CLEANLINESS VERSUS SPECIALNESS

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Abstract. We show that the fundamental group of a geometrically clean graph of finite rank free groups does not need to be virtually compact special, answering a question of Wise. This implies that the class of the virtually $\mathcal{VH}$-clean graphs of finite rank free groups is a proper subclass of the class of virtually geometrically clean graphs of finite rank free groups.

1. Introduction and background

We assume the reader's familiarity with the notions of graphs of spaces and graphs of groups, and refer to [SW79] for the background.

Let Γ be a directed graph, possibly with loops and multiple edges. Let $X(\Gamma)$ be a *graph of graphs* (i.e. a graph of spaces whose vertex and edge spaces are graphs), with vertex graph X_v for $v \in V(\Gamma)$, edge graphs X_e for $e \in E(\Gamma)$, and edge maps $f_e^\iota : X_e \to X_{\iota(e)}$ and $f_e^\tau : X_e \to X_{\tau(e)}$, where $\iota(e), \tau(e)$ denote the initial and terminal vertex of the edge e.

Let $G(\Gamma)$ be the associated graph of groups, i.e. the vertex groups are $G_v = \pi_1(X_v)$ for $v \in V(\Gamma)$, the edge groups are $G_e = \pi_1(X_e)$ for $e \in E(\Gamma)$, and the edge maps are $\phi_e^\iota = (f_e^\iota)_* : G_e \to G_{\iota(e)}$ and $\phi_e^\tau = (f_e^\tau)_* : G_e \to G_{\tau(e)}$. If all X_v, X_e are finite graphs, then all G_v, G_e are finite rank free groups. In such a case, we say that $G(\Gamma)$ is a *graph of finite rank free groups.*

Recall that a map $f : X \to Y$ of graphs is *combinatorial* if it maps vertices to vertices, and each open edge is mapped homeomorphically onto an open edge.

Definition 1. A graph of graphs $X(\Gamma)$ is

- *$\mathcal{VH}$-clean* if each map f_e^ι and f_e^τ is a combinatorial embedding,
- *geometrically clean* if each map f_e^ι and f_e^τ is an embedding,
- *algebraically clean* if each homomorphism ϕ_e^ι and ϕ_e^τ is an injection onto a free factor.

Similarly, we say that a graph of free groups $G(\Gamma)$ is clean in one of the senses above, if it is associated with a graph of graphs with the same property. As usual, we say $G(\Gamma)$ is *virtually* clean in one of the senses above, if its fundamental group has a finite index subgroup that splits as a graph of graphs with that property.

Clearly, every $\mathcal{VH}$-clean graph of groups is geometrically clean, and every geometrically clean graph of groups is algebraically clean. One can easily construct examples of graphs of groups which are geometrically clean but not $\mathcal{VH}$-clean, or algebraically clean but not geometrically clean. However, a more interesting question arises when we consider all possible splittings of a given group as a graph of finite rank free groups, and allow to pass to finite index subgroups.

Question 2. Are the classes of groups that virtually splits as

- $\mathcal{VH}$-clean graphs of finite rank free groups,

- geometrically clean graphs of finite rank free groups,
- algebraically clean graphs of finite rank free groups

distinct?

The total space of a $\mathcal{VH}$-clean graph of graphs can be realized asa clean $\mathcal{VH}$ *square complexes*, as first introduced by Bridson-Wise [BW99] ($\mathcal{VH}$ stands for *vertical* and *horizontal*). Wise further studied and utilized the above notions of the cleanliness in [Wis06, Wis02]. In particular, in [Wis02] he gave the definitions of algebraically clean (referred to as "clean") and geometrically clean graphs of groups, and showed that the fundamental groups of algebraically clean graphs of finite rank free groups are residually finite.

Next, Haglund-Wise showed that $\mathcal{VH}$-cleanliness implies virtual specialness.

Theorem 3 ([HW08, Thm 5.7])**.** Every group which splits as a $\mathcal{VH}$-clean graph of finite rank free groups is virtually compact special.

More recently, the author and Schreve showed that the fundamental groups of algebraically clean graphs of finite rank free groups are also cohomologically good, and for each prime p virtually residually p-finite [JS25]. The author has also showed that many Artin groups virtually split as algebraically clean graphs of finite rank free groups [Jan22, Jan24], however many of them are not virtually compact special [HJP16, Hae21]. Thus the class of groups which virtually split as $\mathcal{VH}$-clean graphs of finite rank free groups is properly contained in the class of groups which virtually split as algebraically clean graphs of finite rank free groups, which partially answer Question 2.

Wise asked[1] whether every geometrically clean graph of finite rank free groups is virtually compact special. We answer this question negatively.

Theorem 4. There exists a geometrically clean graph of finite graphs whose fundamental group is not virtually compact special.

An example of a group satisfying Theorem 4 is the group G which is the fundamental group of the graph of graphs $G(\Theta)$ constructed in Section 2 and illustrated in Figure 1. Algebraically, this group is a free product of two copies of the free group F_4 of rank four, amalgamated three times, each time along a copy of F_3. In Section 2, we also show that this group embeds as a finite index subgroup in a certain Artin group, see Proposition 8. This embedding is similar to the proof that many Artin groups are virtually algebraically clean given in [Jan22, Jan24]. However, there we never explicitly construct algebraically clean graphs of finite rank free groups, only argue that such covers must exist. For the groups considered in [Jan22, Jan24], the algebraically clean graphs fail to be geometrically clean.

As a corollary we obtain a further partial answer to Question 2.

Corollary 5. The class of groups which virtually split as $\mathcal{VH}$-clean graphs of finite rank free groups is properly contained in the class of groups which virtually split as geometrically clean graphs of finite rank free groups.

The following remains open.

Question 6. Does there exist a group that splits as an algebraically clean graph of finite rank free groups, and that does not virtually split as a geometrically clean graph of finite rank free groups?

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[1] Private communication.

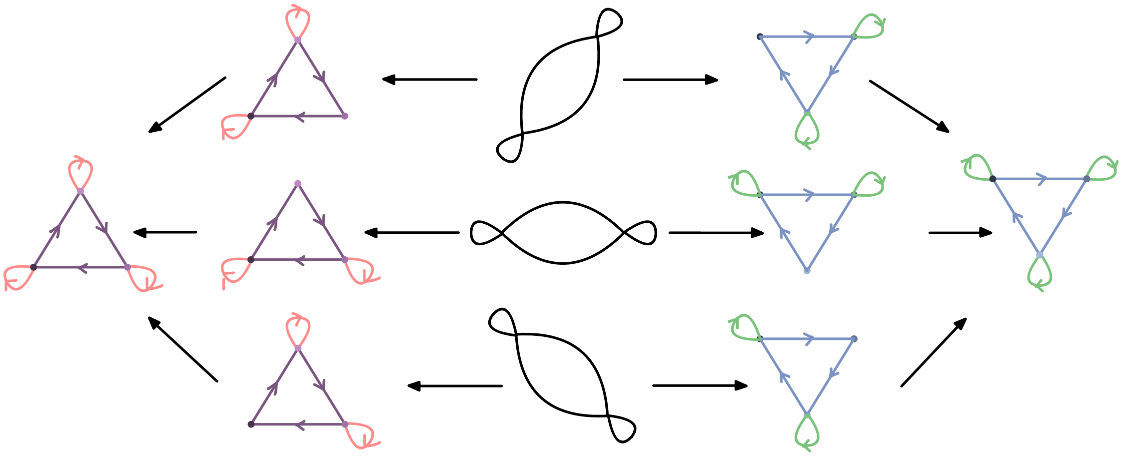

Figure 1. A geometrically clean graph of graphs $X(\Theta)$, whose fundamental group $G(\Theta)$ is not virtually compact special.

2. THE CONSTRUCTION OF THE GRAPH OF GROUPS $G(\Theta)$

Let Θ be the theta-graph, i.e. a graph with two vertices v, w and three edges e_1, e_2, e_3, each with initial vertex v and the terminal vertex w. Let $X(\Theta)$ be a graph of graphs illustrated in Figure 1. I.e. each vertex graph X_v, X_w is a triangle (length 3 cycle) with a loop attached to each vertex, and each edge group X_{e_i} is a bigon (length 2 cycle) with a loop attached to each vertex. The edge maps $f^{\iota}_{e_i}$ and $f^{\tau}_{e_i}$ are (non-combinatorial) embeddings, as illustrated in Figure 1. Let $G(\Theta)$ be the associated graph of groups, i.e. $G(\Theta)$ is the following graph of groups

$$\begin{tikzcd} G(\Theta) = F_4 \arrow[r, dash, bend left=60, "F_3"] \arrow[r, dash, "F_3"] \arrow[r, dash, bend right=60, "F_3"'] & F_4 \end{tikzcd}$$

where the inclusions of the copies of F_3 in F_4 are given by the inclusions of graphs in Figure 1. Let G be the fundamental group of $G(\Theta)$.

The following is immediate.

Proposition 7. The graph of graphs $X(\Theta)$ (and consequently also the graph of groups $G(\Theta)$) is geometrically clean.

Let $A_{2,3,\infty}$ be the Artin group with defining graph a path of length two with labels 2 and 3, i.e.

$$A_{2,3,\infty} = \langle a, b, c \mid ab = ba, bcb = cbc \rangle.$$

Proposition 8. The group G embeds as an index 6 subgroup in the Artin group $A_{2,3,\infty}$.

Proof. First we rewrite the standard presentation of $A_{2,3,\infty}$.

$$A_{2,3,\infty} = \langle a, b, x \mid ab = ba, bxb = x^2 \rangle.$$

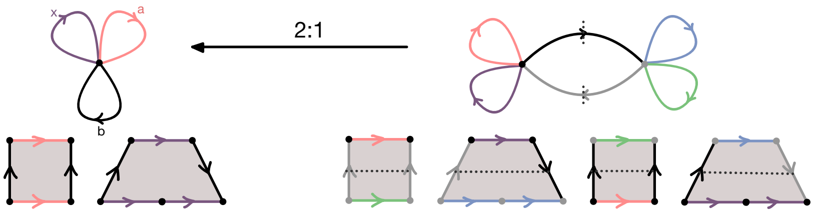

FIGURE 2. On the left: a presentation complex of $A_{2,3,\infty}$. On the right: its double cover whose fundamental group is the kernel of a homomorphism to $\mathbb{Z}/2$.

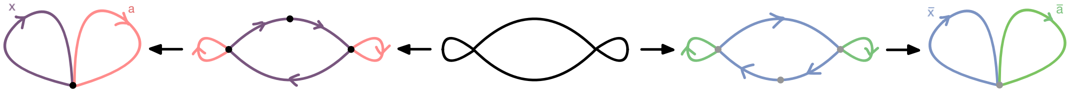

FIGURE 3. The group $\pi_1\hat{Y}$ splits as an amalgamated product $F_2 *_{F_3} F_2$ with the two maps $F_3 \to F_2$ induced by the graph maps above.

Indeed, by setting $x = cb$, the relation $bxb = x^2$ is equivalent to $bcb = cbc$. The presentation complex Y of the new presentation of $A_{2,3,\infty}$ is on the left side of Figure 2. Consider the homomorphism $h : A_{2,3,\infty} \to \mathbb{Z}/2$ sending a, x to 0, and b to 1. The kernel $\ker h$ is an index 2 subgroup of $A_{2,3,\infty}$ whose presentation complex $\hat{Y}$ is the double cover of Y illustrated on the right side of Figure 2.

The complex $\hat{Y}$ can be realized as the total space of the graph of spaces as follows. The underlying graph is a single edge. The vertex spaces are (1) the wedge of the pink and purple loops, denoted by Y_{v_1}, and (2) the wedge of the green and blue loops, denoted by Y_{v_2}. The edge space Y_e is the dotted horizontal graph on the right side of Figure 2, which is homeomorphic to a bigon with a loop at each vertex. Indeed, the complex $\hat{Y}$ is obtained as $(Y_{v_1} \cup Y_e \times [0,1] \cup Y_{v_2})/\sim$ where $(y, 0) \sim f_e^{v_1}(y)$ and $(y, 1) \sim f_e^{v_2}(y)$ for $y \in Y_e$, with the gluing maps $f_e^{v_1}, f_e^{v_2}$ illustrated in Figure 3. The black and gray edges of $\hat{Y}$ (which each map in Y to the loop labelled b) are contained in $Y_e \times [0,1]$ as the product of one of the vertices of Y_e with the interval.

Algebraically, this yields a splitting of $\pi_1\hat{Y}$ as an amalgamated product $F_2 *_{F_3} F_2$. The amalgamating subgroup F_3 maps to the first copy of $F_2 = \langle a, x\rangle$ as $\langle x^3, a, x^{-1}ax\rangle$, and to the second copy of $F_2 = \langle \bar{a}, \bar{x}\rangle$ as $\langle \bar{x}^3, \bar{a}, \bar{x}\bar{a}\bar{x}^{-1}\rangle$, and the identification of these two F_3 subgroups is via an isomorphism sending $x^3 \mapsto \bar{x}^3, a \mapsto \bar{a}, x^{-1}ax \mapsto \bar{x}^{-2}\bar{a}\bar{x}^2$.

In particular, a homomorphism $k : \pi_1\hat{Y} \to \mathbb{Z}/3$ sending $x, \bar{x} \to 1$ and $a, \bar{a} \to 0$ is well-defined, and constant while restricted to the amalgamating subgroup. The kernel $\ker k$ is an index 3 subgroup of $\pi_1\hat{Y}$ which splits as a graph of groups $G(\Theta)$, and the corresponding triple cover of $\hat{Y}$ splits as a graph of graphs $X(\Theta)$, illustrated in Figure 1. This proves that the fundamental group G of $G(\Theta)$ embeds as an index 6 $(= 2 \cdot 3)$ subgroup of $A_{2,3,\infty}$.

□

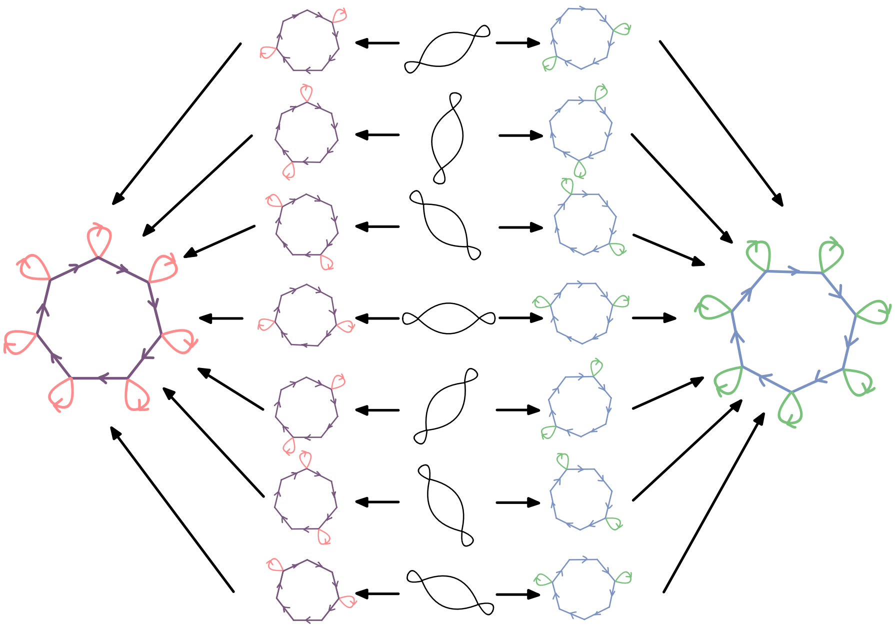

FIGURE 4. Geometrically clean graph of free groups, whose vertex group are copies of F_8, and edge groups are copies of F_3, whose fundamental group embeds as an index 14 subgroup of the Artin group $A_{2,7,\infty}$.

By [HJP16] or [Hae21] $A_{2,3,\infty}$ is not virtually cocompactly cubulated, so in particular not virtually cocompact special. The group G, as a finite index subgroup of $A_{2,3,\infty}$ by Proposition 8, has the same properties.

Corollary 9. The group G is not virtually cocompactly cubulated, so in particular not virtually compact special.

3. AN INFINITE FAMILY OF EXAMPLES

Any finite index subgroup of the group G constructed in the previous section splits as a geometrically clean graph of finite rank free groups, and is not virtually compact special, hence not virtually $\mathcal{VH}$-clean.

More examples arise from Artin groups $A_{2,n,\infty}$ for odd $n \geq 3$. These are also not virtually compact special [HJP16]. The construction presented in the previous section generalizes to all Artin group $A_{2,n,\infty}$ for odd $n \geq 3$, and yields an index $2n$ subgroup of $A_{2,n,\infty}$ which is a geometrically clean graph of free groups. The underlying graphs is a Θ_n-graph (i.e. a graph on two vertices with n edges joining the two vertices) of groups, with vertex groups isomorphic to F_{n+1}, and all n edge groups isomorphic to F_3. See Figure 4 for the resulting graph of groups where $n = 7$.

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