THE BOUNDARY RIGIDITY OF LATTICES IN PRODUCTS OF TREES

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Abstract. We show that every group acting properly and cocompactly by isometries on a product of n bounded valance, bushy trees is boundary rigid. That means that every CAT(0) space that admits a geometric action of any such group has the visual boundary homeomorphic to a join of n copies of the Cantor set.

1. Introduction

A visual boundary is a particular type of compactification of a proper CAT(0) metric space. The boundary is defined as a set of equivalence classes of asymptotic rays endowed with an appropriate topology. For hyperbolic spaces X and Y, any quasi-isometry $X \to Y$ between them extends to a homeomorphism of their visual boundaries. Consequently, the homeomorphism type of the boundary of a hyperbolic group is a well-defined group invariant. This is not true for CAT(0) groups, i.e. groups that act geometrically on CAT(0) spaces.

Bowers-Ruane give an example of a group G acting geometrically on CAT(0) spaces X and Y, such that the associated G-equivariant quasi-isometry between the spaces does not extend to a homoemorphism between their visual boundaries [BR96]. Croke-Kleiner provided an example of a CAT(0) group G and two CAT(0) spaces X, Y, both admitting geometric actions by G such that $\partial_\infty X$ and $\partial_\infty Y$ are non-homeomorphic [CK00]. Wilson further showed that in fact this same G acts geometrically on uncountably many spaces with boundaries of distinct topological type [Wil05]. The group G in the Croke-Kleiner construction is the right-angled Artin group with the defining graph a path on four vertices.

A CAT(0) group G is called *boundary rigid* if the visual boundaries of all CAT(0) spaces admitting a geometric action by G are homeomorphic. As noted above, hyperbolic CAT(0) groups are boundary rigid while not all CAT(0) groups are boundary rigid. Ruane proved that the direct product of hyperbolic groups is boundary rigid [Rua99]. Hosaka extended that to show that any direct product of boundary rigid groups is boundary rigid [Hos03]. Hruska-Kleiner proved that groups acting geometrically on CAT(0) spaces with the isolated flats property are boundary rigid [HK05].

A *bushy tree* is a simplicial tree that is not quasi-isometric to a point or a line. In this note, we study a family of CAT(0) groups acting geometrically on the product of n bushy trees. We will assume the groups preserve the factors, which is always the case after passing to a finite-index subgroup. We refer to such groups as *lattices in a product of trees*. We skip the word *cocompact* even though we are assuming this property throughout the paper.

The simplest example of a lattice in a product of trees is a direct product $F_n \times F_m$ of two finite rank free groups. These are boundary rigid by [Rua99]. However, there exist lattices in product of trees that are *irreducible*, i.e. they do not split as direct products, even after passing to a finite index subgroups. Irreducible lattices in products of trees were first studies by Mozes [Moz92], Burger-Mozes [BM97], [BM00] and by [Wis96]. Burger-Mozes constructed examples of simple lattices in a product of trees, providing the first examples of simple CAT(0) groups, as well as the first examples of simple amalgamated products of free groups.

In this paper, we prove the following.

Theorem 1.1. Let G be a lattice in a product of n trees. Suppose G acts geometrically on a CAT(0) space X. Then $\partial_\infty X$ is the join of n Cantor sets.

Corollary 1.2. Every lattice in a product of n trees is boundary rigid.

Upon the revision of the paper, we learned that Margolis proved a much stronger result than Theorem 1.1 [Mar22, Thm K]. He proves boundary rigidity for finitely generated groups quasi-isometric to a product of finitely many proper cocompact non-elementary hyperbolic spaces.

There are two steps in our proof of Theorem 1.1. We first deduce from the rank-rigidity results of Monod [Mon06] and Caprace–Monod [CM09] that we may assume without loss of generality that X splits as a product of n convex subspaces. This will imply that $\partial_\infty X$ splits as a join of n 0-dimensional topological spaces. In the special case when X is a CAT(0) cube complex and X is either geodesically complete or the action of G on X is essential, we can use rank rigidity for CAT(0) cube complexes, due to Caprace-Sageev [CS11], to obtain a necessary splitting. We mention this case separately to illustrate the usefulness of the Caprace-Sageev Rank Rigidity Theorem in the cube complex setting.

The second step is to prove that the n 0-dimensional subspaces obtained after the first step are indeed homeomorphic to Cantor spaces.

The paper is organized as follows. In section 2 we give the background on lattices in products of trees as well as ends and boundaries of CAT(0) spaces. We prove the main theorem in section 4.

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2. CAT(0) spaces at infinity

2.1. **Ends of a space.** We recall the definitions and relevant facts about the space of ends of a topological space. For more details, see [BH99].

Let X be a topological space. A *ray* in X is a proper map $r : [0, \infty) \to X$. A *ray at* x_0 where x_0 is a point of X, is a ray with $r(0) = x_0$. An *end* e of X is an equivalence class of rays in X where $r_1 \simeq r_2$ if and only if for every compact set $K \subseteq X$ there exists $N \geq 0$ such that $r_1([N, \infty])$ and $r_2([N, \infty])$ are contained in the same connected component of $X - K$. We denote the equivalence class of the ray r by $e(r)$. The set of all ends of X is denoted by $Ends(X)$.

Let U be an open set in X and $e \in Ends(X)$. We use the notation $e < U$ to mean that for any $r : [0, \infty) \to X$ with $e(r) = e$, there exists $N \geq 0$ such that $r([N, \infty)) \subseteq U$.

The set $X \cup Ends(X)$, denoted by $\widehat{X}$, can be endowed with topology that is generated by the basis consisting of the following sets:

- open sets in X,
- sets of the form $U \cup \{e \in Ends(X) \mid e < U\}$ where U is a connected component of $X - K$ for some compact set $K \subseteq X$.

The space $\widehat{X}$ is compact and is called the *end compactification* of X.

2.2. **Visual Boundary.** Assume that X is a metric space with metric d. Two geodesic rays r, r' are *asymptotic*, if there exists a constant $K > 0$ such that $d(r(t), r'(t)) < K$ for all $t \in [0, \infty)$.

The *boundary* of X, denoted ∂X, is the set of equivalence classes of geodesic rays, where two rays are equivalent if they are asymptotic. We denote the equivalence class of a ray r by $r(\infty)$.

When X is a complete CAT(0) space, we can put a topology on ∂X as follows. First fix a basepoint $x_0 \in X$. The *cone topology* on ∂X with respect to x_0 is given by the neighborhood basis $\{U(r, R, \epsilon) : r(\infty) \in \partial X,\ R, \epsilon > 0\}$ where

$$U(r, R, \epsilon) = \{r'(\infty) \in \partial X : r'(0) = x_0, d(r(R), r'(R)) < \epsilon\}.$$

This topology seems to depend on the choice of basepoint $x_0 \in X$ but in fact it does not. There is a well-defined change of basepoint homeomorphism between the topologies determined by different basepoints. This follows from the fact:

Proposition 2.1. [BH99, Prop. II.8.2] If r is a geodesic ray based at x in a complete CAT(0) space X and x' is a point not on this ray, then there exists a unique geodesic ray r' with $r'(0) = x'$ that is asymptotic to r.

The boundary ∂X endowed with the cone topology is called the *visual boundary* of X and we denote it by $\partial_\infty X$. See [BH99, Chap II.8] for more details and properties of the visual boundary.

If X is a proper CAT(0) space, then $\partial_\infty X$ is compact and there is a natural well-defined map $\partial_\infty X \to Ends(Y)$ sending a ray r to $e(r)$. This map does not depend on the choice of a ray in the equivalence class of asymptotic rays. The map is a continuous surjection. [BH99, Rem II.8.10].

One important theorem we will use about the visual boundary is the following theorem of Geoghegan-Ontaneda.

Theorem 2.2 ([GO07])**.** The topological dimension of $\partial_\infty X$ is a quasi-isometry invariant. In particular, if a group G acts geometrically on CAT(0) spaces X and X', then their visual boundaries have the same topological dimension.

We will also use the following Lemma due to Papasoglu-Swenson.

Lemma 2.3 ([PS09, Lem 26])**.** Let G be a group acting geometrically on a CAT(0) space X. Then G virtually stabilizes a finite subset A of the visual boundary of X if and only if G virtually has $\mathbb{Z}$ as a direct factor.

Proof. The contents of [PS09, Lem 26]] proves that if G virtually stabilizes a finite subset of $\partial_\infty X$, then G virtually has $\mathbb{Z}$ as a direct factor. We prove the other direction here.

Suppose G virtually has $\mathbb{Z}$ as direct factor. Thus there is a finite index subgroup of G of the form $H \times \mathbb{Z}$. Let g be the generator of the virtual $\mathbb{Z}$ factor. Notice that $C_G(g)$, the centralizer of g in G, contains this subgroup and is therefore also finite index in G. Then for each $x \in G$, x^n is in $C_G(g)$. Every element in $C_G(g)$ fixes $\{g^{\pm\infty}\}$, the endpoints of g in $\partial_\infty X$. Thus every element of the finite index subgroup $H \times \mathbb{Z}$ fixes this finite set in the boundary as needed.

□

We do not want to assume X is geodesically complete for our theorem so we use work of Caprace–Monod in [CM09] to get around this issue. We explain how to do this here.

An action of a group G on a CAT(0) space X is *minimal* if there does not exists a non-empty G-invariant closed convex subset of X. A CAT(0) space X is *minimal* if the action of its full isometry group is minimal.

A subset $X' \subseteq X$ is *quasi-dense* in X if there exists a $D > 0$ such that each point of X is within distance D of X'. In particular, $\partial X = \partial X'$ as sets. Thus, for purposes of studying the boundary, we can always pass to a quasi-dense subset without losing any information. The following theorem will be used to avoid the extra assumption of geodesic completeness.

Theorem 2.4 ([CM09], see also [Cap14, Ex II.4 and Prop III.10])**.** Let G be a group acting geometrically on a CAT(0) space X. Then G stabilizes a closed, convex, quasi-dense subspace $X' \subseteq X$ such that G acts minimally on X'. In particular, $\partial X = \partial X'$ as sets.

Moreover, $\partial_T X$ splits as a join if and only if X' splits as a product $X_1 \times X_2$.

The subspace X' in the above theorem is not necessarily unique. In fact, every closed, convex, quasi-dense subspace of X on which G acts minimally admits a splitting as a product, provided that $\partial_T X$ splits as a join.

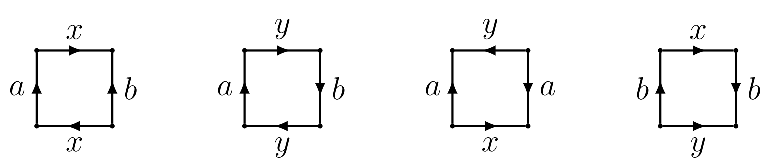
3. Lattices in product of trees

In this section, we summarize facts about lattices in products of trees used in the proof of Theorem 1.1. A simplicial tree T is *bushy*, if it is not quasi-isometric to a point or a line. For example, a regular tree of valence ≥ 3 is bushy.

Standing Assumption. Throughout the paper we assume that all the trees we consider are bushy trees of bounded valence.

We view T as a metric space, with the path metric, where each edge has length 1. The automorphism group $\mathrm{Aut}(T)$ is a group of all isometries $T \to T$, i.e. permutations of the vertex sets that preserve the adjacency. The group $\mathrm{Aut}(T)$ endowed with a compact-open topology, is a locally compact group. We now consider n trees $T_1, \dots, T_n$. The product $T_1 \times \cdots \times T_n$ has a natural structure of a cube complex. It is easy to verify that each vertex link is a complete n-partite graph and, in particular, it is a flag simplicial complex. Thus $T_1 \times \cdots \times T_n$, with the path metric induced by the Euclidean metric on each cube, is a CAT(0) cube complex.

Example 3.1. Let G be a group on four generators a, b, x, y with four relations:



The group G is an irreducible lattice in the product of two copies of a 4-valent tree [JW09].

The presentation complex of G in the example above, and for all torsion-free lattices in product of two trees more generally, is a non-positively curved square complex where each vertex link is a complete bipartite graph. Such square complexes are referred to as *complete square complexes*. We refer the reader to [Wis07] for the theory of complete square complexes. Our approach in the n-factor case is inspired by Wise's work in the two factor case.

Let G be a lattice in $\mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_n)$, and let Z be the orbispace obtained as the quotient of $T_1 \times \cdots \times T_n$ by the action of G. Let $q : T_1 \times \cdots \times T_n \to Z$ denotes the quotient map, which is an orbispace covering map. Let $v \in T_1 \times \cdots \times T_n$ be a basepoint. We identify G with the orbispace fundamental group $\pi_1(Z, \overline{v})$ where $\overline{v} = q(v)$.

All the edges of the product $T_1 \times \cdots \times T_n$ can be labelled by $1, \ldots, n$ indicating in which factor the edge lies. Since the action of G on $T_1 \times \cdots \times T_n$ preserves the factors, the underlying space of Z is a n-dimensional cube complex whose edges can also be labelled by $1, \ldots, n$ according to the label on the lift of the edge. Consider the subspace of Z consisting of all the edges with label i, and denote its connected component containing $\overline{v}$ by Z_i. Note that Z_i is a 1-dimensional cube complex, i.e. Z_i is a graph.

For each $i = 1$, let $pr_i : T_1 \times \cdots \times T_n \to T_1 \times \cdots \times \widehat{T_i} \times \cdots \times T_n$ be the projection onto $n-1$ factors where $\hat{}$ denotes an omission of a factor. Similarly, let $\rho_i : G \to \mathrm{Aut}(T_1) \times \cdots \times \widehat{\mathrm{Aut}}(T_i) \times \cdots \times \mathrm{Aut}(T_n)$ denote the projected action of G on $T_1 \times \cdots \times \widehat{T_i} \times \cdots \times T_n$. For each $i = 1, \ldots, n$ we define a subgroup $G_i \subseteq G$ as the stabilizer of $pr_i(v)$ with respect to the action ρ_i, i.e.

$$(*) \qquad G_i = Stab_{T_1 \times \cdots \times \widehat{T_i} \times \cdots \times T_n}(pr_i(v)).$$

In particular, G_i fixes setwise a copy of T_i in the original action of G on $T_1 \times \cdots \times T_n$. The coordinates other than the i-th one of this copy of T_i are the coordinates of the basepoint v of $T_1 \times \cdots \times T_n$. We abuse notation to denote that copy of T_i by $T_i \times \{pr_i(v)\}$.

We will prove that the action of G_i on $T_i \times \{pr_i(v)\}$ is geometric. We first need the following lemma.

Lemma 3.2. Let T be an unbounded simplicial tree and let H be a group acting geometrically on T. Then H contains an infinite order element.

Proof. Suppose that H contains no infinite order elements. Every finite order isometry of a tree has a fixed point. There cannot exist two elements of H with disjoint sets of fixed points, as their product would have infinite order. Thus the group H has a global fixed point. By the properness of the action, H must be finite, but this is impossible since T is unbounded and the quotient of T by the action of H is finite. □

Lemma 3.3. For each $i = 1, \ldots, n$, the group G_i acts geometrically on $T_i \times \{pr_i(v)\}$, and contains an infinite order element.

Proof. We will prove the statement for $i = 1$. The proof is analgous for every $i = 1, \ldots, n$.

The properness of the action of G_i on $T_i \times \{pr_i(v)\}$ follows from [BH99, Ch. II, Thm 6.10]. In order to show that the action is cocompact, we first consider the restriction $T_1 \times \{pr_1(v)\} \to Z_1$ of the quotient map q, which is also an orbispace covering map. Note that Z_1 is a finite graph, as a subspace of a finite cube complex, and $T_1 \times \{pr_1(v)\}$ is the universal cover of Z_1. In particular, the action of $\pi_1(Z_1, \overline{v})$ on $T_1 \times \{pr_1(v)\}$ is cocompact. Moreover, by Lemma 3.2, the orbispace fundamental group $\pi_1(Z_1, \overline{v})$ has an infinite order element.

Let us now show that $\pi_1(Z_1, \overline{v})$ embeds as a subgroup of G_1. First we check that the map between the orbispace fundamental groups $\pi_1(Z_1, \overline{v}) \to \pi_1(X, \overline{v})$ induced by the inclusion $Z_1 \subseteq X$ is injective. Take $g \in \pi_1(Z_1, \overline{v})$, and denote its image in $\pi_1(X, \overline{v})$ by $\hat{g}$. We can view both $\hat{g}$ and g as Deck transformation of $T_1 \times \cdots \times T_n$ and $T_1 \times \{pr_1(v)\}$ respectively. We have $g = p \circ \hat{g} \circ \iota$, where ι is the inclusion $\iota : T_1 \times \{pr_1(v)\} \hookrightarrow T_1 \times \cdots \times T_n$, and $p : T_1 \times \cdots \times T_n \to T_1 \times \{pr_1(v)\}$ is the projection onto the basepoint in the last $n-1$ trees. In particular, since g is nontrivial, we deduce that $\hat{g}$ is nontrivial. This also shows that $\hat{g}$ maps v to $g(v)$ whose last $n-1$ coordinates are $pr_1(v)$. Thus $\rho_1(\hat{g})$ belongs to the stabilizer of $pr_1(v)$, and so $\hat{g} \in G_1$. Since the action of $\pi_1(Z_1, \overline{v})$ on $T_1 \times \{pr_1(v)\}$ is cocompact, and $\pi_1(Z_1, \overline{v}) \subseteq G_1$, we conclude that the action of G_1 on $T_1 \times \{pr_1(v)\}$ is cocompact. □

Lemma 3.4. A lattice G in $\mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_n)$ has no infinite order central elements. In particular, G does not have a direct $\mathbb{Z}$ factor.

Proof. The second part follows directly from the first part, as if G has a direct $\mathbb{Z}$ factors, then it contains an infinite order central element.

Let g be an infinite order central element. Then the points at infinity $g^{\pm\infty}$ are fixed by G, as G centralizes $\langle g \rangle$. Since G preserves the factors of $T_1 \times \cdots \times T_n$, the existence of a global fixed point in $\partial_\infty T_1 * \cdots * \partial_\infty T_n$ of the action of G implies that G fixes a point in at least one of $\partial_\infty T_i$. By Lemma 3.3, the group G_i has no fixed point in $\partial_\infty T_i$ since the action of G_i on T_i is geometric. In particular, G has no fixed point in $\partial_\infty T_i$. This completes the proof. □

4. Proof of Theorem 1.1

In order to prove the main theorem, we first conclude from Monod [Mon06] and Caprace–Monod [CM09] that X splits as a metric product. Afterwards, we show that the boundary of each factor of the metric product is a Cantor set.

4.1. **Splitting.** Let G be a lattice in a product of n trees acting geometrically on a CAT(0) space X. Recall that we may assume that G acts minimally on X by Theorem 2.4. We conclude the following theorem from the rank-rigidity results of Monod and Caprace–Monod:

Theorem 4.1 ([Mon06, CM09])**.** Let $G < \mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_n)$ be an irreducible lattice in a product of n bushy trees $T_1, \ldots, T_n$. Suppose G acts geometrically and minimally by on a CAT(0) space X. Then X splits as a metric product of n unbounded factors $X_1 \times \cdots \times X_n$, and the action of G preserves the decomposition of X.

Proof. By Theorem 2.2, X has finite-dimensional boundary. By Lemma 3.4 G does not virtually have $\mathbb{Z}$ as a direct factor. In particular, X does not have a Euclidean factor by [CM19, Thm 2(v)]. By [PS09, Lem 26] G acts without a fixed point at infinity. Thus, the rank-rigidity theorem [CM09, Thm 8.4] implies that the G-action of X extends to a continuous $\mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_n)$-action by isometries. By [Mon06, Thm 9] X splits $\mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_n)$-equivalently isometrically as a direct product $X_1 \times \cdots \times X_n$ of $\mathrm{Aut}(T_i)$-spaces X_i.

It remains to show that each factor X_i is unbounded. Let g_i be an infinite order element of G_i, provided by Lemma 3.3. Since the action of G on X is geometric, each element of G acts on X as a semi-simple isometry, and in particular, g_i acts as a hyperbolic isometry. By [BH99, Prop 6.9] g_i acts as a semi-simple isometry of each factor $X_1, \ldots, X_n$. As an element of G_i, g_i acts elliptically on each of the factors $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n$, so g_i must act as a hyperbolic isometry on X_i. Thus X_i contains an axis of g_i, and in particular, is unbounded. □

4.2. **Analyzing the join factors.** Let G be a lattice in a product of n bushy trees. In this subsection, we complete the proof of the main theorem by showing the following proposition.

Proposition 4.2. Let G be a lattice in a product of trees . Suppose G acts geometrically on a CAT(0) space X where $\partial_\infty X = \partial_\infty X_1 * \cdots * \partial_\infty X_n$ for unbounded, closed, convex subspaces X_1,..., X_n in X. Then each $\partial_\infty X_1$, ...,$\partial_\infty X_n$ is homeomorphic to the Cantor set and each subspace X_i is quasi-isometric to a finite valence bushy tree.

The subtlety involved here is that when the group G is *not* (virtually) a product of free groups, we do not have geometric group actions on X_i. Indeed, if G is an irreducible lattice, then the projected action of G on X_i is cocompact but not proper.

Lemma 4.3. Suppose G acts geometrically on a CAT(0) space X so that $\partial_\infty X = \partial_\infty X_1 * \cdots * \partial_\infty X_n$ for closed convex subspaces $X_1, \dots, X_n$ in X. If the topological dimension of $\partial X_1, \dots, \partial X_n$ are zero, then the boundary $\partial_\infty X_i$ is homeomorphic to $\mathrm{Ends}(X_i)$, the ends space of X_i.

Proof. Let us first show that each X_i is hyperbolic. Note that since X is proper, so are $X_i, \dots, X_n$.

First, we show that the subspace X_i for $i = \{1, \dots, n\}$ is a *visibility space*, i.e. given any two distinct points ξ, η in $\partial_\infty X_i$, there is a geodesic line in X_i between them. Suppose there is no geodesic line in X_i between ξ and η. Then by [BH99, Prop. II.9.21(2)], there is a geodesic segment in $\partial_T X_i$ joining them. This segment is an arc in $\partial_T X_i$ which would map to an arc in $\partial_\infty X_i$ via the identity map on ∂X. This contradicts the fact that $\partial_\infty X_i$ is 0-dimensional.

Moreover, the action of G on X_i is cocompact, since so is the action of G on X. Hence, X_i is uniformly visible by [BH99, Prop II.9.32]. Finally, X_i is hyperbolic by [BH99, Prop III.1.4].

Since X_i is a proper hyperbolic space, the natural map $\partial_\infty X_i \to \mathrm{Ends}(X_i)$ is continuous and the fibers of that map are the connected components of $\partial_\infty X_i$ [BH99, Exer III.H.3.9]. Since $\partial_\infty X_i$ is 0-dimensional, the connected components are single points. Thus the map $\partial_\infty X_i \to \mathrm{Ends}(X_i)$ is a continuous bijection. Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism. □

In [Bes96, Def 1.1], Bestvina outlined a set of axioms that a group boundary should have in order to be useful for relating homological invariants of the boundary to cohomological invariants of the group. All of the axioms hold true for a hyperbolic group G acting on $G \cup \partial_\infty G$ and for a CAT(0) group G acting on $X \cup \partial_\infty X$ where G admits a geometric action on the CAT(0) space X. One of the axioms requires the collection of translates of any compact set to form a null set in $X \cup \partial_\infty X$, i.e. for any open cover $\mathcal{U}$ of $X \cup \partial_\infty X$ and any compact set K in X, all but finitely many G-translates of K are contained in an element of $\mathcal{U}$.

The next lemma shows that each $X_i \cup \partial_\infty X_i$ inherits this nullity condition on compact sets from $X \cup \partial_\infty X$ even though we have no geometric group action on X_i.

Lemma 4.4. Suppose G acts geometrically on a CAT(0) space X with $\partial_\infty X = \partial_\infty X_1 * \cdots * \partial_\infty X_n$ for closed convex subspaces $X_1, \dots, X_n$ in X. For every compact set $K \subseteq X_i$ and every open neighborhood U of an end of X_i, there exists $g \in G$ such that $gK \subseteq U$.

Proof. We show that the lemma holds for X_1. The argument for X_i, $i \neq 1$ is identical. Let $K \subseteq X_1$ and $K' \subseteq X_2 \times \cdots \times X_n$ be compact sets. Then $K \times K'$ is a compact set in X. By Lemma 4.3, $\partial_\infty X_1$ is homeomorphic to $\mathrm{Ends}(X_1)$. Let $\xi \in \partial_\infty X_1$, and let U be an open neighborhood of ξ. Then $U \times X_2 \times \cdots \times X_n$ is an open neighborhood of ξ, viewed as a point in $\partial_\infty X = \partial_\infty X_1 * \partial_\infty X_2 * \cdots * \partial_\infty X_n$. Since the action of G on X is geometric, there exists $g \in G$ such that $g(K \times K') \subseteq U \times X_2 \times \dots, \times X_n$ (see e.g. [Bes96, p.124]). It follows that in the action of G on X_1, we have $gK \subseteq U$. □

The following is not stated explicitly, but it is proved in [Hop44]. It is also proved in a similar form in [Ber05]. We include the proof for completeness. We restrict our attention to the case of geodesic metric spaces, but the proposition holds in more general setting.

Proposition 4.5 ([Hop44])**.** Let Y be a geodesic metric space with at least three ends. Let G be a group acting on Y cocompactly so that the following holds: for every compact set $K \subseteq Y$, and every open neighborhood U of an end of Y, there exists $g \in G$ such that $gK \subseteq U$. Then the space of ends $\mathrm{Ends}(Y)$ is perfect.

Proof. Suppose that there exists an end $e \in \mathrm{Ends}(Y)$ that is isolated, i.e. there exists a neighborhood U of e that does not contain any other ends.

First we show that without loss of generality, we can assume that $Y - U$ is connected. Indeed, if $Y - U$ is not connected, we construct a neighborhood $U' \subseteq U$ of e such that $Y - U'$ is connected. Since $\widehat{Y}$ is compact, so is $\widehat{Y} - U$. Let $V_1, \dots, V_n$ be a finite collection of open sets covering $\widehat{Y} - U$ such that each V_i is connected and does not contain e, and its closure $\widehat{V_i}$ in $\widehat{Y}$ is compact. The union $\bigcup_{i=1^n} V_i$ has finitely many components, and since $\mathrm{Ends}(Y)$ is nowhere dense in $\widehat{Y}$, each V_i contains points of Y. Each two points in Y can be joined by a path in Y. In particular, there exists a closed connected set Q which is the union of $\bigcup_{i=1^n} V_i$ and a finite number of paths in Y. Note that Q does not contain e. The set $U' = \widehat{Y} - Q$ is an open neighborhood of e and since $Y - U \subseteq \bigcup V_i \subseteq Q$, we have $U' \subseteq U$. Thus, U' is the neighborhood we were looking for.

By assumption, there are at least three distinct ends e_1, e_2, e_3 in Y. Let K be a compact set in Y such that each of the ends e_1, e_2, e_3 lies in a different connected component Y_1, Y_2, Y_3 of $Y - K$. By assumption, there exists an element $g \in G$ such that $gK \subseteq U$. We claim that for each $i = 1, 2, 3$, either $gY_i \subseteq U$, or $g(Y - Y_i) \subseteq U$. Indeed, otherwise there exists point $p \in gY_i - U$ and $q \in g(Y - Y_i) - U$. Since $Y - U$ is connected, there exists a path γ in $Y - U$ joining p and q. Note that p, q lie in distinct connected component of $Y - gK$, so γ has to pass through gK. This is a contradiction, since $gK \subseteq U$.

Since U contains only one end, we have $g(Y - Y_i) \subseteq U$ for at least two i's among $1, 2, 3$, say 1 and 2. It follows that $Y - U \subseteq gY_i$ for $i = 1, 2$. The subsets Y_1 and Y_2 are disjoint, and so are gY_1 and gY_2. This is a contradiction and therefore the space of $\mathrm{Ends}(Y)$ has no isolated points. □

Lemma 4.6. Suppose G acts geometrically on a CAT(0) space X so that $\partial_\infty X = \partial_\infty X_1 * \cdots * \partial_\infty X_n$ for closed convex subspaces $X_1, \dots, X_n$ in X. Suppose that the topological dimensions of $\partial X_1, \ldots, \partial X_n$ are zero. If $|\partial X_i| \geq 3$, then ∂X_i is homeomorphic to the Cantor set $\mathcal{C}$.

Proof. Since X_i is a proper CAT(0) space, the boundary ∂X_i is compact and metrizable [BH99]. As $\dim \partial_\infty X_i = 0$, the boundary $\partial_\infty X_i$ is totally disconnected. The action of G on X_i is cocompact, since so is the action of G on X. By Lemma 4.4, for every compact set $K \subseteq X_i$ and every open neighborhood $U \subseteq X_i$ of an end of X_i, there exists $g \in G$ such that $gK \subseteq U$. By Proposition 4.5, $\mathrm{Ends}(X_i)$ is a perfect space. This implies that $\partial_\infty X_i$ is a perfect space because $\mathrm{Ends}(X_i) = \partial_\infty X_i$ by Lemma 4.3. By the characterization of the Cantor set, as a non-empty, perfect, totally disconnected, compact metrizable space, we conclude that ∂X_i is the Cantor set. ☐

Proof of Propostion 4.2. By Theorem 2.2, the topological dimension of $\partial_\infty X$ coincides with the topological dimension of $\partial_\infty T_1 * \cdots * \partial_\infty T_n = n-1$, so each $\partial_\infty X_i$ is 0-dimensional.

By Theorem 4.1 the action of G on $X = X_1 \times \cdots \times X_n$ preserves the factors, and in particular, G fixes each $\partial_\infty X_i$ setwise. If for some $i = 1, \ldots, n$ the set $\partial_\infty X_i$ was finite, then G would have a $\mathbb{Z}$ factor, by Lemma 2.2. Since this is not the case for G by Lemma 3.4, we deduce that $\partial_\infty X_i$ is infinite for each $i = 1, \ldots, n$. By Lemma 4.6 each $\partial_\infty X_i$ is homeomorphic to a Cantor set. Consequently, $\partial_\infty X$ is homeomorphic to join of n copies of a Cantor set. We now show that X_i is quasi-isometric to a tree, following [Cha14]. By [Cha14, Lem 5.7], X_i has the bottleneck property, i.e. there is some $\Delta > 0$ so that for all $x, y \in X_i$ there is a midpoint $m = m(x, y)$ with $d(x, m) = d(y, m) = \frac{1}{2} d(x, y)$ and the property that any path from x to y must pass within less than Δ of the point m. By [Man05, Thm 4.6], this property is equivalent to X_i being quasi-isometric to a simplicial tree T_i'. Since X is a proper space, so is each X_i and thus T_i' is finite valence. The boundary ∂X_i is homeomorphic to a Cantor set and thus T_i' is bushy. ☐

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