ANALYTIC PROOFS OF THE "HAIRY BALL THEOREM" AND THE BROUWER FIXED POINT THEOREM

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This note will present strange but quite elementary proofs of two classical theorems of topology, based on a volume computation in Euclidean space and the observation that the function $(1+t^2)^{n/2}$ is not a polynomial when n is odd. The argument was suggested by the methods of Asimov [1]. Familiar proofs of these theorems all use either combinatorial arguments, homology theory, differential forms, or methods from geometric topology. Compare [2], [3], [4], [6], [7].

Here is a preliminary version of the first one.

THEOREM 1. *An even-dimensional sphere does not possess any continuously differentiable field of unit tangent vectors.*

By definition, the *sphere* S^{n-1} is the set of all vectors $\mathbf{u}=(u_1,\ldots,u_n)$ in the Euclidean space $\mathbf{R}^n$ such that the Euclidean length $\|\mathbf{u}\|$ equals 1. A vector $\mathbf{v}(\mathbf{u})$ in $\mathbf{R}^n$ is *tangent* to S^{n-1} at $\mathbf{u}$ if the Euclidean inner product $\mathbf{u}\cdot\mathbf{v}(\mathbf{u})$ is zero.

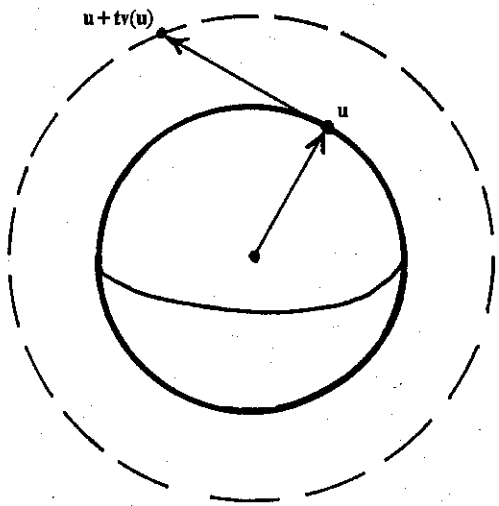

FIG. 1

The hypothesis that the dimension $n-1$ is even is essential. For if $n-1$ is odd then the formula

$$\mathbf{v}(u_1,\ldots,u_n) = (u_2,-u_1,\ldots,u_n,-u_{n-1})$$

defines a differentiable field of unit tangent vectors on S^{n-1}.

The proof of Theorem 1 will depend on two lemmas. The first involves a volume computation. Let A be a compact region in $\mathbf{R}^n$, and let $\mathbf{x}\mapsto\mathbf{v}(\mathbf{x})$ be a continuously differentiable vector field which is defined throughout a neighborhood of A. The values $\mathbf{v}(\mathbf{x})$ can be arbitrary vectors in $\mathbf{R}^n$. For each real number t, consider the function

$$\mathbf{f}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{v}(\mathbf{x})$$

which is defined for all x in A.

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LEMMA 1. *If the parameter t is sufficiently small, then this mapping $\mathbf{f}_t$ is one-to-one and transforms the region A onto a nearby region $\mathbf{f}_t(A)$ whose volume can be expressed as a polynomial function of t.*

Proof. Since A is compact, and since the function $\mathbf{x} \mapsto \mathbf{v}(\mathbf{x})$ is continuously differentiable, there exists a *Lipschitz constant* c so that

$$\|\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y})\| \leqslant c\|\mathbf{x}-\mathbf{y}\| \tag{$*$}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in A. [This is proved as follows. First consider the special case where A is a cube with edges parallel to the coordinate axes. Passing from $\mathbf{x}$ to $\mathbf{y}$ in n steps by changing one coordinate at a time, and applying the Mean Value Theorem of differential calculus, one sees that $|v_i(\mathbf{x})-v_i(\mathbf{y})| \leqslant \Sigma_j c_{ij}|x_j-y_j|$, where

$$c_{ij}=\sup_A |\partial v_i/\partial x_j|.$$

Therefore

$$\|\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y})\| \leqslant \sum_i |v_i(\mathbf{x})-v_i(\mathbf{y})| \leqslant \sum_{i,j} c_{ij}|x_j-y_j| \leqslant \sum_{i,j} c_{ij}\|\mathbf{x}-\mathbf{y}\|,$$

as required. Now an arbitrary compact set A in $\mathbf{R}^n$ can be covered by finitely many open cubes I_α, chosen so that a Lipschitz condition $(*)$ holds whenever $\mathbf{x}$ and $\mathbf{y}$ belong to the same cube. But if $\mathbf{x}$ and $\mathbf{y}$ in A *do not* belong to any common cube I_α, then the distance $\|\mathbf{x}-\mathbf{y}\|$ is bounded away from zero. In fact this expression $\|\mathbf{x}-\mathbf{y}\|$ can be thought of as a continuous and nowhere zero function on the compact set $A \times A - \cup I_\alpha \times I_\alpha$. It is now easy to choose a constant c so that the Lipschitz condition $(*)$ holds uniformly for all $\mathbf{x}$ and $\mathbf{y}$ in A.]

Choose any t with $|t|<c^{-1}$. Then $\mathbf{f}_t$ is one-to-one; for if $\mathbf{f}_t(\mathbf{x})=\mathbf{f}_t(\mathbf{y})$ then $\mathbf{x}-\mathbf{y}=t(\mathbf{v}(\mathbf{y})-\mathbf{v}(\mathbf{x}))$, hence the inequality $\|\mathbf{x}-\mathbf{y}\| \leqslant |t|c\|\mathbf{x}-\mathbf{y}\|$ implies that $\mathbf{x}=\mathbf{y}$.

The matrix of first derivatives of $\mathbf{f}_t$ can be written as $I+t[\partial v_i/\partial x_j]$, where I is the identity matrix. Hence its determinant is a polynomial function of t, of the form $1+t\sigma_1(\mathbf{x})+\cdots+t^n\sigma_n(\mathbf{x})$, the coefficients being continuous functions of $\mathbf{x}$. This determinant is strictly positive for $|t|$ sufficiently small. Integrating over A, we see that the volume of the image region can be expressed as a polynomial function of t,

$$\text{volume } \mathbf{f}_t(A)=a_0+a_1t+\cdots+a_nt^n,$$

with coefficients $a_k=\int\cdots\int_A \sigma_k(\mathbf{x})\,dx_1\cdots dx_n$. ■

Now suppose that the sphere S^{n-1} has a continuously differentiable field $\mathbf{u}\mapsto\mathbf{v}(\mathbf{u})$ of unit tangent vectors. For any real number t, note that the vector $\mathbf{u}+t\mathbf{v}(\mathbf{u})$ has length $\sqrt{1+t^2}$.

LEMMA 2. *If the parameter t is sufficiently small, then the transformation $\mathbf{u}\mapsto\mathbf{u}+t\mathbf{v}(\mathbf{u})$ maps the unit sphere in $\mathbf{R}^n$* **onto** *the sphere of radius $\sqrt{1+t^2}$.*

Assuming this Lemma for the moment, we can now prove Theorem 1. As region A we take the region between two concentric spheres, defined by the inequalities $a \leqslant \|\mathbf{x}\| \leqslant b$. We extend the vector field $\mathbf{v}$ throughout this region by setting $\mathbf{v}(r\mathbf{u})=r\mathbf{v}(\mathbf{u})$ for $a \leqslant r \leqslant b$. It follows that the mapping $\mathbf{f}_t(\mathbf{x})=\mathbf{x}+t\mathbf{v}(\mathbf{x})$ is defined throughout the region A, and maps the sphere of radius r *onto* the sphere of radius $r\sqrt{1+t^2}$ providing that t is sufficiently small. (Note that $\mathbf{f}_t(r\mathbf{u})=r\mathbf{f}_t(\mathbf{u})$.) Hence it maps A onto the region between spheres of radius $a\sqrt{1+t^2}$ and $b\sqrt{1+t^2}$. Evidently

$$\text{volume } \mathbf{f}_t(A) = \left(\sqrt{1+t^2}\,\right)^n \text{volume } (A).$$

Thus, if n is odd, this volume is not a polynomial function of t. Comparing Lemma 1, we obtain a contradiction which proves Theorem 1.

Now we must prove Lemma 2. Here are two alternative arguments, the first based on the "Shrinking Lemma" (see [5]), and the second based on elementary point set topology.

First Proof. Let the region A considered above be defined by the inequalities $1/2 \leqslant \|\mathbf{x}\| \leqslant 3/2$. Choose t small enough so that $|t| < 1/3$ and $|t| < c^{-1}$, where c is a Lipschitz constant for $\mathbf{v}$. Then for each fixed $\mathbf{u}_0$ in S^{n-1} the auxilliary mapping

$$\mathbf{x} \mapsto \mathbf{u}_0 - t\mathbf{v}(\mathbf{x})$$

carries the complete metric space A into itself (since $|t\mathbf{v}(\mathbf{x})| < 1/2$); and satisfies a Lipschitz condition with Lipschitz constant less than 1. Hence, by the Shrinking Lemma, this auxiliary map has a unique fixed point. In other words, the equation $\mathbf{f}_t(\mathbf{x}) = \mathbf{u}_0$ has a unique solution $\mathbf{x}$. Multiplying $\mathbf{x}$ and $\mathbf{u}_0$ by $\sqrt{1+t^2}$, Lemma 2 follows.

Second Proof. We may assume that $n \geqslant 2$. If t is sufficiently small, then the matrix of first derivatives of $\mathbf{f}_t$ is non-singular throughout the compact region A. (Compare the proof of Lemma 1.) Using the Inverse Function Theorem, it follows that $\mathbf{f}_t$ maps open sets in the interior of A to open sets. Hence the image $\mathbf{f}_t(S^{n-1})$ is a relatively open subset of the sphere of radius $\sqrt{1+t^2}$. But this image $\mathbf{f}_t(S^{n-1})$ is also compact and hence closed. Since S^{n-1} is connected, an open and closed subset must be the entire sphere, and the conclusion follows. ∎

A slightly sharper version of Theorem 1, which does not mention differentiability or unit vectors, follows as an immediate corollary.

THEOREM 1′. *An even dimensional sphere does not admit any* **continuous** *field of* **non-zero** *tangent vectors.*

Proof. Suppose that the sphere S^{n-1} possesses a continuous field of non-zero tangent vectors $\mathbf{v}(\mathbf{u})$. Let $m > 0$ be the minimum of $\|\mathbf{v}(\mathbf{u})\|$. By the Weierstrass Approximation Theorem [5], there exists a polynomial mapping $\mathbf{p}$ from S^{n-1} to $\mathbf{R}^n$ satisfying

$$\|\mathbf{p}(\mathbf{u}) - \mathbf{v}(\mathbf{u})\| < m/2$$

for all $\mathbf{u}$. Defining a differentiable vector field $\mathbf{w}(\mathbf{u})$ by the formula

$$\mathbf{w} = \mathbf{p} - (\mathbf{p}\cdot\mathbf{u})\mathbf{u}$$

for every $\mathbf{u}$, the computation $\mathbf{w}\cdot\mathbf{u} = 0$ shows that $\mathbf{w}(\mathbf{u})$ is tangent to S^{n-1} at $\mathbf{u}$, while the computation

$$\|\mathbf{w} - \mathbf{p}\| = |\mathbf{p}\cdot\mathbf{u}| < m/2,$$

together with the triangle inequality, shows that $\mathbf{w} \neq \mathbf{0}$. Therefore, the quotient $\mathbf{w}(\mathbf{u})/\|\mathbf{w}(\mathbf{u})\|$ is an infinitely differentiable field of unit tangent vectors on the sphere S^{n-1}. If $n-1$ is even, this is impossible by Theorem 1. ∎

Starting with Theorem 1′, it is quite easy to prove the Brouwer Fixed Point Theorem:

THEOREM 2. *Every continuous mapping* $\mathbf{f}$ *from the disk* D^n *to itself possesses at least one fixed point.*

Here D^n is defined to be the set of all vectors $\mathbf{x}$ in $\mathbf{R}^n$ with $\|\mathbf{x}\| \leqslant 1$.

Proof. If $\mathbf{f}(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x}$ in D^n, then the formula $\mathbf{v}(\mathbf{x}) = \mathbf{x} - \mathbf{f}(\mathbf{x})$ would define a non-zero vector field $\mathbf{v}$ on D^n which points *outward* everywhere on the boundary, in the sense that $\mathbf{u}\cdot\mathbf{v}(\mathbf{u}) > 0$ for every point $\mathbf{u}$ in S^{n-1}.

With a little care, we can modify this definition to obtain a non-zero vector field $\mathbf{w}$ on D^n which points *directly outward* on the boundary, in the sense that $\mathbf{w}(\mathbf{u}) = \mathbf{u}$ for every $\mathbf{u}$ in S^{n-1}. For example, set

$$\mathbf{w}(\mathbf{x}) = \mathbf{x} - \mathbf{y}(1 - \mathbf{x}\cdot\mathbf{x})/(1 - \mathbf{x}\cdot\mathbf{y}),$$

where $\mathbf{y} = \mathbf{f}(\mathbf{x}) \neq \mathbf{x}$. Evidently $\mathbf{w}(\mathbf{x}) = \mathbf{x}$ whenever $\mathbf{x}\cdot\mathbf{x} = 1$. This expression depends continuously on $\mathbf{x}$, since the denominator never vanishes. It is clearly non-zero whenever $\mathbf{x}$ and $\mathbf{y}$ are linearly independent; while if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent the identity $(\mathbf{x}\cdot\mathbf{x})\mathbf{y} = (\mathbf{x}\cdot\mathbf{y})\mathbf{x}$ implies that $\mathbf{w}(\mathbf{x}) = (\mathbf{x} - \mathbf{y})/(1 - \mathbf{x}\cdot\mathbf{y}) \neq \mathbf{0}$.

Let us transplant this hypothetical vector field $\mathbf{w}(\mathbf{x})$ to the southern hemisphere of the unit sphere S^n in $\mathbf{R}^{n+1}$. Identifying $\mathbf{R}^n$ with the hyperplane $x_{n+1}=0$ which passes through the "equator" of S^n, we will use stereographic projection from the north pole $(0,\ldots,0,1)$ to map each point $\mathbf{x}$ of D^n to a point $\mathbf{s}(\mathbf{x})=\mathbf{u}$ of the southern hemisphere $u_{n+1}<0$. The precise formula is

$$\mathbf{s}(\mathbf{x})=(2x_1,\ldots,2x_n,\mathbf{x}\cdot\mathbf{x}-1)/(\mathbf{x}\cdot\mathbf{x}+1).$$

Applying the derivative of the mapping $\mathbf{s}$ at $\mathbf{x}$ to the vector $\mathbf{w}(\mathbf{x})$, we obtain a corresponding tangent vector $\mathbf{W}(\mathbf{u})$ to S^n at the image point $\mathbf{s}(\mathbf{x})=\mathbf{u}$. (The vector $\mathbf{W}(\mathbf{u})$ can be described as the velocity vector $d\mathbf{s}(\mathbf{x}+t\mathbf{w}(\mathbf{x}))/dt$ of the spherical curve $t\mapsto\mathbf{s}(\mathbf{x}+t\mathbf{w}(\mathbf{x}))$, evaluated at $t=0$.) In this way, we obtain a non-zero tangent vector field $\mathbf{W}$ on the southern hemisphere. At every point $\mathbf{u}=\mathbf{s}(\mathbf{u})$ of the equator, since $\mathbf{w}(\mathbf{u})=\mathbf{u}$ points directly outward, computation shows that the corresponding vector $\mathbf{W}(\mathbf{u})=(0,\ldots,0,1)$ points due north (i.e., away from the southern hemisphere).

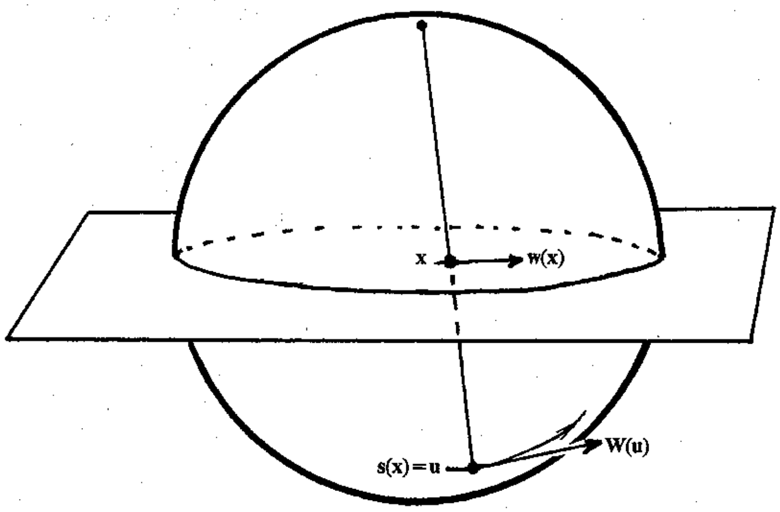

FIG. 2

Similarly, using stereographic projection from the south pole, the vector field $-\mathbf{w}(\mathbf{x})$ corresponds to a vector field on the northern hemisphere which also points due north on the equator. Piecing these two vector fields together, we obtain a non-zero tangent vector field $\mathbf{W}$ which is defined and continuous everywhere on S^n. If n is even, this is impossible by Theorem 1'.

This contradiction proves the Brouwer Fixed Point Theorem for even values of n. But this suffices to prove the Theorem for an odd value $n=2k-1$ also. For any map $\mathbf{f}$ from D^{2k-1} to itself without fixed point would give rise to a map $\mathbf{F}(x_1,\ldots,x_{2k}) = (\mathbf{f}(x_1,\ldots,x_{2k-1}),0)$ from D^{2k} to itself without fixed point. ■

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