1. Introduction

A visual boundary is a particular type of compactification of a proper geodesic metric space. The boundary is defined as a set of equivalence classes of asymptotic rays endowed with an appropriate topology. For hyperbolic spaces $X$ and $Y$, any quasi-isometry $X \to Y$ between them extends to a homeomorphism of their visual boundaries. Consequently, the homeomorphism type of the boundary of a hyperbolic group is a well-defined group invariant. This is not true for CAT(0) groups - i.e. groups that act geometrically on CAT(0) spaces.

Bowers-Ruane give an example of a group $G$ acting geometrically on CAT(0) spaces $X$ and $Y$, such that the associated $G$-equivariant quasi-isometry between the spaces does not extend to a homeomorphism between their visual boundaries [BR96]. Croke-Kleiner provided an example of a CAT(0) group $G$ and two CAT(0) spaces $X, Y$, both admitting geometric actions by $G$ such that $\partial_{\infty} X$ and $\partial_{\infty} Y$ are non-homeomorphic [CK00]. Wilson further showed that in fact this same $G$ acts geometrically on uncountably many spaces with boundaries of distinct topological type [Wil05]. The group $G$ in the Croke-Kleiner construction is the right-angled Artin group with the defining graph a path on four vertices.

A CAT(0) group $G$ is called boundary rigid, if the visual boundaries of all CAT(0) spaces admitting a geometric action by $G$ are homeomorphic. As noted above, hyperbolic CAT(0) groups are boundary rigid while not all CAT(0) groups are boundary rigid. Ruane proved that the direct product of hyperbolic groups is boundary rigid [Rua99]. Hosaka extended that to show that any direct product of boundary rigid groups is boundary rigid [Hos03]. Hruska proved that groups acting geometrically on CAT(0) spaces with the isolated flats property are boundary rigid [Hru05].

In this note, we study a family of CAT(0) groups acting geometrically on the product of two infinite, locally finite, regular trees of valence $\geq 3$. We will assume the groups preserve the factors, which is always the case after passing to an index 2 subgroup. We refer to such groups as lattices in a product of trees. We are interested in the following question:

**Question.** Are lattices in a product of trees boundary rigid?

The simplest example of a lattice in a product of trees is a direct product $F_n \times F_m$ of two finite rank free groups. These are boundary rigid by [Rua99]. However, there exist lattices in product of trees that are irreducible, i.e. they do not split as direct products, even after passing to a finite index subgroups. Irreducible lattices in products of trees were first studies by Mozes [Moz92], Burger-Mozes [BM97], [BM00] and by [Wis96]. Burger-Mozes
constructed examples of simple lattices in a product of trees, providing the first examples of simple CAT(0) groups, as well as the first examples of simple amalgamated products of free groups.

In this paper, we give the positive answer to the above for vertex-transitive lattices.

**Theorem 1.** Let $G$ be a lattice in a product of trees acting freely and vertex-transitively. Suppose $G$ acts geometrically on a CAT(0) space $X$. Then $\partial_\infty X$ is the join $C \ast C$ of two copies of the Cantor set.

Moreover, if $X$ is geodesically complete, then $X$ splits as a product of CAT(0) spaces $X_1 \times X_2$, where $\partial_\infty X_i = C$ for each $i = 1, 2$.

**Remark 2.** We assume $G$ acts freely and vertex-transitively in order to exploit a particular type of presentation for $G$ (see Observation 3 and Proposition 4 here). We suspect these are not necessary assumptions for the proof of our theorem to work but the details are simplified when they are used here.

There are two major steps in the proof of the theorem. We first show that $\partial_\infty X$ splits as a join of two 0-dimensional subspaces and then show that each subspace is homeomorphic to $C$.

To show $\partial_\infty X$ splits as a join, it suffices to show that $\partial_T X$, the Tits boundary of $X$ splits as a metric join of two discrete sets. In our setting, this will provide a quasi-dense subset $X'$ of $X$ which splits isometrically as a product $X_1 \times X_2$. This is enough to conclude that the visual boundary $\partial_\infty X$ splits as $\partial_\infty X_1 \ast \partial_\infty X_2$, with each factor 0-dimensional.

Once we have this information about $\partial_\infty X$, we prove that $\partial_\infty X_i = C$ for $i = 1, 2$. This step is non-trivial when $G$ is irreducible.

For contrast, see Remark 21 for case where $X$ is a CAT(0) cube complex and the action of $G$ is essential (or we can assume $X$ has the geodesic extension property instead of essential action). The Rank Rigidity theorem of [CS11] allows us to get the splitting of $\partial_\infty X$ as a join of 0-dimensional subspaces almost immediately. We mention this case separately to illustrate the usefulness of the Rank Rigidity Theorem in the cube complex setting.

The paper is organized as follows. In section 2 we give the background on lattices in products of trees as well as ends and boundaries of CAT(0) spaces. In section 3 we review the machinery from [GS13] that we use in our proof and we provide a slightly strengthened version of Theorem A from that paper (see Theorem 14 here). In section 4 we prove Theorem 20 which is the first of the two majors steps mentioned above. We finish the proof of the main theorem in section 5.

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## 2. Background

### 2.1. Lattices in products of trees.

Let $T$ be an infinite, regular tree of degree $\geq 3$. We view $T$ as a metric space, with the path metric, where each edge has length 1. The
automorphism group $\text{Aut}(T)$ is a group of all isometries $T \to T$, i.e. permutations of the vertex sets that preserve the adjacency. The group $\text{Aut}(T)$ endowed with a compact-open topology, is a locally compact group. We now consider two infinite regular trees $T_1$ and $T_2$ of finite degrees $n, m \geq 3$ respectively. The product $T_1 \times T_2$ has a natural structure of a square complex. It is easy to verify that each vertex link is a complete bipartite graph $K(n, m)$ and, in particular, it is a flag simplicial complex. Thus $T_1 \times T_2$, with the path metric induced by the Euclidean metric on each square, is a $\text{CAT}(0)$ square complex.

Every lattice in a product of trees acting freely and vertex-transitively on $T_1 \times T_2$ and preserving the factors has a special presentation of the form $\langle A \cup B \mid R \rangle$ where $R$ is a set of relators such that

- $R$ is a collection of relators of the form $a_1b_1a_2b_2$ where $a_1, a_2 \in A^{\pm 1}, b_1, b_2 \in B^{\pm 1}$,
- For every $a_1 \in A^{\pm 1}, b_1 \in B^{\pm 1}$, there exist unique $a_2 \in A^{\pm 1}, b_2 \in B^{\pm 1}$ such that a cyclic permutation of $a_1b_1a_2b_2$ belongs to $R^{\pm 1}$.

Such group presentations are discussed in Caprace in [Cap19, Sec 4.1]. An example is included later in this section. A proof that a lattice acting freely and vertex-transitively on a product of trees admits such a presentations can be found in [Rat04]. The presentation complex of such a presentation is a square complex with a unique vertex, whose link is a complete bipartite graph $K(n, m)$. Such square complexes are examples of complete square complexes. See [Wis07] for more details on complete square complexes. The subgroups $\langle A \rangle$ and $\langle B \rangle$ are free groups. We note the following.

**Observation 3.** For any $b \in B$ we have $b\langle A \rangle \subseteq \bigcup_{b' \in B^{\pm}} \langle A \rangle b'$.

We will also need the following fact:

**Proposition 4 ([Wis05]).** Every lattice in a product of trees acting freely and vertex-transitively contains a $\mathbb{Z}^2$ subgroups generated by some $a \in \langle A \rangle$ and $b \in \langle B \rangle$.

**Example 5.** Let $G$ be a group on four generators $a, b, x, y$ with four relations:

![Diagram of relations in Example 5](attachment:image)

The group $G$ is an irreducible lattice in the product of two copies of a 4-valent tree [JW09].

### 2.2. Ends of a space.

We recall the definitions and relevant facts about the space of ends of a topological space. For more details, see [BH99].

Let $X$ be a topological space. A ray in $X$ is a proper map $r: [0, \infty) \to X$. A ray at $x_0$ where $x_0$ is a point of $X$, is a ray with $r(0) = x_0$. An end $e$ of $X$ is an equivalence class of rays in $X$ where $r_1 \simeq r_2$ if and only if for every compact set $K \subseteq X$ there exists $N \geq 0$ such that $r_1([N, \infty])$ and $r_2([N, \infty])$ are contained in the same connected component of $X - K$. We denote the equivalence class of the ray $r$ by $e(r)$. The set of all ends of $X$ is denoted by $\text{Ends}X$.

Let $U$ be an open set in $X$ and $e \in \text{Ends}(X)$, we use the notation $e < U$ to mean that for any $r : [0, \infty) \to X$ with $e(r) = e$, there exists $N \geq 0$ such that $r([N, \infty)) \subseteq U$.

The set $X \cup \text{Ends}(X)$, denoted by $\hat{X}$, can be endowed with topology that is generated by the basis consisting of the following sets:
• open sets in $X$,
• sets of the form $U \cup \{ e \in \text{Ends}(X) \mid e < U \}$ where $U$ is a connected component of $X - K$ for some compact set $K \subseteq X$.

The space $\hat{X}$ is compact and is called the end compactification of $X$.

2.3. Visual Boundary. Assume that $X$ is a metric space with metric $d$. Two geodesic rays $r, r'$ are asymptotic, if there exists a constant $K > 0$ such that $d(r(t), r'(t)) < K$ for all $t \in [0, \infty)$.

The boundary of $X$, denoted $\partial X$, is the set of equivalence classes of geodesic rays, where two rays are equivalent if they are asymptotic. We denote the equivalence class of a ray $r$ by $r(\infty)$.

When $X$ is a complete CAT(0) space, we can put a topology on $\partial X$ as follows. First fix a basepoint $x_0 \in X$. The cone topology on $\partial X$ with respect to $x_0$ is given by the neighborhood basis

$$U(r, R, \epsilon) = \{ r'(\infty) \in \partial X : \epsilon_0, d(r(R), r'(R)) < \epsilon \}.$$

This topology seems to depend on the choice of basepoint $x_0 \in X$ but in fact it does not. There is a well-defined change of basepoint homeomorphism between the topologies determined by different basepoints. This follows from the fact:

**Proposition 6.** [BH99, Prop. II.8.2] If $r$ is a geodesic ray based at $x$ in a complete CAT(0) space $X$, and $x'$ is a point not on this ray, then there exists a unique geodesic ray $r'$ with $r'(0) = x'$ that is asymptotic to $r$.

The boundary $\partial X$ endowed with the cone topology is called the visual boundary of $X$ and we denote it by $\partial_\infty X$. See [BH99, Chap II.8] for more details and properties of the visual boundary.

If $X$ is a proper CAT(0) space, then $\partial_\infty X$ is compact and there is a natural well-defined map $\partial_\infty X \to \text{Ends}(Y)$ sending a ray $r$ to $e(r)$. This map does not depend on the choice of a ray in the equivalence class of asymptotic rays. The map is a continuous surjection. [BH99, Rem II.8.10].

One important theorem we will use about the visual boundary is the following theorem of Geoghegan-Ontaneda.

**Theorem 7 ([GO07]).** The topological dimension of $\partial_\infty X$ is a quasi-isometry invariant. In particular, if a group $G$ acts geometrically on CAT(0) spaces $X$ and $X'$, then their visual boundaries have the same topological dimension.

2.4. Tits Boundary. A finer topology on $\partial X$ comes from considering a metric induced by angles in the space $X$. The boundary of $X$ with this metric is called the Tits boundary, and is denoted by $\partial_T X$.

In a CAT(0) space, one can measure angles between two geodesics $r, r'$ that emanate from the same point. In fact, this angle can be expressed completely in terms of the metric in $X$. Let $x = r(0) = r'(0)$, then the angle at $x$ between the geodesics $r, r'$ emanating from $x$ can be given by:

$$\angle_x(r, r') = \lim_{t \to 0} 2 \arcsin \frac{1}{2t} d(r(t), r'(t))$$

Note this can be done for both geodesic segments and rays based at $x$. To define an angle metric on $\partial X$, one must view the two points from all possible basepoints $x \in X$. If $\xi, \eta$ are
points of $\partial X$, we can define the angle between them (without reference to a basepoint) as follows:

$$\angle(\xi, \eta) = \sup_{x \in X} \angle_x(\xi, \eta)$$

This formula gives a metric on $\partial X$ called the angle metric. The Tits metric on $\partial X$ is the length metric associated to the angle metric, i.e. the Tits distance between points $\xi, \eta \in \partial X$ is the infimum of the lengths of all rectifiable curves between $\xi$ and $\eta$ where the length of a rectifiable path is measured in the usual way using the angle metric. See [BH99, Chap II.9] for more on the Tits boundary.

A particularly useful property is that the Tits boundary of a non-positively curved space is spherical in the following sense.

**Theorem 8** ([BH99, II.9.20]). If $X$ is a complete CAT(0) space, then $\partial_T X$ is a complete CAT(1) space. Moreover, if $X$ is proper, then any two points $\xi_0, \xi_1 \in \partial X$ with finite Tits distance are joined by a geodesic segment in $\partial_T X$.

2.5. **Relationship between $\partial_\infty X$ and $\partial_T X$.** There is a natural continuous bijection from $\partial_T X \to \partial_\infty X$ induced by the identity map on $\partial X$. One consequence of this is that we can bound the geometric dimension of $\partial_T X$ by the topological dimension of $\partial_\infty X$. Indeed, the geometric dimension of $\partial_T X$ is equal to the topological dimension of $\partial_T X$ by [Kle99].

In order to show $\partial_\infty X$ splits as a join of Cantor sets, it suffices to show $\partial_T X$ splits as a join. Indeed, under the assumption that $X$ is geodesically complete, i.e. if every geodesic segment in $X$ can be extended to a geodesic ray, if $\partial_T X$ splits as a join the space $X$ splits as a metric product $X_1 \times X_2$ for closed, convex subspaces $X_1, X_2$ [BH99, Chap II. Thm 9.24]. It is an elementary fact that the boundary of a metric product is a join - i.e. $\partial_\infty (X_1 \times X_2) = \partial_\infty X_1 \star \partial_\infty X_2$, [BH99, Chap II.8.11(6)].

We do not want to assume $X$ is geodesically complete for our theorem so we use work of Caprace and Monod in [CM09] to get around this issue. We explain how to do this here.

An action of a group $G$ on a CAT(0) space $X$ is minimal if there does not exists a non-empty $G$-invariant closed convex subset of $X$. A CAT(0) space $X$ is minimal if the action of its full isometry group is minimal. A CAT(0) space $X$ is boundary-minimal if it possesses no closed convex subset $X' \subseteq X$ such that $\partial X' = \partial X$. A subset $X' \subseteq X$ is quasi-dense in $X$ if there exists a $D > 0$ such that each point of $X$ is within $D$ of $X'$. In particular, $\partial X = \partial X'$ as sets.

Thus, for purposes of studying the boundary, we can always pass to a quasi-dense subset without losing any information. The following theorem will be used to avoid the extra assumption of geodesic completeness.

**Theorem 9** ([CM09], see also [Cap14, Ex II.4 and Prop III.10]). Let $G$ be a group acting geometrically on a CAT(0) space $X$. Then $G$ stabilizes a closed, convex, quasi-dense subspace $X' \subseteq X$ such that $G$ acts minimally on $X'$. In particular, $\partial X = \partial X'$ as sets.

Moreover, $\partial_T X$ splits as join if and only if $X'$ splits as product $X_1 \times X_2$.

The subspace $X'$ in the above theorem is not necessarily unique. In fact, every closed, convex, quasi-dense subspace of $X$ on which $G$ acts minimally admits a splitting as a product, provided that $\partial_T X$ splits as a join.

2.6. **CAT(0) spaces and their isometries.** Definition and background on CAT(0) spaces can be found in [BH99]. Let $X$ be a CAT(0) space and let $g$ be an isometry of $X$. The
translation length \(|g|\) of \(g\) is the infimum of \(d(gx, x)\) taken over \(x \in X\). The isometry \(g\) is semi-simple if the infimum is the minimum. A semi-simple isometry \(g\) is elliptic if and only if \(\inf d(gx, x) = 0\), i.e. \(g\) has a fixed point in \(X\). Otherwise \(g\) is hyperbolic. Every hyperbolic isometry \(g\) has an axis, i.e. a geodesic line \(\gamma : \mathbb{R} \to X\) such that \(g\cdot \gamma(t) = \gamma(t + |g|)\) for all \(t \in [0, \infty)\).

If \(G\) is a group acting properly and cocompactly on \(X\), then every element of \(G\) is a semi-simple isometry [BH99, Prop II.6.10]. We note that if \(G\) acts properly and cocompactly on \(X\) and \(g \in G\) is an infinite order element, then \(g\) is a hyperbolic isometry.

Let \(g\) be a hyperbolic isometry of \(X\). Then the axis of \(g\) restricted to positive or negative half-line is a geodesic ray. We denote by \(g^\infty\) and \(g^{-\infty}\) the corresponding elements of \(\partial_\infty X\).

Let \(H\) be a subgroup of \(G\). The limit set of \(\Lambda H\) is the intersection of \(\partial_\infty X\) with the closure of an orbit \(Hx_0\) in \(X \cup \partial_\infty X\) for some (any) \(x_0\).

We point out two special cases of interest here.

- The limit set of an infinite cyclic group \(<g>\) where \(g\) is a hyperbolic isometry is the set \(\{g^\infty, g^{-\infty}\}\).
- The Flat Torus Theorem ([BH99, Thm II.7.1]) shows that if \(H = \mathbb{Z}^n\) for \(n > 1\), there exists an isometrically embedded \(n\)-dimensional closed, convex, flat subspace \(F\), in \(X\), on which \(H\) acts geometrically. Thus \(\Lambda H = \partial_\infty F\) which is homeomorphic to an \((n-1)\) sphere \(S^{n-1}\).

3. Limit operators and folding

This section follows [GS13]. We start with standard definitions of ultrafilters, Stone-Čech compactification, \(\omega\)-limits etc. For simplicity, we state those only in the context of a group.

An ultrafilter \(\omega\) on a group \(G\) is a collection of subsets of \(G\) such that

1. \(\emptyset \notin \omega\),
2. if \(A \in \omega\) and \(A \subseteq B \subseteq G\), then \(B \in \omega\),
3. if \(A, B \in \omega\), then \(A \cap B \in \omega\),
4. for every \(A \subseteq G\) either \(A \in \omega\) or \(G - A \in \omega\).

An ultrafilter is principal if there exists \(g \in G\) such that \(\{g\} \in \omega\). We note that by properties (1) and (3) of an ultrafilter, the element \(g\) such that \(\{g\} \in \omega\) is unique. We denote such ultrafilter by \(\rho_g\). We denote the set of all ultrafilters on \(G\) by \(\beta G\). The set \(\beta G\) can be equipped with topology generated by the sets of the form \(\{\omega \in \beta G : A \in \omega\}\) for subsets \(A \subseteq G\). The group \(G\) viewed as a discrete space embeds in \(\beta G\) via \(g \mapsto \rho_g\). The space \(\beta G\) is called the Stone-Čech compactification of \(G\).

Let \(G\) be a group acting on a CAT(0) space \(X\). We describe how the action of \(G\) on \(\overline{X} = X \cup \partial_\infty X\) extends to the action of \(\beta G\) on \(\overline{X}\).

Let \(v = (v_g)_{g \in G} \in \overline{X}^\beta\) be a \(G\)-sequence, i.e. a sequence of elements of \(\overline{X}\) indexed by \(G\), and let \(\omega \in \beta G\). The limit with respect to \(\omega\) is defined as follows.

**Definition 10** ([GS13, Defn 2.1]). We say that \(\xi \in \overline{X}\) is the \(\omega\)-limit of \(v\), and write \(\xi = \lim_\omega v_g\), if for every neighborhood \(U \subseteq \overline{X}\) of \(\xi\) we have \(\{g \in G : v_g \in U\}\) \(\subseteq \omega\).

The \(\omega\)-limit exists and is unique for every \(G\)-sequence \(v\) and every ultrafilter \(\omega\) (see [GS13, Prop 2.2]).

An important special case is when the \(G\)-sequence \(v\) is the orbit of a point, i.e. \(v_g = g \cdot \xi\) for some \(\xi \in \overline{X}\). Then the limit \(\lim_\omega g \cdot \xi\) is denoted by \(T^\omega(\xi)\). If \(\omega = \rho_a\) is the principal
ultrafilter determined by \( a \in G \), then \( T^\rho_\omega(\xi) = a \cdot \xi \) for every \( \xi \in X \). Suppose \( \omega \) is a non-principal ultrafilter. Then \( T^\omega(\xi) \in \partial X \) for all \( \xi \in X \). If \( x \in X \), then the value \( T^\omega(x) \) for \( x \in X \) is independent of the choice of \( x \) (see [GS13]), and we denote the point \( T^\omega(x) \in \partial X \) by \( \omega(\infty) \). Note that \( \omega(\infty) \) is an accumulation point of the \( G \)-orbit of \( x \).

Given an ultrafilter \( \omega \in \beta G \), the antipodal ultrafilter \( S\omega \) is defined as follows: for every \( F \subseteq G \) we have \( F \in S\omega \iff F^{-1} \in \omega \). We denote \( S\omega(\infty) \) by \( \omega(-\infty) \), and write \( n \sim p \) to denote \( \omega(-\infty) = n \) and \( \omega(\infty) = p \). Note that \( T^\omega(\omega(\infty)) = \omega(\infty) \), but in general \( T^\omega(\omega(-\infty)) \) and \( \omega(-\infty) \) might be distinct.

The group structure on \( G \) naturally extends to a semi-group structure on \( \beta G \). See [GS13] for details. We note the following fact.

**Lemma 11** ([GS13, Cor 2.12]). For every \( \xi \in X \) and for every \( \nu, \omega \in \beta G \) one has \( T^\nu T^\omega \xi = T^{\nu \omega} \xi \).

The function \( T^\omega : \partial_T X \to \partial_T X \) is \( 1 \)-Lipschitz in the Tits topology, but generally not continuous in the cone topology \( \partial_T X \). An important special case to consider is when \( G \) is hyperbolic and we construct \( \omega \) with a fixed hyperbolic element in mind.

**Example 12.** Suppose \( G \) is a hyperbolic group acting on a CAT(0) space \( X \), and \( g \in G \) is a hyperbolic element, and \( \omega_g \) is an ultrafilter containing all the sets \( \{ g^n : n \geq N_0 \} \) for all \( N_0 \in \mathbb{Z} \). Then \( T^{\omega_g}(\partial_T X) \) consists of two points \( g^\infty, g^{-\infty} \in \partial_T X \) where \( T^{\omega_g} \xi = g^\infty \) for all \( \xi \in \partial_T X - \{ g^{-\infty} \} \) and \( T^{\omega_g}(g^{-\infty}) = g^{-\infty} \).

Guralnik-Swenson generalize the above example where the two point set \( \{ g^\infty, g^{-\infty} \} \) is replaced by a top dimensional sphere in \( \partial_T X \). Indeed, the Tits boundary of a proper hyperbolic space is 0-dimensional and thus the pair \( \{ g^\infty, g^{-\infty} \} \) is a top-dimensional sphere in that boundary. One could say Theorem 14 is the higher dimensional version of this example.

A round sphere in \( \partial_T X \) is an isometrically embedded copy of \( S^d \) where \( d = \dim \partial_T X \).

**Theorem 13** ([GS13, Thm A]). Suppose \( G \) acts geometrically on a CAT(0) space \( X \) and let \( d \) be the geometric dimension of \( \partial_T X \). Then for every \((d+1)\)-flat \( F_0 \) in \( X \) there exists an ultrafilter \( \omega \in \beta G \) and a possibly different \((d+1)\)-flat \( F \subseteq X \) such that \( T^\omega \) maps \( \partial_T X \) to \( \partial F \) and is an isometry while restricted to \( \partial F_0 \).

We will use a slightly strengthened version of the theorem above in the situation where \( G \) contains a subgroup isomorphic to \( \mathbb{Z}^{d+1} \). Our proof follows quite closely the original proof from [GS13].

**Theorem 14.** Let \( G \) be a group acting geometrically on a CAT(0) space \( X \) and let \( d \) be the geometric dimension of \( \partial_T X \). Suppose that \( G \) has a subgroup \( H \cong \mathbb{Z}^{d+1} \). Then there exists an \( H \)-invariant isometrically embedded \((d+1)\)-flat \( F \) and \( \omega \in \beta G \) such that \( T^\omega(\partial_T X) = \partial F \) and \( T^\omega \) is an identity map while restricted to \( \partial F \).

To prove Theorem 14 we will need the following notion and lemmas from [GS13].

**Definition 15** ([GS13, Def 3.14]). We say \( \omega \in \beta G \) pulls away from a point \( \xi \in \partial_T X \) if there exists \( x \in X \), a ray \( \gamma \) with \( \gamma(\infty) = \xi \) and \( C > 0 \) such that \( A = A_{x,\gamma,M,C} = \{ g \in G \mid g \cdot x \in N_C(\gamma([M,\infty])) \} \) where \( M \in (0, \infty) \).

**Lemma 16** ([GS13, Lem 3.19]). Suppose \( S \subseteq \partial_T X \) is a sphere bounding a flat \( F \), and let \( \ell \subseteq F \) be a line joining points \( \xi, \eta \) in \( S \). If \( \omega \) pulls away from \( \xi \) then \( T^\omega S \) is a sphere bounding a flat isometric to \( F \). In particular, \( \omega(-\infty) = \xi, \omega(\infty) = T^\omega(\eta), \) and \( d_T(T^\omega(\xi), T^\omega(\eta)) = \pi \).
We say a subset $M \subseteq \partial_T X$ is $\pi$-convex if for any $p, q \in M$ such that $d_T(p, q) < \pi$ every geodesic joining $p, q$ is contained in $M$. Let $p, q$ be a pair of points in $\partial_T X$ such that $d_T(p, q) = \pi$. The suspension $\Sigma(p, q)$ is the set \{ $\xi \in \partial_T X \mid d_T(p, \xi) + d_T(\xi, q) = \pi$ \}. The suspension $\Sigma(p, q)$ is closed and $\pi$-convex in $\partial_T X$ [GS13, Lem 3.22]. A suspension point of a subset $M$ is a point $p \in M$ such that there exists $q \in M$ such that $M$ is the union of all geodesics in $M$ joining $p$ and $q$.

**Lemma 17** ([GS13, Cor 3.24]). Suppose $\omega$ pulls away from $\xi$ and there exists $\eta \in \partial_T X$ such that $d(\xi, \eta) = \pi$. Then $T^\omega$ maps $\partial_T X$ into the suspension $\Sigma(T^\omega(\xi), \omega(\infty))$.

We will also need the following corollary of Lemma 17.

**Lemma 18.** Let $a \in G$ be an infinite order element. Let $\omega \in \beta G$ contain all the sets $\{a^n\}_{n \geq N_0}$ for all $N_0$. Then $T^\omega(g^{\pm \infty}) = g^{\pm \infty}$ for all $g \in C(a)$ and $T^\omega$ maps $\partial_T X$ into $\Sigma(a^\infty, a^{-\infty})$.

**Proof.** Let $\omega \in \beta G$ contain all the sets $\{a^n\}_{n \geq N_0}$ for all $N_0$. Let $g \in C(a)$. To show that $T^\omega g^{\pm \infty} = g^{\pm \infty}$ we must check that $\omega$ contains the set $\{h \in G \mid h \cdot g^{\pm \infty} \in U\}$ for any neighborhood $U$ of $g^{\pm \infty}$. Note that $a^n \cdot g^{\pm \infty} = g^{\pm \infty}$ since $[a, g] = 1$. Thus for every neighborhood $U$ of $g^{\pm \infty}$ we have $a^n \cdot g^{\pm \infty} \in U$. It follows that $\{h \in G \mid h \cdot g^{\pm \infty} \in U\} \in \omega$ by Condition (2) from the definition of an ultrafilter. By Lemma 17 $T^\omega$ maps $\partial_T X$ into $\Sigma(a^\infty, a^{-\infty})$ since $T^\omega(a^{-\infty}) = a^{-\infty}$ and $\omega(\infty) = a^\infty$. \qed

We are now ready to prove Theorem 14.

**Proof of Theorem 14.** Let $\{a_0, \ldots, a_d\} \subseteq H$ be a generating set of $H$. By the Flat Torus Theorem ([BH99], Thm II.7.1), we can choose an $H$-invariant $(d + 1)$-flat $F$ on which $H$ acts via a torus action. This implies that for each $h \in H$, $F \subseteq \text{Min}(h)$. Let $S = \partial F$.

Throughout the proof, $\omega_g$ with $g \in G$ denotes an ultrafilter which contains all the sets $\{g^n\}_{n \geq N_0}$ for $N_0 \in \mathbb{Z}$. Let $i \in \{0, \ldots, d\}$. By Lemma 18 $T^{\omega_{a_i}}$ maps $\partial_T X$ into $\Sigma(a^\infty, a_i^{-\infty})$, and $T^{\omega_{a_i}}(h^{\pm \infty}) = h^{\pm \infty}$ for every $h \in H$. In particular, $T^{\omega_{a_i}}$ restricts to identity on $S$.

Let $\omega = \omega_{a_d} \cdots \omega_{a_0}$. Let $M$ denote the image of $\partial_T X$ under $T^\omega = T^{\omega_{a_d}} \cdots T^{\omega_{a_0}}$ (see Lemma 11). Note that $T^\omega$ restricts to identity on $S$. In particular, $S \subseteq M$. Our goal is to prove $M \subseteq S$.

Let $i \in \{0, \ldots, d\}$. We have

$$M = T^\omega(\partial_T X) \subseteq T^{\omega_{a_d} \cdots \omega_{a_i}}(M) \subseteq T^{\omega_{a_d} \cdots \omega_{a_i+1}}(\Sigma(a^\infty, a_i^{-\infty})).$$

We also have

$$T^{\omega_{a_d} \cdots \omega_{a_i+1}}(\Sigma(a^\infty, a_i^{-\infty})) \subseteq \Sigma(T^{\omega_{a_d} \cdots \omega_{a_i+1}}(a^\infty), T^{\omega_{a_d} \cdots \omega_{a_i+1}}(a_i^{-\infty})) = \Sigma(a^\infty, a_i^{-\infty}),$$

so we have shown that $M \subseteq \Sigma(a^\infty, a_i^{-\infty})$ for every $i = 0, \ldots, d$. Let $Y$ be the intersection of all the suspensions $\Sigma(p, q)$ that contain $M$ as a subspace. The space $Y$ is closed, $\pi$-convex and points $a^\pm \infty$ for $i = 0, \ldots, d$ belong to the set of its suspension points. Therefore $S$ belongs to the set of its suspension points. By [GS13, Thm 1.16] $Y = S(Y) \ast E'(Y)$ where $S(Y)$ is the set of suspension points of $Y$. Since $S \subseteq S(Y)$ and since the geometric dimension of $S$ is equal to the geometric dimension of $\partial_T X$, the subspace $E'(Y)$ must be empty. Therefore $M \subseteq Y = S$ which completes the proof. \qed
Corollary 19. Let $G$ be a lattice in a product of trees acting freely and vertex-transitively and let $a \in A, b \in B$ commute where $A, B$ are as in Section 2.1. Then there exists $\omega \in \beta G$ such that $\partial_T X$ folds onto $\Lambda(a, b) \subseteq \partial_T X$.

4. SPLITTING

The goal of this section is to prove the following theorem.

**Theorem 20.** Let $G$ be a lattice in a product of trees acting freely and vertex-transitively. Suppose $G$ acts geometrically on a CAT(0) space $X$. Then a quasi-dense, closed, convex subspace $X' \subseteq X$ splits as a nontrivial product. In particular, $\partial_\infty X$ and $\partial_T X$ split as nontrivial joins.

**Remark 21.** In the special case when $X$ is a cube complex and $X$ is either geodesically complete or the action of $G$ on $X$ is essential, we can use rank rigidity for CAT(0) cube complexes to prove the above theorem.

Every element of $G$ bounds a half-plane in $T_1 \times T_2$, so $G$ contains no rank one elements. By rank rigidity for CAT(0) cube complexes $[CS11, Cor B]$, $X$ is a product of two CAT(0) cube complexes $X_1 \times X_2$. Thus $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$.

As discussed in Section 2.1, $G$ admits a group presentation with the two sets of generators $A, B$ and all the relations of the form $aba'b'$ for some $a, a' \in A$ and $b, b' \in B$.

**Lemma 22.** Let $G$ be a lattice in a product of trees acting freely and vertex-transitively and suppose $G$ act geometrically on a CAT(0) space $X$. Then the limit set $\Lambda_A := \Lambda(A)$ is a closed, non-empty, proper $G$-invariant subset of $\partial_\infty X$.

**Proof.** Limit set of every subgroup is a closed subset of $\partial_\infty X$. For any infinite order element $a \in \langle A \rangle$, the points $a^\pm \infty$ are accumulation points of the set $\{a^k x_0\}_{k \in \mathbb{Z}}$, and hence belong to $\Lambda_A$, showing that the limit set is non-empty.

Let $b \in \langle B \rangle$ be an infinite order element. We claim that $b^{+\infty}$ is not contained in $\Lambda_A$. If a sequence $\{a_n x_0\}_{n \in \mathbb{N}}$ accumulates at $b^{+\infty}$, then it must stay within a finite Hausdorff distance from $\{b^n x_0\}_{n \in \mathbb{N}}$. This implies that in the product of trees $T_1 \times T_2$ (viewed as the Cayley complex of $G$) the sets $\{b^n\}_{n \in \mathbb{N}}$ and $\{a_n\}_{n \in \mathbb{N}}$ have finite Hausdorff distance, but this is a contradiction, since $d(b^n, A) \geq n$ for every $n \in \mathbb{N}$. Thus $\Lambda_A$ is a proper subset of $\partial_\infty X$.

It remains to prove that $\Lambda_A$ is $G$-invariant. It suffices to verify that $b \Lambda_A = \Lambda_A$ for any generator $b \in B$. By Observation 3, $b(A) x_0 \subseteq \bigcup_{y \in B^{\pm 1}} (A) b'y_0$. It follows that $b \Lambda_A \subseteq \Lambda_A$. Analogous argument shows that $b^{-1} \Lambda_A \subseteq \Lambda_A$, which implies $b \Lambda_A = \Lambda_A$. \qed

The following is proven by Ricks in [Ric20] although it is not stated explicitly there. We include the proof referring to theorems in [Ric20].

**Theorem 23 ([Ric20]).** Let $G$ be any group acting geometrically on a CAT(0) space $X$ with 1-dimensional Tits boundary $\partial_T X$ and $|\partial X| \geq 3$. Suppose there exists a closed, non-empty, proper $G$-invariant subset $\Lambda$ of $\partial_\infty X$, and a folded circle $S \subseteq \partial_T X$ such that the intersection $S \cap \Lambda$ consists of at most two antipodal (with respect to $S$) points. Then a quasi-dense, closed, convex subspace $X' \subseteq X$ splits as a nontrivial product. In particular, $\partial_\infty X$ and $\partial_T X$ split as nontrivial spherical joins.

**Proof.** By [Ric20, Cor 22], $\text{radius}_{\partial X}(\Lambda) \leq \frac{\pi}{2}$. If the action of $G$ on $X$ has a global fixed point, then by [PS09, Lem 26], $G$ virtually has $\mathbb{Z}$ as a direct factor. This is a contradiction.
By [Ric20, Corollary B], \( X \) contains a quasi-dense, closed, convex subspace \( X' \subseteq X \) which splits as a nontrivial product, \( X' = X_1 \times X_2 \) for closed, convex subsets \( X_1, X_2 \). Since the boundaries of \( X' \) and \( X \) are set-wise equal, [BH99, Ex II.8.11(6) and Cor II.9.11] imply that \( \partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2 \).

\( \Box \)

**Proof of Theorem 20.** The boundary \( \partial_\infty X \) has topological dimension 1, since the dimension of a visual boundary is a quasi-isometry invariant (see Theorem 7). This implies that the topological dimension of \( \partial_T X \), and by [Kle99] also the geometric dimension of \( \partial_T X \), is at most 1. By Proposition 4, \( G \) contains a \( \mathbb{Z}^2 \) subgroup, so there is a circle contained in \( \partial_T X \). Therefore the dimension of \( \partial_T X \) is 1 and \( \partial_T X \) has at least three points.

By Lemma 22, \( \Lambda_A \) is a closed, non-empty, proper \( G \)-invariant subset of \( \partial_\infty X \). By Proposition 4, there exists \( a \in \langle A \rangle \) and \( b \in \langle B \rangle \) such that \([a, b] = 1\). By Corollary 19, \( S := \Lambda\langle a, b \rangle \) is a round, folded circle. Clearly \( S \cap \Lambda_A = \{ a^{+\infty}, a^{-\infty} \} \) which is a pair of antipodal points on \( S \). Now, the claim follows from Theorem 23.

\( \Box \)

5. Analyzing the join factors

Let \( G \) be a lattice in a product of trees acting freely and vertex-transitively. In Theorem 20 we have shown that if \( G \) acts geometrically on a CAT(0) space \( X \), then \( G \) acts minimally on a quasi-dense convex closed subspace \( X' \subseteq X \) which splits as a product \( X_1 \times X_2 \) where \( X_1, X_2 \) are closed convex subsets of \( X \) with 0-dimensional visual boundaries. In particular, \( \partial_\infty X \) splits as a nontrivial join of the form \( \partial_\infty X_1 \ast \partial_\infty X_2 \). In this section, we complete the proof of the main theorem by proving that \( \partial_\infty X_1 \) are both Cantor sets.

For completeness, we point out the three possibilities that can occur when \( \partial_\infty X \) is the join of 0-dimensional subspaces.

**Proposition 24.** Let \( G \) be any group acting geometrically on a CAT(0) space \( X \) with topological dimension of \( \partial_\infty X = 1 \) and \( \partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2 \) for closed convex subspaces \( X_1, X_2 \) in \( X \). Then exactly one of the following holds:

1. \(|\partial X_i| = 2\) for \( i = 1, 2 \). In this case, \( \partial_\infty X \) is homeomorphic to a circle and \( G \) is either virtually abelian or Fuchsian.
2. \(|\partial X_1| = 2\) and \(|\partial X_2| \geq 3\) (or vice versa). In this case \( \partial_\infty X \) is homeomorphic to the suspension of a Cantor set and \( G \) is virtually \( F \times \mathbb{Z} \) where \( F \) is finitely generated and free.
3. \(|\partial X_i| \geq 3\) for both \( i = 1, 2 \). In this case, \( \partial_\infty X = C \ast C \), the join of two Cantor sets. Each of the subspaces \( X_i \) is quasi-isometric to a simplicial tree.

The first two cases follow directly from [Rua06]. The statements in that paper assume geodesic completeness on \( X \), however one can pass to a quasi-dense subset and the conclusions about the boundary and the groups still hold.

The remainder of this section is devoted to proving that the third case is the only remaining case and that for our groups \( G \), we are in that case.

The subtlety involved here is that when the group \( G \) is not (virtually) a product of free groups, we do not have geometric group actions on \( X_i \).

**Lemma 25.** Suppose \( G \) acts geometrically on a CAT(0) space \( X \) with topological dimension of \( \partial_\infty X = 1 \) and \( \partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2 \) for closed convex subspaces \( X_1, X_2 \) in \( X \). Then the boundary \( \partial_\infty X_i \) is homeomorphic to \( \text{Ends}(X_i) \), the ends space of \( X_i \).
Proof. Let us first show that each $X_i$ is hyperbolic. Note that since $X$ is proper, so are $X_1$ and $X_2$.

First, we show that the subspace $X_i$ for $i = 1, 2$ is a visibility space i.e. given any two distinct points $\xi, \eta$ in $\partial_\infty X_i$, there is a geodesic line in $X_i$ between them. Suppose there is no geodesic line in $X_i$ between $\xi$ and $\eta$. Then by [BH99, Prop. II.9.21(2)], there is a geodesic segment in $\partial_T X_i$ joining them. This segment is an arc in $\partial_T X_i$ which would map to an arc in $\partial_\infty X_i$ via the identity map on $\partial X_i$. This contradicts the fact that $\partial_\infty X_i$ is 0-dimensional.

Moreover, the action of $G$ on $X_i$ is cocompact, since so is the action of $G$ on $X$. Hence, $X_i$ is uniformly visible by [BH99, Prop II.9.32]. Finally, $X_i$ is hyperbolic by [BH99, Prop III.1.4].

Since $X_i$ is a proper hyperbolic space, the natural map $\partial_\infty X_i \to \text{Ends}(X_i)$ is continuous and the fibers of that map are the connected components of $\partial X_i$ [BH99, Exer III.H.3.9]. Since $\partial_\infty X_i$ is 0-dimensional, the connected components are single points. Thus the map $\partial_\infty X_i \to \text{Ends}(X_i)$ is a continuous bijection. Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism. \qed

In [Bes96, Def 1.1], Bestvina outlined a set of axioms that a group boundary should have in order to be useful for relating homological invariants of the boundary to cohomological invariants of the group. All of the axioms hold true for a hyperbolic group $G$ acting on $G \cup \partial_\infty G$ and for a CAT(0) group $G$ acting on $X \cup \partial_\infty X$ where $G$ admits a geometric action on the CAT(0) space $X$. One of the axioms requires the collection of translates of any compact set to form a null set in $X \cup \partial_\infty X$ - i.e. for any open cover $U$ of $X \cup \partial_\infty X$ and any compact set $K$ in $X$, all but finitely many $G$-translates of $K$ are contained in an element of $U$.

The next lemma shows that each $X_1 \cup \partial_\infty X_i$ inherits this nullity condition on compact sets from $X \cup \partial_\infty X$ even though we have no geometric group action on $X_i$.

**Lemma 26.** Suppose $G$ acts geometrically on a CAT(0) space $X$ with $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$ for closed convex subspaces $X_1, X_2$ in $X$. For every compact set $K \subseteq X_i$ and every open neighborhood of an end of $X_i$, there exists $g \in G$ such that $gK \subseteq U$.

**Proof.** We show that the lemma holds for $X_1$. The argument for $X_2$ is identical. Let $K \subseteq X_1$ and $K' \subseteq X_2$ be compact sets. Then $K \times K'$ is a compact set in $X$. By Lemma 25, $\partial_\infty X_1$ is homeomorphic to $\text{Ends}(X_1)$. Let $\xi \in \partial_\infty X_1$, and let $U$ be an open neighborhood of $\xi$. Then $U \times X_2$ is an open neighborhood of $\xi$, viewed as a point in $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$. Since the action of $G$ on $X$ is geometric, there exists $g \in G$ such that $g(K \times K') \subseteq U \times X_2$ (see e.g. [Bes96, p.124]). It follows that in the action of $G$ on $X_1$, we have $gK \subseteq U$. \qed

The following is not stated explicitly, but it is proved in [Hop44]. It is also proved in a similar form in [Ber05]. We include the proof for completeness. We restrict our attention to the case of geodesic metric spaces, but the proposition holds in more general setting.

**Proposition 27** ([Hop44]). Let $Y$ be a geodesic metric space with at least three ends. Let $G$ be a group acting on $Y$ cocompactly so that the following holds: for every compact set $K \subseteq Y$, and every open neighborhood $U$ of an end of $Y$, there exists $g \in G$ such that $gK \subseteq U$. Then the space of ends $\text{Ends}(Y)$ is perfect.

**Proof.** Suppose that there exists an end $e \in \text{Ends}(Y)$ that is isolated, i.e. there exists a neighborhood $U$ of $e$ that does not contain any other ends. First we show that without loss
of generality, we can assume that $Y - U$ is connected. Indeed, if $Y - U$ is not connected, we construct a neighborhood $U' \subseteq U$ of $e$ such that $Y - U'$ is connected. Since $\hat{Y} - U$ is compact, so is $\hat{Y} - U$. Let $V_1, \ldots, V_n$ be a finite collection of open sets covering $\hat{Y} - U$ such that each $V_i$ is connected and does not contain $e$, and its closure $\hat{V}_i$ in $\hat{Y}$ is compact. The union $\bigcup_{i=1}^n V_i$ has finitely many components, and since $\text{Ends}(Y)$ is nowhere dense in $\hat{Y}$, each $V_i$ contains points of $Y$. Each two points in $Y$ can be joined by a path in $Y$. In particular, there exists a closed connected set $Q$ which is the union of $\bigcup_{i=1}^n V_i$ and a finite number of paths in $Y$.

Note that $Q$ does not contain $e$. The set $U' = \hat{Y} - Q$ is an open neighborhood of $e$ and since $Y - U \subseteq \bigcup V_i \subseteq Q$, we have $U' \subseteq U$. Thus, $U'$ is the neighborhood we were looking for.

By assumption, there are at least three distinct ends $e_1, e_2, e_3$ in $Y$. Let $K$ be a compact set in $Y$ such that each of the ends $e_1, e_2, e_3$ lies in a different connected component $Y_1, Y_2, Y_3$ of $Y - K$. By assumption, there exists an element $g \in G$ such that $gK \subseteq U$. We claim that for each $i = 1, 2, 3$, either $gY_i \subseteq U$, or $g(Y - Y_i) \subseteq U$. Indeed, otherwise there exists point $p \in gY_i - U$ and $q \in g(Y - Y_i) - U$. Since $Y - U$ is connected, there exists a path $\gamma$ in $Y - U$ joining $p$ and $q$. Note that $p, q$ lie in distinct connected component of $Y - gK$, so $\gamma$ has to pass through $gK$. This is a contradiction, since $gK \subseteq U$.

Since $U$ contains only one end, we have $g(Y - Y_i) \subseteq U$ for at least two $i$’s among $1, 2, 3$, say $1$ and $2$. It follows that $Y - U \subseteq gY_i$ for $i = 1, 2$. The subsets $Y_1$ and $Y_2$ are disjoint, and so are $gY_1$ and $gY_2$. This is a contradiction and therefore the space of $\text{Ends}(Y)$ has no isolated points.

\begin{lemma}
Suppose $G$ acts geometrically on a CAT(0) space $X$ with dimension of $\partial_\infty X = 1$ and $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$ for closed convex subspaces $X_1, X_2$ in $X$. If $|\partial X_i| \geq 3$, then $\partial X_i$ is homeomorphic to the Cantor set $C$.
\end{lemma}

\begin{proof}
Since $X_i$ is a proper CAT(0) space, the boundary $\partial X_i$ is compact and metrizable [BH99]. As $\dim \partial_\infty X_i = 0$, the boundary $\partial_\infty X_i$ is totally disconnected. The action of $G$ on $X_i$ is cocompact, since so is the action of $G$ on $X$. By Lemma 26, for every compact set $K \subseteq X_i$ and every open neighborhood $U \subseteq X_i$ of an end of $X_i$, there exists $g \in G$ such that $gK \subseteq U$. By Proposition 27, $\text{Ends}(X_i)$ is a perfect space. This implies that $\partial_\infty X_i$ is a perfect space because $\text{Ends}(X_i) = \partial_\infty X_i$ by Lemma 25. By the characterization of the Cantor set, as a non-empty, perfect, totally disconnected, compact metrizable space, we conclude that $\partial X_i$ is the Cantor set.
\end{proof}

\begin{proof}[Proof of Proposition 24]
Since $\partial_\infty X$ has dimension 1, both $\partial X_1$ and $\partial X_2$ are non-empty. Now suppose that $|\partial X_1| = 1$. Note that an $S^1$ in $\partial_\infty X$ would have to have two points in each of $\partial X_1, \partial X_2$. Thus if $|\partial X_1| = 1$, then $\partial_\infty X$ does not contain any copies of $S^1$. By [BH99, Thm III.1.5] $G$ must be hyperbolic and $\partial_\infty X$ is $G$-equivariantly homeomorphic to the Gromov boundary of $G$.

Since $\partial_\infty X = \partial_\infty X_1 \ast \partial_\infty X_2$ with each factor non-empty, $\partial_\infty X$ contains an interval and thus $G$ must be non-elementary. We note that the unique point $\xi$ in $\partial X_1$ is topologically distinguishable, unless $\partial X_2$ also has one point. In either case, $G$ or a subgroup of index $2$ of $G$ must fix $\xi$. This contradicts the convergence group action of $G$ on its boundary. Thus each of $\partial X_1, \partial X_2$ has at least two points.

If $|\partial X_1| = |\partial X_2| = 2$, then clearly $\partial_\infty X \simeq S^1$ and by [Rua06, Thm 3.5] $G$ is either virtually abelian or Fuchsian. We note that the proof of that result follows without assuming geodesic completeness after passing to a closed, convex, quasi-dense subset of $X$. 

\end{proof}
Now assume $|\partial X_1| = 2$ and $|\partial X_2| \geq 3$. By Lemma 28, $\partial_\infty X_2$ is homeomorphic to a Cantor set $C$. Hence $\partial_\infty X$ is a suspension of a Cantor set. By [Rua06, Thm 4.4] $G$ is virtually $F \times \mathbb{Z}$ where $F$ is a finitely generated free group.

Finally, suppose that $|\partial X_i| \geq 3$ for $i = 1, 2$. By Lemma 28, $\partial_\infty X_i$ is homeomorphic to a Cantor set $C$. Consequently, $\partial_\infty X$ is homeomorphic to join of two copies of a Cantor set.

We now show that $X_i$ is quasi-isometric to a tree, following [Cha14]. By [Cha14, Lem 5.7], $X_i$ has the bottleneck property, and by [Man05, Thm 4.6] $X_i$ is quasi-isometric to a simplicial tree.

**Proposition 29.** Let $G$ be a lattice in a product of trees acting freely and vertex-transitively. Suppose $G$ acts geometrically on a CAT(0) space $X$ where $\partial_\infty X = \partial_\infty X_1 \star \partial_\infty X_2$ for non-empty, closed, convex subspaces $X_1, X_2$ in $X$. Then each $\partial_\infty X_1, \partial_\infty X_2$ is homeomorphic to the Cantor set and each subspace $X_i$ is quasi-isometric to a tree.

**Proof.** By Lemma 4, $G$ contains a $\mathbb{Z}^2$ subgroup. By the Flat Torus theorem there is a flat $F$ embedded in $X$ as a convex subspace. Thus, $\partial_\infty F \simeq S^1$ embeds in $\partial_\infty X$. In particular, both $|\partial X_1|, |\partial X_2| \geq 2$. Since $G$ is neither virtually abelian, $F \times \mathbb{Z}$ or Fuchsian, by Proposition 24 we conclude that $\partial_\infty X \simeq C \star C$ and each of the factors $X_i$ is quasi-isometric to a tree.

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