Problem Set 5
Due: Thursday, December 6, 2012

1. Consider the optimal capital income tax problem for the case in which household utility is given by

\[ U_0 = \sum_{t=0}^{\infty} \beta^t \left( \frac{c_t^{\frac{1-\sigma}{\sigma}}}{1-\sigma} + \log \left( 1 - n_t \right) \right) \]

The production function is the standard constant returns to scale one, \( y = f(k) \).

a) Set up the optimization problem for the household with proportional taxes on capital and labor income. Write down the first-order conditions for the household optimum.

b) Set up the optimal tax problem for the government.

c) Derive the first-order conditions for the government’s optimal policy.

d) Show (or explain) that the optimal capital income tax will equal zero for all \( t > 1 \). That is, the only positive tax on capital income for these preferences can be \( \tau_k^t \) if we rule out a capital levy at time 0 (that is, \( \tau_k^0 = 0 \) is fixed).

2. Consider the optimal tax problem outlined in the reading, but assume the production function, \( y_t = wn_t \), where \( w \) is constant labor productivity. There is no capital in this economy, and the only financial asset is government debt. Let household utility be given by

\[ U_0 = \sum_{t=0}^{\infty} \beta^t \left( \log c_t + \log \left( 1 - n_t \right) \right) \]

a) Assume the government only taxes labor. Write down the household problem and the first-order conditions for its optimum.

b) Write down the government’s optimal tax problem (note the simplicity of the resource constraint with no capital). Allow for time-varying expenditures, \( g_t \). Simplify the implementability constraint for this particular utility function and production function.

c) Using your first-order conditions for the optimal policy, the resource constraint and household first-order conditions, you can derive an expression for the optimal labor income tax rate in terms of the multiplier on the implementability condition and leisure consumption. (Hint: The expression you derive will be \( \tau_n^t = \frac{\mu}{\mu + (1-n_t)} \) where \( \mu \) is the multiplier on the implementability constraint.)

d) Suppose \( g_t \) is constant for all \( t \). Is \( \tau_n^t \) constant in optimal tax policy? \( \mu \) is the shadow price of distortionary taxation. Does this rise with \( g \)?
e) Suppose that \( g_t = g \) is constant until date \( T \) when it permanently rises by \( \Delta g \). At date 0, everyone knows that expenditures will rise at \( T \). Explain how public debt, \( b_t \), changes over time and how the tax rate changes at \( T \).

3. Consider an overlapping generations economy with two-period lived households. The utility function for the typical household born in period \( t \) is given by
\[
U_t = \log c_{1t} + \beta \log c_{2t+1},
\]
where \( c_{1t} \) denotes consumption of the generation born at \( t \) in period \( t \), \( c_{2t+1} \) denotes its consumption in period \( t + 1 \). Each generation only works when young supplying a unit of labor inelastically. Output per worker is given by the concave production function,
\[
y_t = f(k_t),
\]
and the rate of depreciation is zero. Let the population grow at the constant proportionate rate \( \eta > 0 \).

a) Find the conditions that determine the decentralized equilibrium given an initial capital stock, \( k_0 > 0 \).

b) Find the steady state for this economy.

c) Introduce an unfunded social security system. Let the lump-sum tax on the young at date \( t \) be given by \( T_{1t} \) and the lump-sum tax imposed on the same generation at time \( t + 1 \) when old be \( T_{2t+1} \). If the old receive a transfer, then \( T_{2t+1} \) is negative. The government budget is balanced for \( g_t = 0 \) if
\[
T_{1t} + \frac{T_{2t}}{1 + \eta} = 0.
\]
Rewrite your equilibrium conditions with this policy included.

d) Let the policy set constant per capita taxes on the young, \( T_{1t} = T_1 \) for all \( t \geq 0 \). Suppose this policy is introduced at time 0 (so that \( T_{1t} = 0 \) for \( t = -1 \)). Show that the introduction of this policy reduces saving by every generation born at time \( t \geq 0 \).

4. Continue with the economy of problem 3.

a) Introduce public debt. Write down the government budget identity, and the conditions that determine the decentralized equilibrium for \( g_t = 0 \) for all \( t \). Use the notation \( b_t \) to denote public debt per capita of the young. This is \( B_t / N_t \) where \( N_t \) is the number of young born at \( t \).

b) Let the economy be in the steady state with \( b_0 = 0 \). The government makes a transfer to the currently old (the generation born at \( t = -1 \) receives the transfer \( -T_{2,0} > 0 \) which is financed by imposing a constant lump-sum tax on the young, \( T_{1t} \), for all \( t \geq 0 \). Let \( T_{2t+1} = 0 \) for all \( t \geq 0 \). The per capita tax, \( T_{1t} \), is set so that per capita public debt stays constant: \( b_{t+1} = b_t \). Show that this policy reduces \( k_{t+1} \) for all \( t \geq 0 \).

c) Show that if the steady state marginal productivity of capital, \( f'(k^*) \), is less than \( \eta \), this policy increases
the utility of the old at time 0 and the utility of the young born at every date, \( t \geq 0 \).

d) Now, let \( T_{1t} = 0 \) for all \( t \geq 0 \) even though the government still gives the transfer, \(-T_{2,0} > 0\), to the
old at time 0. Observe that total public debt \( B_t \) grows at the rate of interest forever so this violates the no
Ponzi condition. If \( f'(k^*) \) is less than \( \eta \), show that this is sustainable. That is, the economy converges to
a steady state in per capita terms. Now, observe that your answer to part c still holds: this policy makes
everyone better off. Can you explain or interpret what is happening?