A Primer on Intertemporal Optimization in Continuous Time

A. Discrete-time optimization

We begin with the discrete-time optimal savings problem for a representative household:

\[
\max_{\{c_t, a_t\}} \sum_{t=0}^{T} \beta^t u(c_t) \tag{1}
\]

subject to

\[
a_{t+1} = (1 + r_t) a_t + w_t - c_t \tag{2}
\]

and

\[
a_{T+1} \prod_{t=1}^{T} \left( \frac{1}{1 + r_t} \right) \geq 0,
\]

given the value of \(a_0\). We also restrict \(c_t\) to be non-negative, \(r_t\) to be greater than \(-1\), \(\beta\) to be between 0 and 1, and \(u(c)\) to be strictly concave and increasing. It is convenient to assume that

\[
\lim_{c \to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \to \infty} u'(c) = 0. \tag{3}
\]

We form the standard Lagrangian,

\[
L = \sum_{t=0}^{T} \left[ \beta^t u(c_t) + \lambda_t \left( (1 + r_t) a_t + w_t - c_t - a_{t+1} \right) \right]. \tag{4}
\]

Note that \(L\) is a function of \(a_0, ..., c_T, a_1, ..., a_{T+1}, \lambda_0, ..., \lambda_T\). Initial financial wealth, \(a_0\), is pre-determined and cannot be chosen to maximize \(L\). The necessary conditions for an optimum are

\[
\frac{\partial L}{\partial c_t} = 0 \quad \Rightarrow \quad \beta^t u'(c_t) - \lambda_t = 0, \quad \text{for} \quad t = 0, ..., T, \tag{5}
\]

\[
\frac{\partial L}{\partial a_t} = 0 \quad \Rightarrow \quad (1 + r_{t+1}) \lambda_{t+1} - \lambda_t = 0, \quad \text{for} \quad t = 0, ..., T - 1, \tag{6}
\]

\[
\frac{\partial L}{\partial \lambda_t} = 0 \quad \Rightarrow \quad (1 + r_t) a_t + w_t - c_t - a_{t+1} = 0, \quad \text{for} \quad t = 0, ..., T, \tag{7}
\]

and

\[
\frac{\partial L}{\partial a_{T+1}} \leq 0, \quad a_{T+1} \prod_{t=1}^{T} \left( \frac{1}{1 + r_t} \right) \geq 0, \quad a_{T+1} \frac{\partial L}{\partial a_{T+1}} = 0 \quad \Rightarrow \quad -\lambda_T a_{T+1} = 0. \tag{8}
\]

This last series of conditions are the Kuhn-Tucker conditions for optimization with respect to \(a_{T+1}\), the stock of financial wealth at the end of period \(T\). In the optimal savings example, we have

\[
\frac{\partial L}{\partial a_{T+1}} = -\lambda_T. \tag{9}
\]
By equation (5) \( \lambda_T = \beta^T u'(c_T) \) is positive under the requirements that marginal utility is positive for all non-negative consumptions, \( \beta \) is positive and \( \lim_{c \to 0} u'(c) = \infty \). This last condition ensures that \( \frac{\partial L}{\partial c_t} = 0 \) has a solution for \( c_t \) positive. It will be helpful for you to remember how the final condition, 

\[-\lambda_T a_{T+1} = 0,\]

was derived for later. In the finite horizon case, this is called the terminal condition. For \( T \) finite, it implies that \( a_{T+1} = 0 \) because \( \beta > 0 \) and \( \lambda_T = \beta^T u'(c_T) > 0 \).

Putting equations (5) and (6) together, we derive the Euler condition:

\[ u'(c_t) = \beta (1 + r_{t+1}) u'(c_{t+1}). \] (10)

Now, we can first extend this to an infinite horizon by simply letting \( T \) go to infinity. The only mathematical detail that needs attention is assure that an optimum exists.

First, the set of all infinite horizon consumption and asset accumulation plans available to the household must be compact and convex (this set is a subset of infinite-dimensional Euclidean space). The intertemporal budget set is given by

\[ \sum_{t=0}^{\infty} R_t c_t \leq (1 + r_0) a_0 + \sum_{t=0}^{\infty} R_t w_t - \lim_{T \to \infty} R_T a_{T+1}, \] (11)

where

\[ R_t \equiv \prod_{s=1}^{t} \left( \frac{1}{1 + r_s} \right). \] (12)

The only condition we need to add is that \( \lim_{T \to \infty} R_T a_{T+1} \) is bounded from below. The natural condition to introduce for the asset accumulation problem is that \( \lim_{T \to \infty} R_T a_{T+1} \geq 0 \). This means that the household is not allowed to borrow indefinitely without paying interest on its debt. This and our restriction that every \( c_t \geq 0 \) assures that the set of all consumption plans, \((c_0, c_1, ...)\), is bounded; it is obviously closed.

Second, the objective function,

\[ U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t), \] (13)

must be continuous over all consumption plans in the budget set. For \( T \) finite, we only needed \( u(c_t) \) to be a continuous function for this to hold. In the infinite horizon case, we need to rule out a possible discontinuity as \( T \) goes to infinity. We need the additional restriction that \( \beta < 1 \). This assures that \( U_0 \) does not diverge to infinity for a constant consumption plan in the budget set.
(Notice that for constant consumption, \(c\),

\[
U_0 = u(c) \sum_{t=0}^{\infty} \beta^t,
\]

which is infinite for \(\beta \geq 1\) and \(u(c) > 0\).)

The necessary conditions for an optimum with an infinite horizon are

\[
\frac{\partial L}{\partial c_t} = 0 \Rightarrow \beta^t u'(c_t) - \lambda_t = 0, \quad \text{for} \quad t \geq 0,
\]

\[
\frac{\partial L}{\partial a_{t+1}} = 0 \Rightarrow (1 + r_{t+1}) \lambda_{t+1} - \lambda_t = 0, \quad \text{for} \quad t \geq 0,
\]

\[
\frac{\partial L}{\partial \lambda_t} = 0 \Rightarrow (1 + r_t) a_t + w_t - c_t - a_{t+1} = 0, \quad \text{for} \quad t \geq 0,
\]

and

\[
\lim_{T \to \infty} \frac{\partial L}{\partial a_{T+1}} \leq 0, \quad \lim_{T \to \infty} R_T a_{T+1} \geq 0, \quad \lim_{T \to \infty} a_{T+1} \frac{\partial L}{\partial a_{T+1}} = 0 \Rightarrow -\lim_{T \to \infty} \lambda_T a_{T+1} = 0.
\]

Eliminating the multiplier \(\lambda\), these simplify to

\[
u'(c_t) = \beta (1 + r_{t+1}) u'(c_{t+1}),
\]

\[
a_{t+1} = (1 + r_t) a_t + w_t - c_t
\]

and

\[
\lim_{T \to \infty} \beta^T u'(c_T) a_{T+1} = 0.
\]

**B. Continuous-time optimization: finite horizon**

We begin by taking the limit of the objective function and resource identity as the time interval between decisions goes to zero. The optimal savings problem is continuous time is

\[
\max_{\{c_t, a_t\}} \int_0^T u(c_t) e^{-\rho t} dt
\]

subject to

\[
\dot{a} = r_t a_t + w_t - c_t
\]

and

\[
e^{-\rho T} a_T \geq 0,
\]

given initial financial wealth, \(a_0\). First, form the Lagrangian,

\[
L = \int_0^T u(c_t) e^{-\rho t} + \lambda_t (r_t a_t + w_t - c_t - \dot{a}) dt.
\]

$L$ is a function of $(c_t, a_t, \dot{a}_t, \lambda_t, t)$ where $c_t$, $a_t$, $\dot{a}_t$ and $\lambda_t$ are each functions of time for $t \in [0, T]$. We need to find the optimal choice of these functions. You need to notice that $a_t$ begins at the pre-determined $a_0$ and varies continuously over time to $T$. The initial choice of consumption, $c_0$, can be chosen, and $c_t$ will be restricted to be a piecewise-continuously differentiable function (a technicality that will be satisfied and implies that $a_t$ is continuous and piecewise-continuously differentiable).

The first approach is heuristic but straightforward. We just differentiate the Lagrangian pointwise with respect to consumption and financial wealth to derive necessary conditions for an optimum. Before doing this, notice that any choice of the function $a_t$ determines the function $\dot{a}_t$. Thus, these cannot be chosen independently of one another. Begin by integrating the last term of the Lagrangian, 

$$-\int_0^T \lambda_t \dot{a}_t dt$$

by parts as

$$-\int_0^T \lambda_t \dot{a}_t dt = -\int_0^T \frac{d}{dt} (\lambda_t a_t) - \dot{\lambda}_t a_t dt = -\left(\lambda_T a_T - \lambda_0 a_0\right) + \int_0^T \dot{\lambda}_t a_t dt.$$

Substituting back into the Lagrangian leads to

$$L = \int_0^T u \left(c_t\right) e^{-\rho t} + \lambda_t (r_t a_t + w_t - c_t) + \dot{\lambda}_t a_t dt - \left(\lambda_T a_T - \lambda_0 a_0\right).$$

(26)

Next, differentiate with respect to $c_t$ and $a_t$ remembering that $a_0$ is given. The first-order conditions are

$$u' \left(c_t\right) e^{-\rho t} - \lambda_t = 0,$$

$$\lambda_t r_t + \dot{\lambda}_t = 0,$$

$$\dot{a}_t = r_t a_t + w_t - c_t$$

and

$$-\lambda_T \leq 0 \quad e^{-rT} a_T \geq 0 \quad \text{and} \quad -\lambda_T a_T = 0.$$

Eliminating $\lambda$, these become

$$\frac{-u'' \left(c_t\right)}{u' \left(c_t\right)} \dot{c}_t = r_t - \rho,$$

$$\dot{a}_t = r_t a_t + w_t - c_t$$

and

$$e^{-rT} a_T \geq 0 \quad \text{and} \quad u' \left(c_T\right) e^{-\rho T} a_T = 0.$$

The general approach for deriving necessary conditions for continuous-time optimization problems is given in Section E below and is optional reading.
C. Continuous-time optimization: infinite horizon

The infinite-horizon optimal savings problem is just

$$\max_{\{c_t, a_t\}} \int_0^\infty u(c_t) e^{-\rho t} dt$$

(27)

subject to

$$\dot{a} = r_t a_t + w_t - c_t$$

(28)

and

$$\lim_{T \to \infty} e^{-\int_0^T r_s ds} a_T \geq 0,$$

(29)

given initial financial wealth, $a_0$. The associated Lagrangian is

$$L = \int_0^\infty u(c_t) e^{-\rho t} + \lambda_t (r_t a_t + w_t - c_t - \dot{a}) dt.$$  

(30)

To assure that a solution exists, we require that $\rho > 0$.

All of the necessary conditions are identical with the exception that we rewrite conditions (81) and (87) as limits, respectively:

$$\lim_{T \to \infty} a_T \lambda_T = 0 \quad \text{and} \quad \lim_{T \to \infty} e^{-\int_0^T r_s ds} a_T \geq 0$$

(31)

and

$$\lim_{T \to \infty} a_T u'(c_T) e^{-\rho T} = 0 \quad \text{and} \quad \lim_{T \to \infty} e^{-\int_0^T r_s ds} a_T \geq 0.$$  

(32)

The condition,

$$\lim_{T \to \infty} a_T \lambda_T = 0,$$

(33)

is called the transversality condition. It is important.

D. Hamilton’s version in continuous time

We can rewrite the Lagrangian and the first-order conditions for an optimum in a thriftier and common form for the case in which the equation of motion (the asset accumulation equation in our example) is additively separable in the time derivative of the state variable. In our example, we have

$$\dot{a} = r_t a_t + w_t - c_t,$$

(34)

and for the neo-classical growth model,

$$\dot{k} = f(k_t) - \eta k_t - c_t.$$  

(35)

These both satisfy the restriction as do almost all applications of optimal control theory in economics.
First, a review of terminology. State variables are the predetermined variables \((a \text{ and } k)\) in these examples, and control variables are the variables that can be chosen by the optimizing agent at each date \((c)\) in each example. The equation of motion tells us how the state variable changes as a function the control variables given the current state.

Now, for the convenient reformulation. The integrand of the Lagrangian is

\[
I = u\left(c_t\right) e^{-\rho t} + \lambda_t \left(r_t a_t + w_t - c_t - \dot{a}_t\right),
\]

and the necessary conditions for an optimum are

\[
\frac{\partial I}{\partial c} = 0, \tag{37}
\]

\[
\frac{\partial I}{\partial a} + \frac{d}{dt} \lambda_t = 0 \tag{38}
\]

\[
\frac{\partial I}{\partial \lambda} = 0 \tag{39}
\]

and

\[
\lim_{T \to \infty} -a_T \lambda_T = 0. \tag{40}
\]

Write the function,

\[
\mathcal{H} = u\left(c_t\right) e^{-\rho t} + \lambda_t \left(r_t a_t + w_t - c_t\right),
\]

by simply dropping off \(-\dot{a}\).

Substituting into the necessary conditions, (37), (38), (39) and (40), using the function \(H\) leads to

\[
\frac{\partial \mathcal{H}}{\partial c} = \frac{\partial I}{\partial c} = 0 \tag{42}
\]

\[
\frac{\partial \mathcal{H}}{\partial a} + \dot{\lambda}_t = \frac{\partial I}{\partial a} + \dot{\lambda}_t = 0 \tag{43}
\]

\[
\frac{\partial \mathcal{H}}{\partial \lambda} - \dot{a}_t = \frac{\partial I}{\partial \lambda} = 0 \tag{44}
\]

and

\[
\lim_{T \to \infty} -a_T \lambda_T = 0. \tag{45}
\]

So, if we use the function \(\mathcal{H}\), called the Hamiltonian, we use the necessary conditions, called Hamilton’s equations,

\[
\frac{\partial \mathcal{H}}{\partial c} = 0 \tag{46}
\]

\[
\frac{\partial \mathcal{H}}{\partial a} = -\lambda_t \tag{47}
\]

and

\[
\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{a}_t \tag{48}
\]
plus the transversality condition,
\[
\lim_{T \to \infty} -a_T \lambda_T = 0.
\] (49)

You should also note that the Lagrange multiplier, \( \lambda_t \), equals \( u'(c_t) e^{-\rho t} \) in the optimum. This is the present value benefit (increase in the objective function) of a one unit increase in the state variable at time \( t, a_t \). As in the static case, the Lagrange multiplier tells us the cost of the constraint in terms of the objective function. It gives a price of the state variable, \( a_t \), in terms of date 0 goods.

We can rewrite the Hamiltonian in terms of “current-time” marginal utility instead of “present-value” marginal utility. This is often very convenient for the economic interpretation of the solutions to express the multiplier as a current price rather than present-value price. We define
\[
q_t \equiv e^{\rho t} \lambda_t \quad \Rightarrow \quad \lambda_t = e^{-\rho t} q_t
\] (50)
so that
\[
\mathcal{H} = u(c_t) e^{-\rho t} + e^{-\rho t} q_t (r_t a_t + w_t - c_t).
\] (51)

We also define a new Hamiltonian, the current-time Hamiltonian, as
\[
H \equiv e^{\rho t} \mathcal{H} = u(c_t) + q_t (r_t a_t + w_t - c_t)
\] (52)

This changes the form of the necessary conditions to
\[
\frac{\partial H}{\partial c} = 0,
\] (53)
\[
\frac{\partial H}{\partial a} = -\lambda_t e^{\rho t} = \left( -e^{-\rho t} q_t + \rho e^{-\rho t} q_t \right) e^{\rho t} = -q_t + \rho q_t,
\] (54)
\[
\frac{\partial H}{\partial q} = \frac{\partial (e^{\rho t} \mathcal{H})}{\partial (e^{\rho t} \lambda)} = \dot{a}_t
\] (55)

and
\[
\lim_{T \to \infty} a_T e^{-\rho T} q_T = 0.
\] (56)

The multipliers, \( q \) and \( \lambda \), are called costate variables. These are prices of the corresponding state variables along the solution path. If we have more than one state variable in an optimization problem, we will have an equal number of equations of motion when our problem is linear in the state variables and an equal number of costate variables.

E. Background: A brief introduction to Calculus of Variations

A formal and general method for maximizing the Lagrangian is the calculus of variations. This is a method for maximizing the Lagrangian with respect to the functions, \( c_t, a_t \) and \( \lambda_t \) over a compact set in which each point is a triple of continuous functions over the interval \([0, T]\). The idea is that
an infinitesimal change in each function will not change $L$ in an arbitrarily small neighborhood of the optimum. To do this, let $a_t$ be the solution for financial wealth over time. Take any other continuous function, $h_t$, that satisfies the condition, $h_0 = 0$, and form the new function,

$$\tilde{a}_t = a_t + \eta h_t.$$  \hfill (57)

The time derivative of $\tilde{a}_t$ is

$$\frac{d}{dt} \tilde{a}_t = \dot{a}_t + \eta \dot{h}_t,$$  \hfill (58)

and

$$\tilde{a}_0 = a_0.$$  \hfill (59)

Similarly, we add functions to $c_t$ and $\lambda_t$ to form

$$\tilde{c}_t = c_t + \eta g_t$$  \hfill (60)

and

$$\tilde{\lambda}_t = \lambda_t + \eta \theta_t,$$  \hfill (61)

where we do not need to impose restrictions on $g_0$ or $\theta_0$.

Rewriting the Lagrangian allows a general derivation:

$$L = \int_{0}^{T} I (c_t, a_t, \dot{a}_t, \lambda_t, t) \, dt,$$  \hfill (62)

where $I (c_t, a_t, \dot{a}_t, \lambda_t, t) \equiv u (c_t) e^{-\rho t} + \lambda_t (r_t a_t + w_t - c_t - \dot{a}_t)$ for the standard savings problem.

Substituting our test functions, we have

$$L = \int_{0}^{T} I (c_t + \eta g_t, a_t + \eta h_t, \dot{a}_t + \eta \dot{h}_t, \lambda_t + \eta \theta_t, t) \, dt$$  \hfill (63)

which we differentiate with respect to $\eta$ and evaluate for $\eta = 0$.

Maximizing $L$ with respect to $\eta$ requires taking the derivative inside the integral:

$$\frac{dL}{d\eta} = \int_{0}^{T} \frac{d}{d\eta} I (c_t + \eta g_t, a_t + \eta h_t, \dot{a}_t + \eta \dot{h}_t, \lambda_t + \eta \theta_t, t) \, dt = 0.$$  \hfill (64)

Taking derivatives with respect to $\eta$, we get

$$\int_{0}^{T} \left( g \frac{\partial I}{\partial c} + h \frac{\partial I}{\partial a} + \dot{h} \frac{\partial I}{\partial \dot{a}} + \theta \frac{\partial I}{\partial \lambda} \right) \, dt = 0.$$  \hfill (65)

Since the functions, $g$, $h$ and $\theta$, are all chosen independently of each other, this first-order condition becomes

$$\int_{0}^{T} g \frac{\partial I}{\partial c} \, dt + \int_{0}^{T} \left( h \frac{\partial I}{\partial a} + \dot{h} \frac{\partial I}{\partial \dot{a}} \right) \, dt + \int_{0}^{T} \theta \frac{\partial I}{\partial \lambda} \, dt = 0.$$  \hfill (66)
Before proceeding further, we need to integrate the middle integral by parts. Note that
\[ \dot{h} \frac{\partial I}{\partial \dot{a}} = \frac{d}{dt} \left( h \frac{\partial I}{\partial a} \right) - h \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{a}} \right), \quad (67) \]
and substitute to get
\[ \int_0^T \left( h \frac{\partial I}{\partial a} + \dot{h} \frac{\partial I}{\partial \dot{a}} \right) dt = \int_0^T \left( h \frac{\partial I}{\partial a} + \frac{d}{dt} \left( h \frac{\partial I}{\partial \dot{a}} \right) - h \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{a}} \right) \right) dt, \quad (68) \]
which we then can partly integrate:
\[ \int_0^T \left( h \frac{\partial I}{\partial a} + \dot{h} \frac{\partial I}{\partial \dot{a}} \right) dt = \int_0^T \left( h \frac{\partial I}{\partial a} - h \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{a}} \right) \right) dt + \int_0^T \frac{d}{dt} \left( h \frac{\partial I}{\partial \dot{a}} \right) dt \quad (69) \]
Remember that \( h_0 = 0 \), so that \( \left( h \frac{\partial I}{\partial \dot{a}} \right)_0 = 0 \). Our derivative is now
\[ \int_0^T g \frac{\partial I}{\partial c} dt + \int_0^T \left( h \frac{\partial I}{\partial a} - h \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{a}} \right) \right) dt + \left( h \frac{\partial I}{\partial \dot{a}} \right)_T + \int_0^T \theta \frac{\partial I}{\partial \lambda} dt = 0. \quad (71) \]
Because this must hold for arbitrary choices of the functions, \( g, h \) and \( \theta \), the only way that we can assure that
\[ \int_0^T g \frac{\partial I}{\partial c} dt = 0, \quad (72) \]
for example, is if \( g \frac{\partial I}{\partial c} = 0 \) for all \( t \in [0, T] \). Our necessary conditions for an optimum are
\[ \int_0^T g \frac{\partial I}{\partial c} dt = 0 \Rightarrow g \frac{\partial I}{\partial c} = 0 \Rightarrow \frac{\partial I}{\partial c} = 0, \quad (73) \]
\[ \int_0^T h \frac{\partial I}{\partial a} - h \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{a}} \right) dt \Rightarrow h \frac{\partial I}{\partial a} - h \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{a}} \right) = 0 \Rightarrow \frac{\partial I}{\partial a} - \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{a}} \right) = 0, \quad (74) \]
\[ \int_0^T \theta \frac{\partial I}{\partial \lambda} dt = 0 \Rightarrow \theta \frac{\partial I}{\partial \lambda} = 0 \Rightarrow \frac{\partial I}{\partial \lambda} = 0 \quad (75) \]
and
\[ h_T \frac{\partial I}{\partial \dot{a}_T} = 0. \quad (76) \]
The second necessary condition, \( \frac{\partial I}{\partial \dot{a}} - \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{a}} \right) = 0 \), is known as the Euler-Lagrange condition.

Now, applying these necessary conditions to the optimal savings problem, we have that
\[ \frac{\partial I}{\partial c} = u' \left( c_t \right) e^{-\rho t} - \lambda_t = 0, \quad (77) \]
\[ \frac{\partial I}{\partial a} - \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{a}} \right) = \lambda_t r_t + \frac{d}{dt} \lambda_t = \lambda_t r_t + \dot{\lambda}_t = 0 \quad (78) \]
\[ \frac{\partial I}{\partial \lambda} = r_t a_t + w_t - c_t - \dot{a} = 0 \quad (79) \]
and
\[ h \frac{\partial I}{\partial \dot{a}_T} = -h_T \lambda_T = 0. \quad (80) \]
The constraint $e^{-\tau T} a_T \geq 0$ implies that $e^{-\tau T} (a_T + \eta h_T) \geq 0$ for all $\eta > \eta > 0$ for some $\eta > 0$ ($\eta$ defines the neighborhood of variations of $a_t$ around the optimum). Thus, the necessary condition requires that $-h_T \lambda_T = 0$ for all $h_T \geq \frac{a_T}{\tau}$. This last necessary condition for an optimum implies that

$$a_T \lambda_T = 0. \quad (81)$$

For our optimal savings problem, the necessary conditions for an optimum are

$$u'(c_t) e^{-\rho t} = \lambda_t, \quad (82)$$

$$\begin{align*}
\lambda_t r_t + \frac{d}{dt} \lambda_t &= r_t u'(c_t) e^{-\rho t} + \frac{d}{dt} [u'(c_t) e^{-\rho t}] = 0 \\
&= e^{-\rho t} [r_t u'(c_t) + \dot{c}_t u''(c_t) - \rho u'(c_t)] = 0 \\
&\Rightarrow \dot{c}_t \left(\frac{-u''(c_t)}{u'(c_t)}\right) = r_t - \rho, \\
\dot{a} &= r_t a_t + \omega_t - c_t \quad (85)
\end{align*}$$

and

$$a_T u'(c_T) e^{-\rho T} = 0 \quad \text{and} \quad a_T \geq 0. \quad (87)$$