THE BOUNDARY RIGIDITY OF LATTICES IN PRODUCTS OF TREES

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Abstract. We show that every group acting freely and cocompactly by isometries on a product of two bushy trees is boundary rigid. That means that every CAT(0) space that admits a geometric action of any such group has the visual boundary homeomorphic to a join of two copies of the Cantor set.

1. Introduction

A visual boundary is a particular type of compactification of a proper geodesic metric space. The boundary is defined as a set of equivalence classes of asymptotic rays endowed with an appropriate topology. For hyperbolic spaces $X$ and $Y$, any quasi-isometry $X \to Y$ between them extends to a homeomorphism of their visual boundaries. Consequently, the homeomorphism type of the boundary of a hyperbolic group is a well-defined group invariant. This is not true for CAT(0) groups, i.e. groups that act geometrically on CAT(0) spaces. Bowers-Ruane give an example of a group $G$ acting geometrically on CAT(0) spaces $X$ and $Y$, such that the associated $G$-equivariant quasi-isometry between the spaces does not extend to a homeomorphism between their visual boundaries [BR96]. Croke-Kleiner provided an example of a CAT(0) group $G$ and two CAT(0) spaces $X, Y$, both admitting geometric actions by $G$ such that $\partial \infty X$ and $\partial \infty Y$ are non-homeomorphic [CK00]. Wilson further showed that in fact this same $G$ acts geometrically on uncountably many spaces with boundaries of distinct topological type [Wil05]. The group $G$ in the Croke-Kleiner construction is the right-angled Artin group with the defining graph a path on four vertices.

A CAT(0) group $G$ is called boundary rigid if the visual boundaries of all CAT(0) spaces admitting a geometric action by $G$ are homeomorphic. As noted above, hyperbolic CAT(0) groups are boundary rigid while not all CAT(0) groups are boundary rigid. Ruane proved that the direct product of hyperbolic groups is boundary rigid [Rua99]. Hosaka extended that to show that any direct product of boundary rigid groups is boundary rigid [Hos03]. Hruska-Kleiner proved that groups acting geometrically on CAT(0) spaces with the isolated flats property are boundary rigid [HK05].

A bushy tree is an infinite tree of bounded valence with no terminal vertices. In this note, we study a family of CAT(0) groups acting geometrically on the product of two bushy trees. We will assume the groups preserve the factors, which is always the case after passing to an index 2 subgroup. We refer to such groups as (cocompact) lattices in a product of trees. We will skip the word cocompact even though we are assuming this property throughout the paper. We are interested in the following question:

Question. Are lattices in a product of trees boundary rigid?

The simplest example of a lattice in a product of trees is a direct product $F_n \times F_m$ of two finite rank free groups. These are boundary rigid by [Rua99]. However, there exist lattices in product of trees that are irreducible, i.e. they do not split as direct products, even
after passing to a finite index subgroups. Irreducible lattices in products of trees were first studied by Mozes [Moz92], Burger-Mozes [BM97], [BM00] and by [Wis96]. Burger-Mozes constructed examples of simple lattices in a product of trees, providing the first examples of simple CAT(0) groups, as well as the first examples of simple amalgamated products of free groups.

In this paper, we give the positive answer to the above for torsion-free lattices.

**Theorem 1.1.** Let $G$ be a torsion-free lattice in a product of trees. Suppose $G$ acts geometrically on a CAT(0) space $X$. Then $\partial_\infty X$ is the join $C * C$ of two copies of the Cantor set.

Moreover, if $X$ is geodesically complete, then $X$ splits as a product of CAT(0) spaces $X_1 \times X_2$, where $\partial_\infty X_i = C$ for each $i = 1, 2$.

If a lattice $G$ in a product of trees $T_1 \times T_2$ is vertex-transitive, i.e. its action on $T_1 \times T_2$ is vertex-transitive, then $G$ admits a particularly nice presentation and it is always virtually torsion-free. See [Cap19, Sec 4.1] or [Rat04] for details.

**Corollary 1.2.** Every (not necessarily torsion-free) vertex-transitive lattice in a product of trees is boundary rigid.

There are two major steps in the proof of the theorem. We first show that $\partial_\infty X$ splits as a join of two 0-dimensional subspaces and then show that each subspace is homeomorphic to $C$.

To show $\partial_\infty X$ splits as a join, it suffices to show that $\partial_T X$, the Tits boundary of $X$ splits as a metric join of two discrete sets. In our setting, this will provide a quasi-dense subset $X'$ of $X$ which splits isometrically as a product $X_1 \times X_2$. This is enough to conclude that the visual boundary $\partial_\infty X$ splits as $\partial_\infty X_1 * \partial_\infty X_2$, with each factor 0-dimensional.

Once we have this information about $\partial_\infty X$, we prove that $\partial_\infty X_i = C$ for $i = 1, 2$. This step is non-trivial when $G$ is irreducible.

For contrast, see Remark 5.2 for case where $X$ is a CAT(0) cube complex and the action of $G$ is essential (or we can assume $X$ has the geodesic extension property instead of essential action). The Rank Rigidity theorem of [CS11] allows us to get the splitting of $\partial_\infty X$ as a join of 0-dimensional subspaces almost immediately. We mention this case separately to illustrate the usefulness of the Rank Rigidity Theorem in the cube complex setting.

The paper is organized as follows. In section 2 we give the background on lattices in products of trees as well as ends and boundaries of CAT(0) spaces. In section 4 we review the machinery from [GS13] that we use in our proof and we provide a slightly strengthened version of Theorem A from that paper (see Theorem 4.5 here). In section 5 we prove Theorem 5.1 which is the first of the two majors steps mentioned above. We finish the proof of the main theorem in section 6.

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2. CAT(0) spaces at infinity

2.1. Ends of a space. We recall the definitions and relevant facts about the space of ends of a topological space. For more details, see [BH99].

Let $X$ be a topological space. A ray in $X$ is a proper map $r : [0, \infty) \to X$. A ray at $x_0$ where $x_0$ is a point of $X$, is a ray with $r(0) = x_0$. An end $e$ of $X$ is an equivalence class of rays in $X$ where $r_1 \simeq r_2$ if and only if for every compact set $K \subseteq X$ there exists $N \geq 0$ such that $r_1([N, \infty])$ and $r_2([N, \infty])$ are contained in the same connected component of $X - K$. We denote the equivalence class of the ray $r$ by $e(r)$. The set of all ends of $X$ is denoted by $\text{Ends}(X)$.

Let $U$ be an open set in $X$ and $e \in \text{Ends}(X)$. We use the notation $e < U$ to mean that for any $r : [0, \infty) \to X$ with $e(r) = e$, there exists $N \geq 0$ such that $r([N, \infty)) \subseteq U$.

The set $X \cup \text{Ends}(X)$, denoted by $\hat{X}$, can be endowed with topology that is generated by the basis consisting of the following sets:

- open sets in $X$,
- sets of the form $U \cup \{e \in \text{Ends}(X) \mid e < U\}$ where $U$ is a connected component of $X - K$ for some compact set $K \subseteq X$.

The space $\hat{X}$ is compact and is called the end compactification of $X$.

2.2. Visual Boundary. Assume that $X$ is a metric space with metric $d$. Two geodesic rays $r, r'$ are asymptotic, if there exists a constant $K > 0$ such that $d(r(t), r'(t)) < K$ for all $t \in [0, \infty)$.

The boundary of $X$, denoted $\partial X$, is the set of equivalence classes of geodesic rays, where two rays are equivalent if they are asymptotic. We denote the equivalence class of a ray $r$ by $r(\infty)$.

When $X$ is a complete CAT(0) space, we can put a topology on $\partial X$ as follows. First fix a basepoint $x_0 \in X$. The cone topology on $\partial X$ with respect to $x_0$ is given by the neighborhood basis $\{U(r, R, \epsilon) : r(\infty) \in \partial X, R, \epsilon > 0\}$ where

$$U(r, R, \epsilon) = \{r'(\infty) \in \partial X : r'(0) = x_0, d(r(R), r'(R)) < \epsilon\}.$$ 

This topology seems to depend on the choice of basepoint $x_0 \in X$ but in fact it does not. There is a well-defined change of basepoint homeomorphism between the topologies determined by different basepoints. This follows from the fact:

**Proposition 2.1.** [BH99, Prop. II.8.2] If $r$ is a geodesic ray based at $x$ in a complete CAT(0) space $X$ and $x'$ is a point not on this ray, then there exists a unique geodesic ray $r'$ with $r'(0) = x'$ that is asymptotic to $r$.

The boundary $\partial X$ endowed with the cone topology is called the visual boundary of $X$ and we denote it by $\partial_\infty X$. See [BH99, Chap II.8] for more details and properties of the visual boundary.

If $X$ is a proper CAT(0) space, then $\partial_\infty X$ is compact and there is a natural well-defined map $\partial_\infty X \to \text{Ends}(Y)$ sending a ray $r$ to $e(r)$. This map does not depend on the choice of a ray in the equivalence class of asymptotic rays. The map is a continuous surjection. [BH99, Rem II.8.10].

One important theorem we will use about the visual boundary is the following theorem of Geoghegan-Ontaneda.
**Theorem 2.2** ([GO07]). The topological dimension of \( \partial_\infty X \) is a quasi-isometry invariant. In particular, if a group \( G \) acts geometrically on CAT(0) spaces \( X \) and \( X' \), then their visual boundaries have the same topological dimension.

2.3. **Tits Boundary.** A finer topology on \( \partial X \) comes from considering a metric induced by angles in the space \( X \). The boundary of \( X \) with this metric is called the **Tits boundary**, and is denoted by \( \partial_T X \).

In a CAT(0) space, one can measure angles between two geodesics \( r, r' \) that emanate from the same point. In fact, this angle can be expressed completely in terms of the metric in \( X \). Let \( x = r(0) = r'(0) \), then the angle at \( x \) between the geodesics \( r, r' \) emanating from \( x \) can be given by:

\[
\angle_x(r, r') = \lim_{t \to 0} \frac{1}{2t} \arcsin \frac{d(r(t), r'(t))}{2t}
\]

Note this can be done for both geodesic segments and rays based at \( x \). To define an angle metric on \( \partial X \), one must view the two points from all possible basepoints \( x \in X \). If \( \xi, \eta \) are points of \( \partial X \), we can define the angle between them (without reference to a basepoint) as follows:

\[
\angle(\xi, \eta) = \sup_{x \in X} \angle_x(\xi, \eta)
\]

This formula gives a metric on \( \partial X \) called the **angle metric**. The Tits metric on \( \partial X \) is the length metric associated to the angle metric, i.e. the Tits distance between points \( \xi, \eta \in \partial X \) is the infimum of the lengths of all rectifiable curves between \( \xi \) and \( \eta \) where the length of a rectifiable path is measured in the usual way using the angle metric. See [BH99, Chap II.9] for more on the Tits boundary.

A particularly useful property is that the Tits boundary of a non-positively curved space is spherical in the following sense.

**Theorem 2.3** ([BH99, II.9.20]). If \( X \) is a complete CAT(0) space, then \( \partial_T X \) is a complete CAT(1) space. Moreover, if \( X \) is proper, then any two points \( \xi_0, \xi_1 \in \partial X \) with finite Tits distance are joined by a geodesic segment in \( \partial_T X \).

2.4. **Relationship between \( \partial_\infty X \) and \( \partial_T X \).** There is a natural continuous bijection from \( \partial_T X \to \partial_\infty X \) induced by the identity map on \( \partial X \). One consequence of this is that we can bound the geometric dimension of \( \partial_T X \) by the topological dimension of \( \partial_\infty X \). Indeed, the geometric dimension of \( \partial_T X \) is equal to the topological dimension of \( \partial_T X \) by [Kle99].

In order to show \( \partial_\infty X \) splits as a join of Cantor sets, it suffices to show \( \partial_T X \) splits as a join. Indeed, under the assumption that \( X \) is **geodesically complete**, i.e. if every geodesic segment in \( X \) can be extended to a geodesic ray, if \( \partial_T X \) splits as a join the space \( X \) splits as a metric product \( X_1 \times X_2 \) for closed, convex subspaces \( X_1, X_2 \) [BH99, Chap II. Thm 9.24]. It is an elementary fact that the boundary of a metric product is a join - i.e. \( \partial_\infty(X_1 \times X_2) = \partial_\infty X_1 \ast \partial_\infty X_2 \), [BH99, Chap II.8.11(6)].

We do not want to assume \( X \) is geodesically complete for our theorem so we use work of Caprace and Monod in [CM09] to get around this issue. We explain how to do this here.

An action of a group \( G \) on a CAT(0) space \( X \) is **minimal** if there does not exists a non-empty \( G \)-invariant closed convex subset of \( X \). A CAT(0) space \( X \) is **minimal** if the action of its full isometry group is minimal. A CAT(0) space \( X \) is **boundary-minimal** if it possesses no closed convex subset \( X' \subseteq X \) such that \( \partial X' = \partial X \).
A subset $X' \subseteq X$ is quasi-dense in $X$ if there exists a $D > 0$ such that each point of $X$ is within distance $D$ of $X'$. In particular, $\partial X = \partial X'$ as sets.

Thus, for purposes of studying the boundary, we can always pass to a quasi-dense subset without losing any information. The following theorem will be used to avoid the extra assumption of geodesic completeness.

**Theorem 2.4** ([CM09], see also [Cap14, Ex II.4 and Prop III.10]). Let $G$ be a group acting geometrically on a CAT(0) space $X$. Then $G$ stabilizes a closed, convex, quasi-dense subspace $X' \subseteq X$ such that $G$ acts minimally on $X'$. In particular, $\partial X = \partial X'$ as sets.

Moreover, $\partial_T X$ splits as a join if and only if $X'$ splits as a product $X_1 \times X_2$.

The subspace $X'$ in the above theorem is not necessarily unique. In fact, every closed, convex, quasi-dense subspace of $X$ on which $G$ acts minimally admits a splitting as a product, provided that $\partial_T X$ splits as a join.

### 2.5. CAT(0) spaces and their isometries

Definition and background on CAT(0) spaces can be found in [BH99]. Let $X$ be a CAT(0) space and let $g$ be an isometry of $X$. The translation length $|g|$ of $g$ is the infimum of $d(gx, x)$ taken over $x \in X$. The isometry $g$ is semi-simple if the infimum is the minimum. A semi-simple isometry $g$ is elliptic if and only if $\inf d(gx, x) = 0$, i.e. $g$ has a fixed point in $X$. Otherwise $g$ is hyperbolic. Every hyperbolic isometry $g$ has an axis, i.e. a geodesic line $\gamma : \mathbb{R} \to X$ such that $g, \gamma(t) = \gamma(t + |g|)$ for all $t \in [0, \infty)$.

If $G$ is a group acting properly and cocompactly on $X$, then every element of $G$ is a semi-simple isometry [BH99, Prop II.6.10]. We note that if $G$ acts properly and cocompactly on $X$ and $g \in G$ is an infinite order element, then $g$ is a hyperbolic isometry.

Let $g$ be a hyperbolic isometry of $X$. Then the axis of $g$ restricted to positive or negative half-line is a geodesic ray. We denote by $g^\infty$ and $g^{-\infty}$ the corresponding elements of $\partial_\infty X$.

Let $H$ be a subgroup of $G$. The limit set of $\Lambda H$ is the intersection of $\partial_\infty X$ with the closure of an orbit $Hx_0$ in $X \cup \partial_\infty X$ for some (any) $x_0$. Note that if $H_1, H_2$ are commensurable, then $\Lambda H_1 = \Lambda H_2$.

We point out two special cases of interest here.

- The limit set of an infinite cyclic group $\langle g \rangle$ where $g$ is a hyperbolic isometry is the set $\{g^\infty, g^{-\infty}\}$.
- The Flat Torus Theorem ([BH99, Thm II.7.1]) shows that if $H = \mathbb{Z}^n$ for $n > 1$, there exists an isometrically embedded $n$-dimensional closed, convex, flat subspace $F$, in $X$, on which $H$ acts geometrically. Thus $\Lambda H = \partial_\infty F$ which is homeomorphic to an $(n-1)$ sphere $S^{n-1}$.

### 3. Lattices in product of trees

Let $T$ be a bushy tree, i.e. an infinite tree of bounded valence and no terminal vertices. We view $T$ as a metric space, with the path metric, where each edge has length 1. The automorphism group $\text{Aut}(T)$ is a group of all isometries $T \to T$, i.e. permutations of the vertex sets that preserve the adjacency. The group $\text{Aut}(T)$ endowed with a compact-open topology, is a locally compact group. We now consider two bushy trees $T_1$ and $T_2$. The product $T_1 \times T_2$ has a natural structure of a square complex. It is easy to verify that each vertex link is a complete bipartite graph and, in particular, it is a flag simplicial complex.
Thus $T_1 \times T_2$, with the path metric induced by the Euclidean metric on each square, is a CAT(0) square complex.

**Example 3.1.** Let $G$ be a group on four generators $a, b, x, y$ with four relations:

\[
\begin{array}{c}
\text{a} \quad \text{b} \\
\text{x} \quad \text{y}
\end{array}
\quad
\begin{array}{c}
\text{a} \quad \text{b} \\
\text{x} \quad \text{y}
\end{array}
\quad
\begin{array}{c}
\text{a} \quad \text{a} \\
\text{x} \quad \text{y}
\end{array}
\quad
\begin{array}{c}
\text{b} \quad \text{b} \\
\text{x} \quad \text{y}
\end{array}
\]

The group $G$ is an irreducible lattice in the product of two copies of a 4-valent tree [JW09].

The presentation complex of $G$ in the example above, and for all torsion-free lattices in product of trees more generally, is a non-positively curved square complex where each vertex link is a complete bipartite graph. Such square complexes are referred to as complete square complexes. We recall some definitions and facts about V$H$-complexes and complete square complexes from [Wis07]. We also give a description of the graph of group decomposition of the fundamental group of a V$H$-complex, as in [Wis07].

3.1. $VH$-complexes. A square complex $X$ is a V$H$-complex if the 1-cells of $X$ are partitioned into two classes $V$ and $H$ called vertical and horizontal edges respectively and the attaching map of each 2-cell of $X$ alternates between edges in $V$ and $H$. Note that the link of each 0-cell in $X$ is a bipartite graph. By [Wis07, Lem 3.4] every complete square complex has a double cover that is a V$H$-complex.

Let $V_X = V \cup X^0$ denote the vertical 1-skeleton and $H_X = H \cup X^0$ the horizontal 1-skeleton. For a 0-cell $x \in X^0$, $V_x$ denotes the component of $V_X$ containing $x$. $H_x$ is defined similarly. By [Wis07, Thm 3.8] any elevations $\tilde{V}_x$ and $\tilde{H}_x$ in the universal cover $\tilde{X}$ of $X$ are trees, and $\tilde{X}$ is isomorphic to $\tilde{V}_x \times \tilde{H}_x$.

One can define a vertical foliation on the V$H$-complex $X$. The unit square $I \times I$ is foliated by vertical line segments. Similarly, the image of a square in $X$ is foliated by vertical segments parallel to the pair of vertical edges on its boundary. For an arbitrary point $x \in X$ we define the leaf $V_x$ to be the smallest subset of $X$ having the property that $x \in V_x$ and that $V_x$ contains any vertical segment which intersects it. This definition of $V_x$ coincides with the above definition for $x \in X^0$.

Suppose $H_X$ is a directed graph. The V$H$-structure on $X$ is horizontally (resp. vertically) directed if the horizontal (resp. vertical) edges have the same direction (i.e. both are directed “left” or both are directed “right”, resp. “up” and “down”). By possibly subdividing the squares in a V$H$-complex, we can always assume that it is both horizontally and vertically directed.

3.2. Graph of groups decompositions. Given a directed V$H$-complex $X$, there is a graph of spaces decomposition of $X$ as described in [Wis07, Def 2.14]. We denote the underlying graph by $\Gamma_X$. The vertex spaces are connected components of $V_X$, i.e. each vertex space arises as $V_x$ for some 0-cell $x \in X^0$.

The edges of $\Gamma_X$ correspond to the connected components of $X - V_X$. If $x$ and $y$ are in the same component, then $V_x$ and $V_y$ are isomorphic graphs. Note that each such component is of the form $V_x \times (0,1)$ for some $x$ in a connected component of $X - V_X$ (in particular $x \notin V_X$). We identify the edge space with such $V_x$. 

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We now describe the attaching maps. Note that \( V_x \times (0,1) \) can be thought of as a subspace of \( V_x \times [0,1] \) which is a square complex. For each edge space, the inclusion \( V_x \times (0,1) \hookrightarrow X \) uniquely extends to a combinatorial map \( V_x \times [0,1] \to X \). The attaching maps are the restrictions \( V_x \times \{0\} \to V_X \) and \( V_x \times \{1\} \to V_X \). The images of the attaching maps are two connected components of \( V_X \) and correspond to two vertices of \( \Gamma_X \) that are connected by an edge in \( \Gamma_X \). Since \( X \) is non-positively curved these maps are locally injective and hence \( \pi_1 \)-injective.

The graph of spaces decomposition of \( X \) induces a graph of groups decomposition of \( \pi_1 X \), where each vertex group is a finite rank free group \([\text{Wis07, Thm 2.16}]\). We note the following.

**Lemma 3.2.** The attaching maps in the graph of groups decomposition of \( \pi_1 X \) are the inclusions of the edge groups into the adjacent vertex groups as finite index subgroups. In particular, the groups \( \pi_1 V_x \) and \( \pi_1 V_{x'} \) are commensurable for any \( x, x' \in X^0 \).

**Proof.** By \([\text{Wis07, Lem 3.11}]\) the attaching maps in the vertical (resp. horizontal) graph of spaces decomposition are covering maps. Since all edge and vertex spaces are finite graphs, the attaching maps must be finite covering maps. \( \square \)

Let us denote the vertex groups of \( \pi_1 X \) by \( G_v \) where \( v \in V(\Gamma_X) \), and the edge groups by \( G_e \) where \( e \in E(\Gamma_X) \). The attaching maps are \( \phi_0^e : G_e \to G_{i(e)} \) and \( \phi_1^e : G_e \to G_{t(e)} \) where \( i(e), t(e) \) are the initial and the terminal vertices of \( e \in E(\Gamma_X) \).

Let \( T \subseteq E(\Gamma_X) \) form a spanning tree of \( \Gamma_X \). By \([\text{Ser80, Sec 5.1}]\) \( \pi_1 X \) has a presentation where it is generated by the \( G_v \) for every \( v \in V(\Gamma_X) \) and extra generators \( t_e \) for every \( e \in E(\Gamma_X) \) such that

- \( t_e \phi_0^e(g)t_e^{-1} = \phi_1^e(g) \) for every \( g \in G_e \), and
- \( t_e = 1 \) for all \( e \in T \).

**Lemma 3.3.** Let \( x \in X^0 \). For every \( e \in E(\Gamma_X) \) there exist finite index subgroups \( K_0, K_1 \subseteq \pi_1 V_x \) such that \( t_e K_0 t_e^{-1} = K_1 \).

**Proof.** By Lemma 3.2, \( \phi_0^e(G_e) \) has finite index in \( G_{i(e)} \) and \( \phi_1^e(G_e) \) has finite index in \( G_{t(e)} \). In particular, \( \phi_0^e(G_e) \) and \( \phi_1^e(G_e) \) are both commensurable with \( \pi_1 V_x \), also by Lemma 3.2. Let \( L_i = \pi_1 V_x \cap \phi_i^e(G_e) \) for \( i = 0,1 \). Note that \( L_i \) has finite index in \( \pi_1 V_x \) by Lemma 3.2. Note that \( t_e^{-1}L_1t_e \subseteq \phi_0^e(G_e) \) is also a finite index subgroup. Set \( K_0 = L_0 \cap t_e^{-1}L_1t_e \) and \( K_1 = t_e K_0 t_e^{-1} = t_e L_0 t_e^{-1} \cap L_1 \). As the intersection of finite index subgroups of \( \pi_1 V_x \), \( K_i \) has finite index in \( \pi_1 V_x \) for \( i = 0,1 \).

\( \square \)

We will also need the following fact:

**Proposition 3.4** ([\text{Wis05, Lem 7.7}]). Let \( x \in X^0 \). Every torsion-free lattice in a product of trees contains a \( \mathbb{Z}^2 \) subgroups generated by some \( a \in \pi_1 V_x \) and \( b \in \pi_1 H_x \).

4. Limit operators and folding

In this section, we will follow the work of \([\text{GS13}]\) which outlines a strategy for studying dynamical properties of the action of a CAT(0) group on its boundary. Thus we only need to assume \( G \) is a group acting geometrically on a CAT(0) space in this section.

We start with standard definitions of ultrafilters, Stone-Čech compactification, \( \omega \)-limits etc. For simplicity, we state those only in the context of a group.
An ultrafilter $\omega$ on a group $G$ is a collection of subsets of $G$ such that

1. $\emptyset \notin \omega$,
2. if $A \in \omega$ and $A \subseteq B \subseteq G$, then $B \in \omega$,
3. if $A, B \in \omega$, then $A \cap B \in \omega$,
4. for every $A \subseteq G$ either $A \in \omega$ or $G - A \in \omega$.

An ultrafilter is **principal** if there exists $g \in G$ such that $\{g\} \in \omega$. We note that by properties (1) and (3) of an ultrafilter, the element $g$ such that $\{g\} \in \omega$ is unique. We denote such an ultrafilter by $\rho_g$. We denote the set of all ultrafilters on $G$ by $\beta G$. The set $\beta G$ can be equipped with topology generated by the sets of the form $\{\omega \in \beta G : A \in \omega\}$ for subsets $A \subseteq G$. The group $G$ viewed as a discrete space embeds in $\beta G$ via $g \mapsto \rho_g$. The space $\beta G$ is called the **Stone-Čech compactification** of $G$.

We describe how the action of $G$ on $\overline{X} = X \cup \partial_\infty X$ extends to the action of $\beta G$ on $\overline{X}$. Let $v = (v_g)_{g \in G} \in \overline{X}^G$ be a $G$-sequence, i.e. a sequence of elements of $\overline{X}$ indexed by $G$, and let $\omega \in \beta G$. The limit with respect to $\omega$ is defined as follows.

**Definition 4.1** ([GS13, Defn 2.1]). We say that $\xi \in \overline{X}$ is the $\omega$-limit of $v$, and write $\xi = \lim_\omega v_g$, if for every neighborhood $U \subseteq \overline{X}$ of $\xi$ we have $\{g \in G : v_g \in U\} \in \omega$.

The $\omega$-limit exists and is unique for every $G$-sequence $v$ and every ultrafilter $\omega$ (see [GS13, Prop 2.2]).

An important special case is when the $G$-sequence $v$ is the orbit of a point, i.e. $v_g = g \cdot \xi$ for some $\xi \in \overline{X}$. Then the limit $\lim_\omega g \cdot \xi$ is denoted by $T^\omega(\xi)$. If $\omega = \rho_a$ is the principal ultrafilter determined by $a \in G$, then $T^{\rho_a}(\xi) = a \cdot \xi$ for every $\xi \in \overline{X}$. Suppose $\omega$ is a non-principal ultrafilter. Then $T^\omega(\xi) \in \partial X$ for all $\xi \in \overline{X}$. If $x \in X$, then the value $T^\omega(x)$ is independent of the choice of $x$ (see [GS13]), and we denote the point $T^\omega(x) \in \partial X$ by $\omega(\infty)$. Note that $\omega(\infty)$ is an accumulation point of the $G$-orbit of $x$.

Given an ultrafilter $\omega \in \beta G$, the **antipodal ultrafilter** $S\omega$ is defined as follows: for every $F \subseteq G$ we have $F \in S\omega \iff F^{-1} \subseteq \omega$. We denote $S\omega(\infty)$ by $\omega(-\infty)$, and write $n \cdot \omega \cdot p$ to denote $\omega(-\infty) = n$ and $\omega(\infty) = p$. Note that $T^\omega(\omega(\infty)) = \omega(\infty)$, but in general $T^\omega(\omega(-\infty))$ and $\omega(-\infty)$ might be distinct.

The group structure on $G$ naturally extends to a semi-group structure on $\beta G$. See [GS13] for details. We note the following fact.

**Lemma 4.2** ([GS13, Cor 2.12]). For every $\xi \in \overline{X}$ and for every $\nu, \omega \in \beta G$ one has $T^{\nu \cdot \omega} \xi = T^\nu T^\omega \xi$.

The function $T^\omega : \partial_T X \to \partial_T X$ is 1-Lipschitz in the Tits topology, but generally not continuous in the cone topology $\partial_\infty X$. An important special case to consider is when $G$ is hyperbolic and we construct $\omega$ with a fixed hyperbolic element in mind.

**Example 4.3.** Suppose $G$ is a hyperbolic group acting on a CAT(0) space $X$, and $g \in G$ is a hyperbolic element, and $\omega_g$ is an ultrafilter containing all the sets $\{g^n : n \geq N_0\}$ for all $N_0 \in \mathbb{Z}$. Then $T^{\omega_g}(\partial_T X)$ consists of two points $g^\infty, g^{-\infty} \in \partial_T X$ where $T^{\omega_g} \xi = g^\infty$ for all $\xi \in \partial_T X - \{g^{-\infty}\}$ and $T^{\omega_g}(g^{-\infty}) = g^{-\infty}$. 

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Guralnik-Swenson generalize the above example where the two point set \( \{g^\infty, g^{-\infty}\} \) is replaced by a top dimensional sphere in \( \partial_T X \). Indeed, the Tits boundary of a proper hyperbolic space is 0-dimensional and thus the pair \( \{g^\infty, g^{-\infty}\} \) is a top-dimensional sphere in that boundary. One could say Theorem 4.5 is the higher dimensional version of this example.

A round sphere in \( \partial_T X \) is an isometrically embedded copy of \( S^d \) where \( d = \dim \partial_T X \).

**Theorem 4.4** ([GS13, Thm A]). Suppose \( G \) acts geometrically on a CAT(0) space \( X \) and let \( d \) be the geometric dimension of \( \partial_T X \). Then for every \((d + 1)\)-flat \( F_0 \) in \( X \) there exists an ultrafilter \( \omega \in \beta G \) and a possibly different \((d + 1)\)-flat \( F \subseteq X \) such that \( T^\omega \) maps \( \partial_T X \) to \( \partial F \) and is an isometry while restricted to \( \partial F_0 \).

If the conclusion of the above theorem holds, we say that \( \omega \) folds \( \partial_T X \) onto the sphere \( \partial F \), and that \( \partial F \) is a folded sphere.

We will use a slightly strengthened version of the theorem above in the situation where \( G \) contains a subgroup isomorphic to \( \mathbb{Z}^{d+1} \). Our proof follows quite closely the original proof from [GS13].

**Theorem 4.5.** Let \( G \) be a group acting geometrically on a CAT(0) space \( X \) and let \( d \) be the geometric dimension of \( \partial_T X \). Suppose that \( G \) has a subgroup \( H \simeq \mathbb{Z}^{d+1} \). Then there exists an \( H \)-invariant isometrically embedded \((d + 1)\)-flat \( F \) and \( \omega \in \beta G \) such that \( T^\omega(\partial_T X) = \partial F \) and \( T^\omega \) is an identity map while restricted to \( \partial F \).

To prove Theorem 4.5 we will need the following notion and lemmas from [GS13].

**Definition 4.6** ([GS13, Def 3.14]). We say \( \omega \in \beta G \) pulls away from a point \( \xi \in \partial_T X \) if there exists \( x \in X \), a ray \( \gamma \) with \( \gamma(\infty) = \xi \) and \( C > 0 \) such that \( A = A_{x, \gamma, M, C} = \{ g \in G \mid g \cdot x \in N_C(\gamma([M, \infty))) \} \) where \( M \in (0, \infty) \).

**Lemma 4.7** ([GS13, Lem 3.19]). Suppose \( S \subseteq \partial_T X \) is a sphere bounding a flat \( F \), and let \( \ell \subseteq F \) be a line joining points \( \xi, \eta \) in \( S \). If \( \omega \) pulls away from \( \xi \) then \( T^\omega S \) is a sphere bounding a flat isometric to \( F \). In particular, \( \omega(-\infty) = \xi, \omega(\infty) = T^\omega(\eta) \), and \( d_T(T^\omega(\xi), T^\omega(\eta)) = \pi \).

We say we subset \( M \subseteq \partial_T X \) is \( \pi \)-convex if for any \( p, q \in M \) such that \( d_T(p, q) < \pi \) every geodesic joining \( p, q \) is contained in \( M \). Let \( p, q \) be a pair of points in \( \partial_T X \) such that \( d_T(p, q) = \pi \). The suspension \( \Sigma(p, q) \) is the set \( \{ \xi \in \partial_T X \mid d_T(p, \xi) + d_T(\xi, q) = \pi \} \). The suspension \( \Sigma(p, q) \) is closed and \( \pi \)-convex in \( \partial_T X \) [GS13, Lem 3.22]. A suspension point of a subset \( M \) is a point \( p \in M \) such that there exists \( q \in M \) such that \( M \) is the union of all geodesics in \( M \) joining \( p \) and \( q \).

**Lemma 4.8** ([GS13, Cor 3.24]). Suppose \( \omega \) pulls away from \( \xi \) and there exists \( \eta \in \partial_T X \) such that \( d_T(\xi, \eta) = \pi \). Then \( T^\omega \) maps \( \partial_T X \) into the suspension \( \Sigma(T^\omega(\xi), \omega(\infty)) \).

We will also need the following corollary of Lemma 4.8.

**Lemma 4.9.** Let \( a \in G \) be an infinite order element. Let \( \omega \in \beta G \) contain all the sets \( \{a^n\}_{n \geq N_0} \) for all \( N_0 \). Then \( T^\omega(g^{\pm \infty}) = g^{\pm \infty} \) for all infinite-order elements \( g \) in the centralizer \( C(a) \) of \( a \) and \( T^\omega \) maps \( \partial_T X \) into \( \Sigma(a^{\infty}, a^{-\infty}) \).

**Proof.** Let \( \omega \in \beta G \) contain all the sets \( \{a^n\}_{n \geq N_0} \) for all \( N_0 \). Let \( g \in C(a) \). To show that \( T^\omega g^{\pm \infty} = g^{\pm \infty} \) we must check that \( \omega \) contains the set \( \{h \in G \mid h \cdot g^{\pm \infty} \in U \} \) for any neighborhood \( U \) of \( g^{\pm \infty} \). Note that \( a^n \cdot g^{\pm \infty} = g^{\pm \infty} \) since \( [a, g] = 1 \). Thus for every
neighborhood $U$ of $g^{±∞}$ we have $a^n \cdot g^{±∞} \in U$. It follows that \{ $h \in G \mid h \cdot g^{±∞} \in U$ \} ∈ $ω$ by Condition (2) from the definition of an ultrafilter. By Lemma 4.8 $T^ω$ maps $\partial_T X$ into $Σ(a^∞, a^{−∞})$ since $T^ω(a^{−∞}) = a^{−∞}$ and $ω(∞) = a^{∞}$.

We are now ready to prove Theorem 4.5.

**Proof of Theorem 4.5.** Let \{ $a_0, \ldots, a_d$ \} ⊆ $H$ be a generating set of $H$. By the Flat Torus Theorem ([BH99], Thm II.7.1), we can choose an $H$-invariant $(d + 1)$-flat $F$ on which $H$ acts via a torus action. This implies that for each $h \in H$, $F \subset \text{Min}(h)$. Let $S = ∂F$.

Throughout the proof, $ω_g$ with $g ∈ G$ denotes an ultrafilter which contains all the sets \{ $g^n$ \} \text{for} $n \geq N_0$ for $N_0 \in \mathbb{Z}$. Let \{ $i \in \{0, \ldots, d\}$ \}. By Lemma 4.9 $T^{ω_{a_i}}$ maps $\partial_T X$ into $Σ(a_i^∞, a_i^{−∞})$, and $T^{ω_{a_i}}(h^{±∞}) = h^{±∞}$ for every $h \in H$. In particular, $T^{ω_{a_i}}$ restricts to identity on $S$.

Let $ω = ω_{a_d} \cdots ω_{a_0}$. Let $M$ denote the image of $\partial_T X$ under $T^ω = T^{ω_{a_d}} \cdots T^{ω_{a_0}}$ (see Lemma 4.2). Note that $T^ω$ restricts to identity on $S$. In particular, $S \subseteq M$. Our goal is to prove $M \subseteq S$.

Let $i \in \{0, \ldots, d\}$. We have
\[
M = T^ω(\partial_T X) \subseteq T^{ω_{a_d} \cdots ω_{a_i}}(\partial_T X) \subseteq T^{ω_{a_d} \cdots ω_{a_i+1}}(Σ(a_i^∞, a_i^{−∞})).
\]
We also have
\[
T^{ω_{a_d} \cdots ω_{a_i+1}}(Σ(a_i^∞, a_i^{−∞})) \subseteq Σ(T^{ω_{a_d} \cdots ω_{a_i+1}}(a_i^∞), T^{ω_{a_d} \cdots ω_{a_i+1}}(a_i^{−∞})) = Σ(a_i^∞, a_i^{−∞}),
\]
so we have shown that $M \subseteq Σ(a_i^∞, a_i^{−∞})$ for every $i = 0, \ldots, d$. Let $Y$ be the intersection of all the suspensions $Σ(p, q)$ that contain $M$ as a subspace. The space $Y$ is closed, π-convex and points $a_i^{±∞}$ for $i = 0, \ldots d$ belong to the set of its suspension points. Therefore $S$ belongs to the set of its suspension points. By [GS13, Thm 1.16] $Y = S(Y) * E'(Y)$ where $S(Y)$ is the set of suspension points of $Y$. Since $S \subseteq S(Y)$ and since the geometric dimension of $S$ is equal to the geometric dimension of $S$, the subspace $E'(Y)$ must be empty. Therefore $M \subseteq Y = S$ which completes the proof.

**Corollary 4.10.** Let $G$ be a torsion-free lattice in a product of trees and let $a \in π_1V_{x_0}, b \in π_1H_{x_0}$ be commuting elements as in Lemma 3.4. Then there exists $ω ∈ βG$ that folds $\partial_T X$ onto $Λ(a, b) ⊆ ∂_T X$.

## 5. Splitting

The goal of this section is to prove the following theorem.

**Theorem 5.1.** Let $G$ be a torsion-free lattice in a product of trees. Suppose $G$ acts geometrically complete or the action of $G$ on $X$ is essential, we can use rank rigidity for CAT(0) cube complexes to prove the above theorem. Every element of $G$ acts a half-plane in $T_1 \times T_2$, so $G$ contains no rank one elements. By rank rigidity for CAT(0) cube complexes [CS11, Cor B], $X$ is a product of two CAT(0) cube complexes $X_1 \times X_2$. Thus $∂_∞ X = ∂_∞ X_1 * ∂_∞ X_2$. Thus
Theorem 5.5. Let $G$ be a torsion-free lattice in a product of trees and suppose $G$ acts geometrically on a CAT(0) space $X$. Let $a \in \pi_1 V_x$ and $b \in \pi_1 H_x$ commute where $b$ is an infinite order element. Let $S := \Lambda(a, b) \subseteq \partial X$ and $\Lambda = \Lambda(\pi_1 V_x) \subseteq \partial X$. Then $S \cap \Lambda = \{a^{+\infty}, a^{-\infty}\}$.

Proof. Let $p \in X$ and $\xi \in S$. By possibly replacing $a$ and/or $b$ with their inverses, we can assume that $\xi$ is an accumulation point of a sequence $\{a^{l_n}b^{m_n}p\}_{n \in \mathbb{N}}$ where $l_n$ and $m_n$ are non-decreasing sequences of natural numbers.

Suppose that a sequence $\{a_n p\}_{n \in \mathbb{N}}$ accumulates at $\xi$ where each $a_n \in \pi_1 V_x$. Then $\{a_n p\}_{n \in \mathbb{N}}$ must stay within a finite Hausdorff distance from $\{a^{l_n}b^{m_n}p\}_{n \in \mathbb{N}}$. This implies that in the product of trees $\tilde{V}_x \times \tilde{H}_x$, the sets $\{a^{l_n}b^{m_n}\tilde{x}\}_{n \in \mathbb{N}}$ and $\{a_n \tilde{x}\}_{n \in \mathbb{N}}$ have finite Hausdorff distance, where $\tilde{x}$ is the basepoint of $\tilde{X}$ lying in the intersection $\tilde{V}_x \cap \tilde{H}_x$. Let $d$ denote the $L^1$ metric on the product of trees $\tilde{V}_x \times \tilde{H}_x$.

Note that $d(a^{l_n}b^{m_n}\tilde{x}, \pi_1 V_x \tilde{x}) \geq d(a^{l_n}b^{m_n}\tilde{x}, \tilde{V}_x) \geq m_n$ for every $n \in \mathbb{N}$. Since $m_n$ is a non-decreasing sequence and the Hausdorff distance between $\{a^{l_n}b^{m_n}p\}_{n \in \mathbb{N}}$ and $\tilde{H}_x$ is finite, $m_n$ must be bounded. Thus $\xi = a^{\pm \infty}$.

Since $a^{+\infty}, a^{-\infty} \in \Lambda$, we have $S \cap \Lambda = \{a^{+\infty}, a^{-\infty}\}$. In particular, $(ab)^{\pm \infty} \notin \Lambda(\pi_1 V_x)$. □

Lemma 5.4. Let $G$ be a torsion-free lattice in a product of trees and suppose $G$ acts geometrically on a CAT(0) space $X$. Then the limit set $\Lambda := \Lambda(\pi_1 V_x)$ is a closed, non-empty, proper $G$-invariant subset of $\partial X$.

Proof. The limit set of every subgroup is a closed subset of $\partial X$. Let $x_0 \in X$ be any point. For any infinite order element $a \in \pi_1 V_x$, the points $a^{\pm \infty}$ are accumulation points of the set $\{a^k x_0\}_{k \in \mathbb{Z}}$, and hence belong to $\Lambda$, showing that the limit set is non-empty.

By Proposition 3.4 there exist nontrivial elements $a \in \pi_1 V_x$ and $b \in \pi_1 H_x$ that commute. By Lemma 5.3 $b^{+\infty}$ is not contained in $\Lambda$. Thus $\Lambda$ is a proper subset of $\partial X$.

It remains to prove that $\Lambda$ is $G$-invariant. Using the presentation of $G$ decomposed as the graph of groups, as discussed in Section 3.2, we see that $G$ is generated by elements of the vertex groups and generators $t_e$ for $e \in E(\Gamma_X)$. By Lemma 3.2, any two vertex groups are commensurable, so $g\Lambda = \Lambda$ for each $g$ in a vertex group. By Lemma 3.3 $t_e \Lambda = t_e \Lambda(K_0) = \Lambda(t_e K_0 t_e^{-1}) = \Lambda(K_1) = \Lambda$.

□

The following is proven by Ricks in [Ric20] although it is not stated explicitly there. We include the proof referring to theorems in [Ric20].

Theorem 5.5 ([Ric20]). Let $G$ be any group acting geometrically on a CAT(0) space $X$ with 1-dimensional Tits boundary $\partial_T X$ and $|\partial X| \geq 3$. Suppose there exists a closed, non-empty, proper $G$-invariant subset $\Lambda$ of $\partial X$, and a folded circle $S \subseteq \partial X$ such that the intersection $S \cap \Lambda$ consists of at most two antipodal (with respect to $S$) points. Then a quasi-dense, closed, convex subspace $X' \subseteq X$ splits as a nontrivial product. In particular, $\partial_X X$ and $\partial_T X$ split as nontrivial spherical joins.
**Proof.** By [Ric20, Cor 22], radius$^{\partial X}(\Lambda) \leq \frac{\pi}{2}$. If the action of $G$ on $X$ has a global fixed point, then by [PS09, Lem 26], $G$ virtually has $\mathbb{Z}$ as a direct factor. This is a contradiction.

By [Ric20, Corollary B], $X$ contains a quasi-dense, closed, convex subspace $X' \subseteq X$ which splits as a nontrivial product, $X' = X_1 \times X_2$ for closed, convex subsets $X_1, X_2$. Since the boundaries of $X'$ and $X$ are set-wise equal, [BH99, Ex II.8.11(6) and Cor II.9.11] imply that $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$.

**Proof of Theorem 5.1.** The boundary $\partial_\infty X$ has topological dimension 1, since the dimension of a visual boundary is a quasi-isometry invariant (see Theorem 2.2). This implies that the topological dimension of $\partial_T X$, and by [Kle99] also the geometric dimension of $\partial_T X$, is at most 1. By Proposition 3.4, $G$ contains a $\mathbb{Z}^2$ subgroup, so there is a circle contained in $\partial_T X$. Therefore the dimension of $\partial_T X$ is 1 and $\partial_T X$ has at least three points.

Let $x \in X^0/G$ be a basepoint. By Lemma 5.4, $\Lambda = \Lambda(\pi_1 V_x)$ is a closed, non-empty proper $G$-invariant subset of $\partial_\infty X$. By Proposition 3.4, there exists $a \in \pi_1 V_x$ and $b \in \pi_1 H_x$ such that $[a, b] = 1$. By Corollary 4.10, $S := \Lambda\langle a, b \rangle$ is a round, folded circle. By Lemma 5.3 $S \cap \Lambda = \{a^{+\infty}, a^{-\infty}\}$ which is a pair of antipodal points on $S$. The claim follows from Theorem 5.5.

**6. Analyzing the join factors**

Let $G$ be a torsion-free lattice in a product of trees. In Theorem 5.1 we have shown that if $G$ acts geometrically on a CAT(0) space $X$, then $G$ acts minimally on a quasi-dense convex closed subspace $X' \subseteq X$ which splits as a product $X_1 \times X_2$ where $X_1, X_2$ are closed convex subsets of $X$ with 0-dimensional visual boundaries. In particular, $\partial_\infty X$ splits as a nontrivial join of the form $\partial_\infty X_1 \ast \partial_\infty X_2$. In this section, we complete the proof of the main theorem by proving that $\partial_\infty X_i$ are both Cantor sets.

For completeness, we point out the three possibilities that can occur when $\partial_\infty X$ is the join of 0-dimensional subspaces.

**Proposition 6.1.** Let $G$ be any group acting geometrically on a CAT(0) space $X$ with topological dimension of $\partial_\infty X = 1$ and $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$ for closed convex subspaces $X_1, X_2$ in $X$. Then exactly one of the following holds:

1. $|\partial X_i| = 2$ for $i = 1, 2$. In this case, $\partial_\infty X$ is homeomorphic to a circle and $G$ is either virtually abelian or fuchsian.
2. $|\partial X_1| = 2$ and $|\partial X_2| \geq 3$ (or vice versa). In this case $\partial_\infty X$ is homeomorphic to the suspension of a Cantor set and $G$ is virtually $F \times \mathbb{Z}$ where $F$ is finitely generated and free.
3. $|\partial X_i| \geq 3$ for both $i = 1, 2$. In this case, $\partial_\infty X = C \ast C$, the join of two Cantor sets. Each of the subspaces $X_i$ is quasi-isometric to a simplicial tree.

The first two cases follow directly from [Rua06]. The statements in that paper assume geodesic completeness on $X$, however one can pass to a quasi-dense subset and the conclusions about the boundary and the groups still hold.

The remainder of this section is devoted to proving that the third case is the only remaining case and that for our groups $G$, we are in that case.

The subtlety involved here is that when the group $G$ is not (virtually) a product of free groups, we do not have geometric group actions on $X_i$.  

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Lemma 6.2. Suppose $G$ acts geometrically on a $\text{CAT}(0)$ space $X$ with topological dimension of $\partial_\infty X = 1$ and $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$ for closed convex subspaces $X_1, X_2$ in $X$. Then the boundary $\partial_\infty X_i$ is homeomorphic to $\text{Ends}(X_i)$, the ends space of $X_i$.

Proof. Let us first show that each $X_i$ is hyperbolic. Note that since $X$ is proper, so are $X_1$ and $X_2$.

First, we show that the subspace $X_i$ for $i = 1, 2$ is a visibility space, i.e. given any two distinct points $\xi, \eta$ in $\partial_\infty X_i$, there is a geodesic line in $X_i$ between them. Suppose there is no geodesic line in $X_i$ between $\xi$ and $\eta$. Then by [BH99, Prop. II.9.21(2)], there is a geodesic segment in $\partial_T X_i$ joining them. This segment is an arc in $\partial_T X_i$ which would map to an arc in $\partial_\infty X_i$ via the identity map on $\partial X$. This contradicts the fact that $\partial_\infty X_i$ is 0-dimensional.

Moreover, the action of $G$ on $X_i$ is cocompact, since so is the action of $G$ on $X$. Hence, $X_i$ is uniformly visible by [BH99, Prop II.9.32]. Finally, $X_i$ is hyperbolic by [BH99, Prop III.1.4].

Since $X_i$ is a proper hyperbolic space, the natural map $\partial_\infty X_i \to \text{Ends}(X_i)$ is continuous and the fibers of that map are the connected components of $\partial_\infty X_i$ [BH99, Exer III.H.3.9]. Since $\partial_\infty X_i$ is 0-dimensional, the connected components are single points. Thus the map $\partial_\infty X_i \to \text{Ends}(X_i)$ is a continuous bijection. Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism. □

In [Bes96, Def 1.1], Bestvina outlined a set of axioms that a group boundary should have in order to be useful for relating homological invariants of the boundary to cohomological invariants of the group. All of the axioms hold true for a hyperbolic group $G$ acting on $G \cup \partial_\infty G$ and for a $\text{CAT}(0)$ group $G$ acting on $X \cup \partial_\infty X$ where $G$ admits a geometric action on the $\text{CAT}(0)$ space $X$. One of the axioms requires the collection of translates of any compact set to form a null set in $X \cup \partial_\infty X$, i.e. for any open cover $U$ of $X \cup \partial_\infty X$ and any compact set $K$ in $X$, all but finitely many $G$-translates of $K$ are contained in an element of $U$.

The next lemma shows that each $X_i \cup \partial_\infty X_i$ inherits this nullity condition on compact sets from $X \cup \partial_\infty X$ even though we have no geometric group action on $X_i$.

Lemma 6.3. Suppose $G$ acts geometrically on a $\text{CAT}(0)$ space $X$ with $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$ for closed convex subspaces $X_1, X_2$ in $X$. For every compact set $K \subseteq X_i$ and every open neighborhood of an end of $X_i$, there exists $g \in G$ such that $gK \subseteq U$.

Proof. We show that the lemma holds for $X_1$. The argument for $X_2$ is identical. Let $K \subseteq X_1$ and $K' \subseteq X_2$ be compact sets. Then $K \times K'$ is a compact set in $X$. By Lemma 6.2, $\partial_\infty X_1$ is homeomorphic to $\text{Ends}(X_1)$. Let $\xi \in \partial_\infty X_1$, and let $U$ be an open neighborhood of $\xi$. Then $U \times X_2$ is an open neighborhood of $\xi$, viewed as a point in $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$. Since the action of $G$ on $X$ is geometric, there exists $g \in G$ such that $g(K \times K') \subseteq U \times X_2$ (see e.g. [Bes96, p.124]). It follows that in the action of $G$ on $X_i$, we have $gK \subseteq U$. □

The following is not stated explicitly, but it is proved in [Hop44]. It is also proved in a similar form in [Ber05]. We include the proof for completeness. We restrict our attention to the case of geodesic metric spaces, but the proposition holds in more general setting.

Proposition 6.4 ([Hop44]). Let $Y$ be a geodesic metric space with at least three ends. Let $G$ be a group acting on $Y$ cocompactly so that the following holds: for every compact
set \( K \subseteq Y \), and every open neighborhood \( U \) of an end of \( Y \), there exists \( g \in G \) such that \( gK \subseteq U \). Then the space of ends \( \text{Ends}(Y) \) is perfect.

**Proof.** Suppose that there exists an end \( e \in \text{Ends}(Y) \) that is isolated, i.e. there exists a neighborhood \( U \) of \( e \) that does not contain any other ends. First we show that without loss of generality, we can assume that \( Y - U \) is connected. Indeed, if \( Y - U \) is not connected, we construct a neighborhood \( U' \subseteq U \) of \( e \) such that \( Y - U' \) is connected. Since \( \hat{Y} \) is compact, so is \( \hat{Y} - U \). Let \( V_1, \ldots, V_n \) be a finite collection of open sets covering \( \hat{Y} - U \) such that each \( V_i \) is connected and does not contain \( e \), and its closure \( \hat{V}_i \) in \( \hat{Y} \) is compact. The union \( \bigcup_{i=1}^n V_i \) has finitely many components, and since \( \text{Ends}(Y) \) is nowhere dense in \( \hat{Y} \), each \( V_i \) contains points of \( Y \). Each two points in \( Y \) can be joined by a path in \( Y \). In particular, there exists a closed connected set \( Q \) which is the union of \( \bigcup_{i=1}^n V_i \) and a finite number of paths in \( Y \). Note that \( Q \) does not contain \( e \). The set \( U' = \hat{Y} - Q \) is an open neighborhood of \( e \) and since \( Y - U \subseteq \bigcup V_i \subseteq Q \), we have \( U' \subseteq U \). Thus, \( U' \) is the neighborhood we were looking for.

By assumption, there are at least three distinct ends \( e_1, e_2, e_3 \) in \( Y \). Let \( K \) be a compact set in \( Y \) such that each of the ends \( e_1, e_2, e_3 \) lies in a different connected component \( Y_1, Y_2, Y_3 \) of \( Y - K \). By assumption, there exists an element \( g \in G \) such that \( gK \subseteq U \). We claim that for each \( i = 1, 2, 3 \), either \( gY_i \subseteq U \), or \( g(Y - Y_i) \subseteq U \). Indeed, otherwise there exists point \( p \in gY_i - U \) and \( q \in g(Y - Y_i) - U \). Since \( Y - U \) is connected, there exists a path \( \gamma \) in \( Y - U \) joining \( p \) and \( q \). Note that \( p, q \) lie in distinct connected component of \( Y - gK \), so \( g \) has to pass through \( gK \). This is a contradiction, since \( gK \subseteq U \).

Since \( U \) contains only one end, we have \( g(Y - Y_i) \subseteq U \) for at least two \( i \)'s among 1, 2, 3, say 1 and 2. It follows that \( Y - U \subseteq gY_i \) for \( i = 1, 2 \). The subsets \( Y_1 \) and \( Y_2 \) are disjoint, and so are \( gY_1 \) and \( gY_2 \). This is a contradiction and therefore the space of \( \text{Ends}(Y) \) has no isolated points. \( \square \)

**Lemma 6.5.** Suppose \( G \) acts geometrically on a CAT(0) space \( X \) with dimension of \( \partial_\infty X = 1 \) and \( \partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2 \) for closed convex subspaces \( X_1, X_2 \) in \( X \). If \( |\partial X_i| \geq 3 \), then \( \partial X_i \) is homeomorphic to the Cantor set \( C \).

**Proof.** Since \( X_i \) is a proper CAT(0) space, the boundary \( \partial X_i \) is compact and metrizable [BH99]. As \( \dim \partial_\infty X_i = 0 \), the boundary \( \partial_\infty X_i \) is totally disconnected. The action of \( G \) on \( X_i \) is cocompact, since so is the action of \( G \) on \( X \). By Lemma 6.3, for every compact set \( K \subseteq X_i \) and every open neighborhood \( U \subseteq X_i \) of an end of \( X_i \), there exists \( g \in G \) such that \( gK \subseteq U \). By Proposition 6.4, \( \text{Ends}(X_i) \) is a perfect space. This implies that \( \partial_\infty X_i \) is a perfect space because \( \text{Ends}(X_i) = \partial_\infty X_i \) by Lemma 6.2. By the characterization of the Cantor set, as a non-empty, perfect, totally disconnected, compact metrizable space, we conclude that \( \partial X_i \) is the Cantor set. \( \square \)

**Proof of Proposition 6.1.** Since \( \partial_\infty X \) has dimension 1, both \( \partial X_1 \) and \( \partial X_2 \) are non-empty. Now suppose that \( |\partial X_1| = 1 \). Note that an \( S^1 \) in \( \partial_\infty X \) would have to have two points in each of \( \partial X_1, \partial X_2 \). Thus if \( |\partial X_1| = 1 \), then \( \partial_\infty X \) does not contain any copies of \( S^1 \). By [BH99, Thm III.1.5] \( G \) must be hyperbolic and \( \partial_\infty X \) is \( G \)-equivariantly homeomorphic to the Gromov boundary of \( G \).

Since \( \partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2 \) with each factor non-empty, \( \partial_\infty X \) contains an interval and thus \( G \) must be non-elementary. We note that the unique point \( \xi \) in \( \partial X_1 \) is topologically distinguishable, unless \( \partial X_2 \) also has one point. In either case, \( G \) or a subgroup of index 2
of $G$ must fix $\xi$. This contradicts the convergence group action of $G$ on its boundary. Thus each of $\partial X_1$, $\partial X_2$ has at least two points.

If $|\partial X_1| = |\partial X_2| = 2$, then clearly $\partial_\infty X \simeq S^1$ and by [Rua06, Thm 3.5] $G$ is either virtually abelian or fuchsian. We note that the proof of that result follows without assuming geodesic completeness after passing to a closed, convex, quasi-dense subset of $X$.

Now assume $|\partial X_1| = |\partial X_2| \geq 3$. By Lemma 6.5, $\partial_\infty X$ is homeomorphic to a Cantor set $C$. Hence $\partial_\infty X$ is a suspension of a Cantor set. By [Rua06, Thm 4.4] $G$ is virtually $F \times \mathbb{Z}$ where $F$ is a finitely generated free group.

Finally, suppose that $|\partial X_i| \geq 3$ for $i = 1, 2$. By Lemma 6.5, $\partial_\infty X_i$ is homeomorphic to a Cantor set $C$. Consequently, $\partial_\infty X$ is homeomorphic to join of two copies of a Cantor set.

We now show that $X_i$ is quasi-isometric to a tree, following [Cha14]. By [Cha14, Lem 5.7], $X_i$ has the bottleneck property, and by [Man05, Thm 4.6] $X_i$ is quasi-isometric to a simplicial tree $T_i$. □

Proposition 6.6. Let $G$ be a torsion-free lattice in a product of trees. Suppose $G$ acts geometrically on a CAT(0) space $X$ where $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$ for non-empty, closed, convex subspaces $X_1, X_2$ in $X$. Then each $\partial_\infty X_1, \partial_\infty X_2$ is homeomorphic to the Cantor set and each subspace $X_i$ is quasi-isometric to a tree.

Proof. By Lemma 3.4, $G$ contains a $\mathbb{Z}^2$ subgroup. By the Flat Torus theorem there is a flat $F$ embedded in $X$ as a convex subspace. Thus, $\partial_\infty F \simeq S^1$ embeds in $\partial_\infty X$. In particular, both $|\partial X_1|, |\partial X_2| \geq 2$. Since $G$ is neither virtually abelian, $F \times \mathbb{Z}$ or fuchsian, by Proposition 6.1 we conclude that $\partial_\infty X \simeq C \ast C$ and each of the factors $X_i$ is quasi-isometric to a tree. □

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