1. A divergent series whose general term approaches zero.

The harmonic series $\sum 1/n$.

2. A convergent series $\sum a_n$ and a divergent series $\sum b_n$ such that $a_n \geq b_n$, $n = 1, 2, \ldots$.

Let $a_n = 0$ and $b_n = -1/n$, $n = 1, 2, \ldots$.

3. A convergent series $\sum a_n$ and a divergent series $\sum b_n$ such that $|a_n| \geq |b_n|$, $n = 1, 2, \ldots$.

Let $\sum a_n$ be the conditionally convergent alternating harmonic series $\sum (-1)^{n+1}/n$, and let $\sum b_n$ be the divergent harmonic series $\sum 1/n$.

4. For an arbitrary given positive series, either a dominated divergent series or a dominating convergent series.

A non-negative series $\sum a_n$ is said to dominate a series $\sum b_n$ iff $a_n \geq |b_n|$, $|n = 1, 2, \ldots$ . If the given positive series is $\sum b_n$, let $a_n = b_n$, for $n = 1, 2, \ldots$ Then if $\sum b_n$ diverges it dominates the divergent series $\sum a_n$, and if $\sum b_n$ converges it is dominated by the convergent series $\sum a_n$. The domination inequalities can all be made strict by means of factors $\frac{1}{2}$ and $2$.

This simple result can be framed as follows: There exists no positive series that can serve simultaneously as a comparison test series for convergence and as a comparison test series for divergence. (Cf. Example 19, below.)

5. A convergent series with a divergent rearrangement.

With any conditionally convergent series $\sum a_n$, such as the alternating harmonic series $\sum (-1)^{n+1}/n$, the terms can be rearranged in such a way that the new series is convergent to any given sum, or is divergent. Divergent rearrangements can be found so that the sequence $\{s_n\}$ of partial sums has the limit $+\infty$, the limit $-\infty$, or no limit at all. In fact, the sequence $\{s_n\}$ can be determined in such a way that its set of limits points is an arbitrary given closed interval, bounded or not (cf. Example 2, Chapter 5). The underlying reason that this is possible is that the series of positive terms of $\sum a_n$ and the series of negative terms of $\sum a_n$ are both divergent.
1. Functions of a Real Variable

Conditionally convergent series of vectors in a finite-dimensional space, then the sums obtainable by all possible rearrangements constitute a set that is some linear variety in the space (cf. [47]).

6. For an arbitrary conditionally convergent series \( \sum a_n \) and an arbitrary real number \( x \), a sequence \( \{ x_n \} \), where \( |x_n| = 1 \) for \( n = 1, 2, \ldots \), such that \( \sum x_n a_n = x \).

The procedure here is similar to that employed in Example 5. Since \( \sum |a_n| = +\infty \), we may attach factors \( x_n \) of absolute value 1 in such a fashion that \( x_1 a_1 + \cdots + x_n a_n = |a_1| + \cdots + |a_n| > x \). Let \( n_1 \) be the least value of \( n \) that ensures this inequality. We then provide factors \( x_n \) of absolute value 1, for the next terms in order to obtain (for the least possible \( n_2 \)):

\[
\begin{align*}
|a_1| &+ \cdots + |a_{n_1}| - |a_{n_1+1}| - \cdots - |a_{n_2}| < x.
\end{align*}
\]

If this process is repeated, with partial sums alternately greater than \( x \) and less than \( x \), a series \( \sum x_n a_n \) is obtained which, since \( a_n \to 0 \) as \( n \to +\infty \), must converge to \( x \).

7. Divergent series satisfying any two of the three conditions of the standard alternating series theorem.

The alternating series theorem referred to states that the series \( \sum x_n c_n \) where \( |x_n| = 1 \) and \( c_n > 0 \), \( n = 1, 2, \ldots \), converges provided

(i) \( x_n = (-1)^{n+1} \), \( n = 1, 2, \ldots \),

(ii) \( x_n \) is bounded,

(iii) \( \lim_{n \to \infty} x_n = 0 \).

No two of these three conditions by themselves imply convergence: that is, no one can be omitted. The following three examples demonstrate this fact:

(i): Let \( x_n = 1 \), \( c_n = 1/n \), \( n = 1, 2, \ldots \). Alternatively, for an example that is, after a fashion, an "alternating series" let \( \{ x_n \} \) be the sequence repeating in triplets: 1, 1, -1, 1, 1, -1, \ldots.

(ii): Let \( x_n = 1/n \) if \( n \) is odd, and let \( x_n = 1/n^2 \) if \( n \) is even.

(iii): Let \( x_n = (n + 1)/n \) (or, more simply, let \( x_n = 1 \)), \( n = 1, 2, \ldots \).

8. A divergent series whose general term approaches zero and which, with a suitable introduction of parentheses, becomes convergent to an arbitrary sum.

Introduction of parentheses in an infinite series means grouping of consecutive finite sequences of terms (each such finite sequence consisting of at least one term) to produce a new series, whose sequence of partial sums is therefore a subsequence of the sequence of partial sums of the original series. For example, one way of introducing parentheses in the alternating harmonic series gives the series

\[
(1 - \frac{1}{2}) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2n-1) \cdot 2n} + \cdots
\]

Any series derived from a convergent series by means of introduction of parentheses is convergent, and has a sum equal to that of the given series.

The final rearranged series described under Example 5 has the stated property since, for an arbitrary real number, a suitable introduction of parentheses gives a convergent series whose sum is the given number.

9. For a given positive sequence \( \{ b_n \} \) with limit inferior zero, a positive divergent series \( \sum a_n \) whose general term approaches zero and such that \( \lim_{n \to \infty} a_n/b_n = 0 \).

Choose a subsequence \( \{ b_{n_1}, b_{n_2}, \ldots \} \) of \( \{ b_n \} \) such that \( \lim_{n \to \infty} b_{n_k} = 0 \), and let \( a_{n_k} = b_{n_k}^k \) for \( k = 1, 2, \ldots \). For every other value of \( n = m_1, m_2, m_3, \ldots, m_j, \ldots \), let \( a_{m_j} = 1/j \). Then \( a_n \to 0 \) as \( n \to +\infty \), \( \sum a_n \) diverges, and \( a_n/b_n = b_n \to 0 \) as \( k \to +\infty \).

This example shows (in particular) that no matter how rapidly a positive sequence \( \{ b_n \} \) may converge to zero, there is a positive sequence \( \{ c_n \} \) that converges to zero slowly enough to ensure divergence of the series \( \sum c_n \), and yet has a subsequence converging to zero more rapidly than the corresponding subsequence of \( \{ b_n \} \).

10. For a given positive sequence \( \{ b_n \} \) with limit inferior zero, a positive convergent series \( \sum a_n \) such that \( \lim_{n \to \infty} a_n/b_n = +\infty \).
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preceding, let \( \{i_n\} \) and \( \{j_k\} \) be strictly increasing sequences of positive integers chosen so that the related properties hold as follows: We choose \( i_1 \) and \( j_1 \) so that the related properties hold for \( T_1 \); then \( i_2 \) and \( j_2 \) so that the related properties hold for both \( T_1 \) and \( T_2 \); etc. Let the sequence \( \{a_n\} \) be defined as in Example 21. For any fixed \( m \) the sequence \( \{a_n\} \) is transformed into a sequence \( \{b_m\} \) and, since the numbers \( i_n \) and \( j_k \) for \( k > m \) constitute sequences valid for the counterexample technique applied to \( T_m \), it follows that \( \lim_{n \to \infty} b_m \) does not exist for any \( m \).

22. A power series convergent at only one point. (Cf. Example 24.)

The series \( \sum_{n=0}^{+\infty} n! x^n \) converges for \( x = 0 \) and diverges for \( x \neq 0 \).

23. A function whose Maclaurin series converges everywhere but represents the function at only one point.

The function

\[
f(x) = \begin{cases} 
  e^{-1/4} & \text{if } x \neq 0, \\
  0 & \text{if } x = 0 
\end{cases}
\]

is infinitely differentiable, all of its derivatives at \( x = 0 \) being equal to 0 (cf. Example 10, Chapter 3). Therefore its Maclaurin series

\[
\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{+\infty} 0
\]

converges for all \( x \) to the function that is identically zero, and hence represents (converges to) the given function \( f \) only at the single point \( x = 0 \).

24. A function whose Maclaurin series converges at only one point.

A function with this property is described in [10], p. 153. The function

\[
f(x) = \sum_{n=0}^{+\infty} e^{-n} \cos n^2 x,
\]

because of the factors \( e^{-n} \) present in all of the series obtained by successive term-by-term differentiation (which therefore all converge uniformly), is an infinitely differentiable function. Its Maclaurin series has only terms of even degree, and the absolute value of the term of degree \( 2k \) is

\[
\sum_{n=0}^{+\infty} \frac{x^{2k} e^{-n} n^{4k}}{(2k)!} > \left( \frac{n^2 x}{2k} \right)^{2k} e^{-n}
\]

for every \( n = 0, 1, 2, \ldots \), and in particular for \( n = 2k \). For this value of \( n \) and, in terms of any given nonzero \( x \), with \( k \) any integer greater than \( e/2x \), we have

\[
\left( \frac{n^2 x}{2k} \right)^{2k} e^{-n} = \left( \frac{2kx}{e} \right)^{2k} > 1.
\]

This means that for any nonzero \( x \) the Maclaurin series for \( f \) diverges.

The series \( \sum_{n=0}^{+\infty} n! x^n \) was shown in Example 22 to be convergent at only one point, \( x = 0 \). It is natural to ask whether this series is the Maclaurin series for some function \( f(x) \), since an affirmative answer would provide another example of a function of the type described in the present instance. We shall now show that it is indeed possible to produce an infinitely differentiable function \( f(x) \) having the series given above as its Maclaurin series. To do this, let \( \phi_{n0}(x) \) be defined as follows: For \( n = 1, 2, \ldots \), let

\[
\phi_{n0}(x) = \begin{cases} 
  ((n-1)!)^2 & \text{if } 0 \leq |x| \leq 2^{-n/(n!)^2} \\
  0 & \text{if } |x| \geq 2^{-n+1/(n!)^2} 
\end{cases}
\]

where, by means of the type of "bridging functions" constructed for Example 12, Chapter 3, \( \phi_{n0}(x) \) is made infinitely differentiable everywhere. Let \( f_1(x) = \phi_{10}(x) \), and for \( n = 2, 3, \ldots \), let

\[
\phi_{n1}(x) = \int_0^x \phi_{n0}(t) \, dt,
\]

\[
\phi_{n2}(x) = \int_0^x \phi_{n1}(t) \, dt,
\]

\[
\vdots
\]

\[
f_n(x) = \phi_{n,n-1}(x) = \int_0^x \phi_{n,n-2}(t) \, dt.
\]
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Thus \( f_n(x) = \phi_{n-1}(x), f_n^2(x) = \phi_{n-2}(x), \ldots, f_n^{(n-1)}(x) = \phi_0(x), \)
\( f_n^{(n)}(x) = \phi_0(0). \) For any \( x \) and \( 0 \leq k \leq n - 2, \)
\( f_n^{(k)}(x) = \sum_{n-k+1}^{n-1} \frac{1}{(n-k-1)!} x^{n-k-1}. \)

For any \( x \) and \( 0 \leq k \leq n - 2, \)
\( f_n^{(k)}(x) = \sum_{n-k+1}^{n-1} \frac{1}{(n-k-1)!} x^{n-k-1}. \)

The series \( \sum_{n=1}^{\infty} f_n^{(k)}(x), \) for each \( k = 0, 1, 2, \ldots, \) converges uniformly in every closed finite interval. Indeed, if \( |x| \leq K, \)
\( \sum_{n=2}^{\infty} |f_n^{(k)}(x)| \leq \sum_{n=2}^{\infty} \frac{K^{n-k-1}}{n^2 2^{n-1} (n-k-2)!}, \)
and uniform convergence follows from the Weierstrass M-test (cf. [34], p. 445). Hence we see that
\( f(x) = \sum_{n=1}^{\infty} f_n(x) \)
is an infinitely differentiable function such that for \( k = 0, 1, 2, \ldots, \)
\( f_n^{(k)}(x) = \sum_{n=1}^{\infty} f_n^{(k)}(x). \)

For \( k \geq 1, f_n^{(0)}(0) = \phi_0(0) = 0. \) For \( n \geq 1 \) and \( k = n - 1, \)
\( f_n^{(n)}(0) = \phi_0(0) = ((n-1)!)^2. \) For \( 0 \leq k < n - 1, f_n^{(k)}(0) = 0. \)
Thus the Maclaurin series for \( f(x) \) is \( \sum_{n=0}^{\infty} n! x^n. \)

23. A convergent trigonometric series that is not a Fourier series.

We shall present two examples, one in case the integration involved is that of Riemann, and one in case the integration is that of Lebesgue.

The series \( \sum_{n=1}^{\infty} \frac{\sin \alpha x}{n^a}, \) where \( 0 < \alpha \leq \frac{1}{2}, \) converges for every real number \( x, \) as can be seen (cf. [34], p. 533) by an application of a convergence test due to N. H. Abel (Norwegian, 1802–1829). However, this series cannot be the Fourier series of any Riemann-integrable function \( f(x) \) since, by Bessel’s inequality (cf. [34], p. 532),
for \( n = 1, 2, \ldots, \)
\( \frac{1}{2^a} + \frac{1}{2^{2a}} + \cdots + \frac{1}{n^a} \leq \frac{1}{\pi} \int_0^\pi [f(x)]^2 dx. \)
Since \( f(x) \) is Riemann-integrable so is \( [f(x)]^2, \) and the right-hand side of the preceding inequality is finite, whereas if \( \alpha \leq \frac{1}{2}, \) the left-hand side is unbounded as \( n \to +\infty. \) (Contradiction.)

The series \( \sum_{n=1}^{\infty} \frac{\sin \alpha x}{n^a} \) also converges for every real number \( x. \) Let
\( f(x) = \sum_{n=1}^{\infty} \frac{\sin \alpha x}{n^a}. \)
If \( f(x) \) is Lebesgue-integrable, then the function
\( F(x) = \int_0^x f(t) \, dt \)
is both periodic and absolutely continuous. Since \( f(x) \) is an odd function \( (f(x) = -f(-x)), \) we see that \( F(x) \) is an even function \( (F(x) = F(-x)) \) and thus the Fourier series for \( F(x) \) is of the form
\( \sum_{n=0}^{\infty} a_n \cos nx, \)
where \( a_0 = \frac{1}{\pi} \int_0^\pi F(x) \, dx, \) and for \( n \geq 2, \)
\( a_n = \frac{2}{\pi} \int_0^\pi F(x) \cos nx \, dx \)
\( = \frac{2}{\pi} F(x) \sin \frac{\pi x}{n} \bigg|_0^\pi = \frac{2}{\pi} F'(x) \sin \frac{\pi x}{n} \, dx \)
\( = -\frac{2}{\pi} \int_0^\pi f(x) \sin \frac{\pi x}{n} \, dx = -\frac{1}{n} \sin \frac{\pi x}{n}. \)
(F’(x) exists and is equal to \( f(x) \) almost everywhere.) Since \( F(x) \) is of bounded variation, its Fourier series converges at every point, and in particular at \( x = 0, \) from which we infer that \( \sum_{n=1}^{\infty} a_n \) converges. But since \( a_0 = -1/(n \ln n), \) and \( \sum_{n=2}^{\infty} (-1/(n \ln n)) \) diverges, we have a contradiction to the assumption that \( f(x) \) is Lebesgue-integrable.
Chapter 7
Uniform Convergence

Introduction
The examples of this chapter deal with uniform convergence — and convergence that is not uniform — of sequences of functions on certain sets. The basic definitions and theorems will be assumed to be known (cf. [34], pp. 441-462, [36], pp. 270-292).

1. A sequence of everywhere discontinuous functions converging uniformly to an everywhere continuous function.

\[ f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \]

Clearly, \( \lim_{n \to \infty} f_n(x) = 0 \) uniformly for \(-\infty < x < +\infty\).

This simple example serves to illustrate the following general principle: Uniform convergence preserves good behavior, not bad behavior. This same principle will be illustrated repeatedly in future examples.

2. A sequence of infinitely differentiable functions converging uniformly to zero, the sequence of whose derivatives diverges everywhere.

If \( f_n(x) = (\sin nx)/\sqrt{n} \), then since \( |f_n(x)| \leq 1/\sqrt{n} \) this sequence converges uniformly to 0. To see that the sequence \( \{f_n'(x)\} \) converges nowhere, let \( x \) be fixed and consider

\[ b_n = f_n'(x) = \sqrt{n} \cos nx. \]

If \( x = 0 \), \( b_n = \sqrt{n} \to +\infty \) as \( n \to +\infty \). We shall show that for any \( x \neq 0 \) the sequence \( \{b_n\} \) is unbounded, and hence diverges, by showing that there are arbitrarily large values of \( n \) such that \( |\cos nx| \geq \frac{1}{2} \).

Indeed, for any positive integer \( m \) such that \( |\cos mx| < \frac{1}{2} \),

\[ |\cos 2mx| = |2 \cos^2 mx - 1| = 1 - 2 \cos^2 mx \geq \frac{1}{2}, \]

so that there exists an \( n > m \) such that \( |\cos nx| > \frac{1}{2} \).

3. A nonuniform limit of bounded functions that is not bounded.

Each function

\[ f_n(x) = \begin{cases} \min \left( \frac{1}{n}, \frac{1}{x} \right) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0 \end{cases} \]

is bounded on the closed interval \([0, 1]\), but the limit function \( f(x) \), equal to \( 1/x \) if \( 0 < x \leq 1 \) and equal to 0 if \( x = 0 \), is unbounded there.

Let it be noted that for this example to exist, the limit cannot be uniform.

4. A nonuniform limit of continuous functions that is not continuous.

A trivial example is given by

\[ f_n(x) = \begin{cases} \min (1, nx) & \text{if } x \geq 0, \\ \max (-1, nx) & \text{if } x < 0, \end{cases} \]

whose limit is the signum function (Example 3, Chapter 3), which is discontinuous at \( x = 0 \).

A more interesting example is given by use of the function \( f \) (cf. Example 15, Chapter 2) defined:

\[ f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms, where } p \text{ and } q \text{ are integers} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \]
For an arbitrary positive integer $n$, define $f_n(x)$ as follows: According to each point $(\frac{p}{q}, \frac{1}{q})$, where $1 \leq q < n$, $0 \leq p \leq q$, in each interval of the form $\left(\frac{p}{q}, \frac{p+1}{2n^2}, \frac{p}{q}\right)$ define

$$f_n(x) = \min\left(\frac{1}{n}, \frac{1}{q} + 2n^2\left(x - \frac{p}{q}\right)\right);$$

in each interval of the form $\left(\frac{p}{q}, \frac{p+1}{2n^2}\right)$ define

$$f_n(x) = \max\left(\frac{1}{n}, \frac{1}{q} - 2n^2\left(x - \frac{p}{q}\right)\right);$$

and at every point $x$ of $[0, 1]$ at which $f_n(x)$ has not already been defined, let $f_n(x) = 1/n$. Outside $[0, 1]$ $f_n(x)$ is defined so as to be periodic with period one. The graph of $f_n(x)$, then, consists of an infinite polygonal arc made up of segments that either lie along the horizontal line $y = 1/n$ or rise with slope $\pm 2n^2$ to the isolated points of the graph of $f$. (Cf. Fig. 2.) As $n$ increases, these "spikes" sharpen, and the base approaches the $x$ axis. As a consequence, for each $x \in \mathbb{Q}$ and $n = 1, 2, \ldots$,

$$f_n(x) \leq f_{n+1}(x),$$

and

$$\lim_{n \to \infty} f_n(x) = f(x),$$

as defined above. Each function $f_n$ is everywhere continuous, but the limit function $f$ is discontinuous on the dense set $\mathbb{Q}$ of rational numbers. (Cf. Example 24, Chapter 2.)

5. A nonuniform limit of Riemann-integrable functions that is not Riemann-integrable. (Cf. Example 33, Chapter 8.)

Each function $g_n$, defined for Example 24, Chapter 2, when restricted to the closed interval $[0, 1]$ is Riemann-integrable there, since it is bounded there and has only a finite number of points of discontinuity. The sequence $\{g_n\}$ is an increasing sequence $(g_n(x) \leq g_{n+1}(x)$ for each $x$ and $n = 1, 2, \ldots)$ converging to the function $f$ of Example 1, Chapter 4, that is equal to 1 on $\mathbb{Q} \cap [0, 1]$ and equal to 0 on $[0, 1] \setminus \mathbb{Q}$.
1. Functions of a Real Variable

Another example is the sequence \( \{f_n(x)\} \) where \( f_n(x) = nx^{n}e^{-nx} \), \( 0 \leq x \leq 1 \).

A more extreme case is given by

\[
f_n(x) = \begin{cases} 
2n^2 & \text{if } 0 \leq x \leq \frac{1}{2n}, \\
n^2 - 2n^2(x - \frac{1}{2n}) & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{n}, \\
0 & \text{if } \frac{1}{n} \leq x \leq 1,
\end{cases}
\]

in which case, for any \( b \in (0, 1) \)

\[
\lim_{n \to +\infty} \int_{0}^{b} f_n(x) \, dx = \lim_{n \to +\infty} \frac{n}{2} = +\infty,
\]

while

\[
\int_{0}^{b} \lim_{n \to +\infty} f_n(x) \, dx = \int_{0}^{b} 0 \, dx = 0.
\]

7. A sequence of functions for which the limit of the derivatives is not equal to the derivative of the limit.

If \( f_n(x) = x/(1 + n^2x^2) \) for \( -1 \leq x \leq 1 \) and \( n = 1, 2, \ldots \), then \( f(x) = \lim_{n \to \pm \infty} f_n(x) \) exists and is equal to 0 for all \( x \in [-1, 1] \) (and this convergence is uniform since the maximum and minimum values of \( f_n(x) \) on \([-1, 1]\) are \( \pm 1/2n \)). The derivative of the limit is identically equal to 0. However, the limit of the derivatives is

\[
\lim_{n \to +\infty} f'_n(x) = \lim_{n \to +\infty} \frac{1 - n^2x^2}{(1 + n^2x^2)^2} = \begin{cases} 
1 & \text{if } x = 0, \\
0 & \text{if } 0 < |x| \leq 1.
\end{cases}
\]

8. Convergence that is uniform on every closed subinterval but not uniform on the total interval.

Let \( f_n(x) = x^n \) on the open interval \( (0, 1) \).

9. A sequence \( \{f_n\} \) converging uniformly to zero on \([0, +\infty)\) but such that \( \int_{0}^{+\infty} f_n(x) \, dx \to 0 \).

Let \( f_n(x) = \begin{cases} 
1/n & \text{if } 0 \leq x \leq n, \\
0 & \text{if } x > n.
\end{cases} \)

Then \( f_n \) converges uniformly to 0 on \([0, +\infty)\), but

\[
\int_{0}^{+\infty} f_n(x) \, dx = 1 \to 1.
\]

A more extreme case is given by

\[
f_n(x) = \begin{cases} 
1/n & \text{if } 0 \leq x \leq n^2, \\
0 & \text{if } x > n^2.
\end{cases}
\]

Then \( \int_{0}^{+\infty} f_n(x) \, dx = n \to +\infty. \)

10. A series that converges nonuniformly and whose general term approaches zero uniformly.

The series \( \sum_{n=1}^{+\infty} x^n/n \) on the half-open interval \([0, 1)\) has these properties. Since the general term is dominated by \( 1/n \) on \([0, 1)\) its uniform convergence to zero there follows immediately. The convergence of the series follows from its domination by the series \( \sum x^n \) which converges on \([0, 1)\). The nonuniformity of this convergence is a consequence of the fact that the partial sums are not uniformly bounded (the harmonic series diverges; cf. [34], p. 447, Exs. 31, 32).

11. A sequence converging nonuniformly and possessing a uniformly convergent subsequence.

On the real number system \( \mathbb{R} \), let

\[
f_n(x) = \begin{cases} 
x & \text{if } n \text{ is odd,} \\
1/n & \text{if } n \text{ is even.}
\end{cases}
\]

The convergence to zero is nonuniform, but the convergence of the subsequence \( \{f_{2n}(x)\} = \{1/2n\} \) is uniform.

12. Nonuniformly convergent sequences satisfying any three of the four conditions of Dini's theorem.

Dini's theorem states that if \( \{f_n\} \) is a sequence of functions defined on a set \( A \) and converging on \( A \) to a function \( f \), and if

(i) \( f_n \) is continuous on \( A \), \( n = 1, 2, \ldots \),

(ii) \( f_n \) converges uniformly to \( f \) on \( A \),

(iii) \( f \) is bounded on \( A \),

(iv) \( A \) is compact,

then \( \{f_n\} \) converges uniformly to \( f \) on \( A \).