Problem Set 1
Due: Tuesday, October 9, 2012

1. Consider the simple command economy with capital in which the planner seeks to maximize the utility function

\[ U_0 = \sum_{t=0}^{T} \beta^t u(c_t) \]

given the resource identity

\[ k_{t+1} - k_t = f(k_t) - \delta k_t - c_t \]
given the initial capital-labor ratio, \( k_0 \).

(a) Begin by setting up a Lagrangian and deriving the necessary conditions for an optimum.

(b) Now, let \( T \) go to infinity. What changes appear in the necessary conditions?

(c) Find the conditions that determine the steady state.

(d) Let \( u(c) = \log c \). Write out the dynamics for \( k_t \) and \( c_t \) and linearize these around the steady state. Write out expressions (not approximations) for the eigenvalues.

(e) Find the eigenvectors associated with each eigenvalue (please use \( \lambda \) to denote an eigenvalue rather than repeat a messy expression). Is the steady state saddle-path stable? Explain.

(f) Show that the capital accumulation path given by choosing \( c_0 \) so that \( (c_0, k_0) \) is on the stable saddle-path for the linearized solution satisfies all of the necessary conditions for an optimum.

2. Continue using this model, but change the production function to

\[ y_t = A_t f(k_t) \]

for \( A_t > 0 \).

(a) Use algebra and a phase diagram to show how a permanent increase in \( A_t \) from 1 to a constant \( A > 1 \) at time 0 when the economy is in the steady state affects consumption and investment.

(b) Suppose that \( A_t = 1 \) for all \( t \) except \( t = 0 \) and \( A_0 = A > 1 \). Let the economy be in the steady state at \( t = -1 \) (thus, \( k_0 = k^* \)). Does the steady state for this economy change? Explain.

(c) Use the Euler condition and resource identity to show how \( c_0 \) differs from \( c_{-1} \) and how \( k_1 \) differs from \( k_0 \). Is \( \Delta k_1 \) positive?

(d) How do \( c_1 \) and \( \Delta k_2 \) respond to the one-time increase in output at time 0?
3. Consider the simple resource identity,

\[ k_t - k_{t-1} = Ak_{t-1} - c_{t-1}, \]

for \( A > 0 \) and constant.

(a) Write down the same relationship for \( k_{t-1} \), then substitute this expression in the one given above for \( k_t \). By repeated substitution, you will arrive at a relationship between \( k_t \) and \( k_0 \) and a series in \( c_s \) for \( s = 0, \ldots, t - 1 \).

(b) Use the same equation, but solve it forward in time. Start with

\[ k_{t+1} - k_t = Ak_t - c_t, \]

where \( k_t \) is now given, and rearrange as

\[ \frac{k_{t+1}}{1 + A} = k_t - \frac{c_t}{1 + A}. \]

Iterating forward in \( t \), you solve for \((1 + A)^{-T} k_{t+T}\) for any \( T > 0 \).

(c) Let \( T \) go to infinity. Show that you now have a constraint for the planner’s consumption over time given that \( k_t \geq 0 \) for all \( t \geq 0 \).

4. Next, consider an economy in which the planner’s utility is given by

\[ U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t) \]

and the resource identity by

\[ k_{t+1} - k_t = Ak_t - c_t. \]

The initial capital stock is given by \( k_0 \).

(a) Use the Lagrangian to find the necessary conditions for the optimum.

(b) Let \( u(c_t) = \log c_t \) and solve out for consumption at \( t = 0 \), \( c_0 \).

(c) Explain where you use the transversality condition in part (b).