt{2}, -\sqrt{2}, -2\sqrt{2} \rangle.
\]

Therefore the parametric equations of the tangent line are

\[
\begin{align*}
x &= 3\sqrt{2} - 3\sqrt{2}t \\
y &= -\sqrt{2} - \sqrt{2}t \\
z &= \sqrt{2} - 2\sqrt{2}t.
\end{align*}
\]

15. Find the curvature of the graph given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$. 
Solution
Recall that the curvature, $\kappa(t)$, of a curve $\mathbf{r}(t)$ is given by the formula

$$
\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.
$$

First we will need to the value of $t$ such that $\mathbf{r}(t) = (1, 1, 1)$. If $\mathbf{r}(t) = \langle t^2, t^3, t^3 \rangle = (1, 1, 1)$ then, equating $x$ components, we find $t = 1$. Therefore, since

$$
\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle,
$$

$$
\mathbf{r}''(t) = \langle 0, 2, 6t \rangle,
$$

we find that

$$
\kappa(1) = \frac{\|\mathbf{r}'(1) \times \mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|^3}
= \frac{\|\langle 1, 2, 3 \rangle \times \langle 0, 2, 6 \rangle\|}{\|\langle 1, 2, 3 \rangle\|^3}
= \frac{\|\langle 6, -6, 2 \rangle\|}{\left(\sqrt{1^2 + 2^2 + 3^2}\right)^3}
= \frac{\sqrt{76}}{14\sqrt{14}}
= \frac{\sqrt{19}}{7\sqrt{14}} = \frac{\sqrt{266}}{98}.
$$

(Any of these last three answers are fine.)

16. Suppose $\mathbf{a}(t) = \langle e^t, 6t^2, \sin t \rangle$, $\mathbf{v}(0) = \langle 3, 2, 1 \rangle$ and $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$.

Solution
Recall that

$$
\mathbf{v}(t) = \int \mathbf{a}(t) \, dt
$$
\[ \mathbf{r}(t) = \int \mathbf{v}(t) \, dt. \]

Therefore

\[ \mathbf{v}(t) = \langle \int e^t \, dt, \int 6t^2 \, dt, \int \sin t \, dt \rangle = \langle e^t + C_1, 2t^3 + C_2, -\cos t + C_3 \rangle. \]

Since \( \mathbf{v}(0) = \langle 3, 2, 1 \rangle \), we find

\[ \mathbf{v}(0) = \langle C_1 + 1, C_2, C_3 - 1 \rangle = \langle 3, 2, 1 \rangle. \]

Hence \( C_1 = 2, \ C_2 = 2, \ C_3 = 2 \) and

\[ \mathbf{v}(t) = \langle e^t + 2, 2t^3 + 2, 2 - \cos t \rangle. \]

Similarly

\[ \mathbf{r}(t) = \left\langle \int e^t + 2 \, dt, \int 2t^3 + 2 \, dt, \int 2 - \cos t \, dt \right\rangle = \left\langle e^t + 2t + D_1, \frac{t^4}{2} + 2t + D_2, 2t - \sin t + D_3 \right\rangle, \]

and since \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \), we find

\[ \mathbf{r}(0) = \langle 1 + D_1, D_2, D_3 \rangle = \langle 1, 1, 1 \rangle. \]

Hence \( D_1 = 0, \ D_2 = 10, \ D_3 = 1 \) and

\[ \mathbf{r}(t) = \langle e^t + 2t, t^4/2 + 2t + 1, 2t - \sin t + 1 \rangle. \]