Present value calculations for a long-term bond

Start with a nominal bond paying a coupon of $z$ for every period starting next year (in year $t+1$) (this is a consol):

$$V_t = \frac{z}{1+i} + \frac{z}{(1+i)(1+i)} + \frac{z}{(1+i)(1+i)(1+i)} + \ldots$$

Now, let the interest rate be constant:

$$V_t = \frac{z}{1+i} + \frac{z}{(1+i)^2} + \frac{z}{(1+i)^3} + \ldots$$

A really useful way to look at the price is to notice that if you hold the bond for one period, receive the coupon $z$ and then sell it, the price at time $t+1$ equals

$$V_{t+1} = \frac{z}{1+i} + \frac{z}{(1+i)^2} + \frac{z}{(1+i)^3} + \ldots$$

which is exactly the same as $V_t$. This means you can write the value of the bond at time $t$ as $V$

$$V = \frac{z + V}{1+i}$$

(The economics helps you understand this equation – you hold the bond for one period and then sell it.) A little algebra,

$$V - \frac{1}{1+i}V = V\left(1 - \frac{1}{1+i}\right) = V\left(\frac{i}{1+i}\right) = \frac{z}{1+i},$$

leads to

$$V = \frac{z}{i}.$$  

To price a bond that pays $z$ for ten periods, we can subtract all the payments made after the first ten from the present value calculation. Start with the price of the ten-year bond:

$$V_{10} = \frac{z}{1+i} + \frac{z}{(1+i)^2} + \frac{z}{(1+i)^3} + \ldots + \frac{z}{(1+i)^{10}},$$

I added the superscript so you will not confuse this with the price of the consol, $V$. Now, this is
\[ SV_{10}^{i} = \frac{S_z}{1+i} + \frac{S_z}{(1+i)^2} + \frac{S_z}{(1+i)^3} + \cdots + \frac{S_z}{(1+i)^{10}} = SV - \left( \frac{1}{1+i} \right)^{10} \left( \frac{S_z}{1+i} + \frac{S_z}{(1+i)^2} + \frac{S_z}{(1+i)^3} + \cdots \right) \]

That is, I subtract off all the payments from year 11 on. Since

\[ SV = \left( \frac{S_z}{1+i} + \frac{S_z}{(1+i)^2} + \frac{S_z}{(1+i)^3} + \cdots \right) \]

our new equation for the price of the ten-year bond is

\[ SV_{10}^{i} = \frac{S_z}{1+i} + \frac{S_z}{(1+i)^2} + \frac{S_z}{(1+i)^3} + \cdots + \frac{S_z}{(1+i)^{10}} = SV - \left( \frac{1}{1+i} \right)^{10} \]

\[ SV_{10}^{i} = SV \left( 1 - \left( \frac{1}{1+i} \right)^{10} \right) \]

\[ SV_{10}^{i} = \frac{S_z}{i} \left( 1 - \left( \frac{1}{1+i} \right)^{10} \right) \]