Robust Control Barrier and Control Lyapunov Functions with Fixed-Time Convergence Guarantees

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Abstract—This paper studies control synthesis for a general class of nonlinear, control-affine dynamical systems under additive disturbances and state-estimation errors. We enforce forward invariance of static and dynamic safe sets and convergence to a given goal set within a user-defined time in the presence of input constraints. We use robust variants of control barrier functions (CBF) and fixed-time control Lyapunov functions (FxT-CLF) to incorporate a class of additive disturbances in the system dynamics, and state-estimation errors. To solve the underlying constrained control problem, we formulate a quadratic program and use the proposed robust CBF-FxT-CLF conditions to compute the control input. We showcase the efficacy of the proposed method on a numerical case study involving multiple underactuated marine vehicles.

I. INTRODUCTION

With the advent of complex missions that require multi-robot systems to execute various tasks in parallel, the need for a systematic synthesis of algorithms that enable the underlying objectives has emerged. Standard objectives in such missions include, but are not limited to, requiring each robot to stay within a given subset of the state space for a given time duration, while keeping a point of interest in its field of view, and reaching a destination within a given time horizon. It is also important that each robot always maintains a safe distance from stationary and moving objects or other robots in the environment. In problems where the objective is to stabilize the closed-loop trajectories to a given desired point or a set, control Lyapunov functions (CLFs) are very commonly used to design the control input [1], [2]. Temporal constraints, i.e., constraints pertaining to convergence within a fixed time, appear in time-critical applications, for instance when a task must be completed within a given time interval. The use of fixed-time stability (FxTS) [3] has enabled the synthesis of controllers guaranteeing finite- or fixed-time reachability to the desired point or a set [4]. Similarly, safety or containment of the closed-loop trajectories in a subset of the state-space can be enforced using control barrier functions (CBFs). Traditionally, CBFs have been used to encode safety with respect to static safe sets arising due to the presence of stationary obstacles or unsafe regions in the state-space (see [1], [5]) and with respect to dynamically-changing safe sets, such as in multi-agent systems ( [6], [7]).

The development of fast optimization solvers has enabled the online control synthesis using quadratic programs (QPs), where CLF and CBF conditions are encoded as linear constraints, while the objective is to minimize the norm of the control input [2], [8] or the deviation of the control input from a nominal controller [6], [9]. Most of the prior work on QP-based control design enforces the safety constraint with one fix CBF condition and uses a slack term in the CLF condition to guarantee that the QP is feasible in the absence of input constraints. However, control input constraints should be also considered in the design step, otherwise, the derived control input might not be realizable due to actuator limits, and might lead to violation of the safety requirements. In the prior work [10], we considered an ideal case without any disturbances, and proposed a QP with feasibility guarantees that achieve forward invariance of a safe set and reachability to a goal set, even in the presence of control input constraints.

Encoding safety in the presence of disturbances can be done using robust CBFs [11], [12]. While the aforementioned work considers bounded additive disturbance in the system dynamics, it is generally assumed that the system states are available without any errors. In their majority, earlier work in the literature on multi-agent collision avoidance using CBFs [13], [14], [8] assumes perfect knowledge of the states of the agents and no state-estimation errors. In this paper, apart from additive disturbances in the system dynamics, we also consider bounded state-estimations errors and incorporate them in the robust CBF design to guarantee forward invariance of the safe sets.

This paper studies QP-based control synthesis for multi-task problems involving agents of nonlinear, control-affine dynamics, with the following objectives for the closed-loop trajectories: (i) remain inside a static safe set, (ii) remain inside a time-varying safe set (arising for instance due to the presence of moving obstacles or neighboring agents), and (iii) reach a given goal set within a user-defined time. We first present robust CBF conditions to guarantee forward invariance while incorporating both the disturbance in the system dynamics as well as the state-estimation error. Then, utilizing the fixed-time stability conditions from [15], we propose a robust fixed-time CLF condition to guarantee convergence to the desired goal set within the user-defined time, extending the prior results in [6], [4], [10]. Finally, we merge the presented robust CBF-FxT-CLF conditions in a QP formulation, show its feasibility, and discuss the conditions under which the control input defined as the solution of the QP solves the multi-task problem. We showcase the efficacy of the proposed method via a multi-agent case study involving under-actuated marine vehicles.

II. MATHEMATICAL PRELIMINARIES

Notations: In the rest of the paper, \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{R}_+ \) denotes the set of non-negative real numbers. We use \( \|x\| \) to denote the Euclidean norm of a vector \( x \in \mathbb{R}^n \). \( |x| \) denotes the absolute value when \( x \in \mathbb{R} \), and cardinality, or the number of elements, when \( x \in \mathbb{Z} \) is a set, for some positive integer \( N \). We use \( \partial S \) to denote the boundary of a closed set \( S \subset \mathbb{R}^n \) and \( \text{int}(S) \) to denote its interior, and \( \|x\|_S = \inf_{y \in S} \|x - y\| \), to denote the distance...
of $x \in \mathbb{R}^n$ from the set $S$. We use $B_\epsilon$ to denote a ball of radius $\epsilon > 0$ centered at the origin.

**System model:** In this work, we consider a multi-task problem for the dynamical system given as:

$$\dot{x}(t) = f(x(t)) + g(x(t))u + d(t, x), \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ are the state and the control input vectors, respectively, with $U$ the control constraint set. $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are continuous functions and $d : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is an unknown additive disturbance. We make the following assumption.

**Assumption 1.** There exists $\gamma > 0$ such that for all $t \geq 0$ and $x \in D \subset \mathbb{R}^n$, $\|d(t, x)\| \leq \gamma$.

Assumption 1 implies that the disturbance $d$ is uniformly bounded in the domain $D$. This is a standard model to account for various types of uncertainties, environmental noises, and external disturbances (see, e.g., [12]). Furthermore, we assume that the state $x$ is not perfectly known, to account for sensor noises and uncertainties. More specifically, we consider that only an estimate of the system state, denoted as $\hat{x}$, is available, that satisfies:

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + d(t, \hat{x}), \quad (2)$$

and make the following assumption.

**Assumption 2.** There exists an $\epsilon > 0$ such that $\|\dot{x}(t) - x(t)\| \leq \epsilon$, for all $t \geq 0$.

We now define some notations and functions necessary to state the main problem. Let $h_S : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function defining the static safe set $S_S = \{x \mid h_S(x) \leq 0\}$. The system trajectories might also need to maintain safety with respect to a dynamically-changing safe set, for instance due to the presence of moving obstacles or other agents in a multi-agent scenario. In such a case, a centralized collision avoidance scheme would require each agent $i$ to be in a safe set defined as $\{x_i(t) \mid h(x_i(t), x_j(t)) \leq 0\}$ for all $j \neq i$, where $x_i, x_j \in \mathbb{R}^n$ are the states of agent $i$ and $j$ (see Section IV for a multi-agent case study). In particular, if $x_i$ represents the position of the agent $i$, then the function $h$ can be chosen as $h(x_i(t), x_j(t)) = d_i^2 - |x_i(t) - x_j(t)|^2$, where $d_i > 0$ is the safety distance. In this case, we can define $h_T(t, x_i) = \max_{j \neq i} h(x_i(t), x_j(t))$ so that it encodes safety with respect to all other agents. To encode safety with respect to a general time-varying safe set, such as the one discussed above, let $h_T : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function defining the time-varying safe set $S_T(t) = \{x \mid h_T(t, x) \leq 0\}$. Finally, let $h_G : \mathbb{R}^n \to \mathbb{R}$ a continuously differentiable function defining the goal set $S_G = \{x \mid h_G(x) \leq 0\}$. The problem formulation provides.

**Problem 1.** Find a control input $u(t) \in U = \{v \in \mathbb{R}^m \mid u_{j,min} \leq v \leq u_{j,max}, j = 1, 2, \ldots, m\}$, $t \geq 0$, and a set $D$, such that for all $x(0) \in D \subset S_S \cap S_T(0)$, the closed-loop trajectories of (1) satisfy

(i) $x(T) \in S_G$ for some user-defined $T > 0$;
(ii) $x(t) \in S_S$ for all $t \geq 0$;
(iii) $x(t) \in S_T$ for all $t \geq 0$.

One can use a smooth approximation for the max function, e.g., $h_T = \log(\sum_j e^{h_j})$, so that the resulting function $h_T$ is continuously differentiable (see [7]).

Here, $U$ is box-constraint set where $u_{j,min} < u_{j,max}$ are the lower and upper bounds on the individual control input $v_j$ for $j = 1, 2, \ldots, m$, respectively. Input constraints of this form are very commonly considered in the literature [2], [12]. We can write $U$ in a compact form as $\mathcal{U} = \{v \mid A_Uv \leq b_u\}$ where $A_U \in \mathbb{R}^{2m \times m}$, $b_u \in \mathbb{R}^{2m}$.

**Forward invariance:** We first review a sufficient condition for guaranteeing forward invariance of a set in the absence of the disturbances and noises. Define $S(t) = \{x \mid h(t, x) \leq 0\}$ for some continuously differentiable $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

**Lemma 1.** Let $d \equiv 0$ and the solution $x(t)$ of (1) exist and be unique in forward time. Then, the set $S(t) = \{x \mid h(t, x) \leq 0\}$ is forward invariant for the trajectories of (1) for all $x(0) \in S(0)$ if the following condition holds:

$$\inf_{u \in \mathcal{U}} \left\{ L_fh(t, x) + L_gh(t, x)u + \frac{\partial h}{\partial t}(t, x) \right\} \leq \alpha(-h(t, x)), \quad (3)$$

for all $x \in S(t)$, $t \geq 0$ where $\alpha$ is a locally Lipschitz class-$K$. Furthermore, if $h(0, x(0)) < 0$, then for any $T \geq 0$, $h(t, x(t)) < 0$ for all $0 \leq t \leq T$.

A function that satisfies (3) is called a valid CBF by the authors in [7], and a zeroing-CBF by the authors in [2].

**Fixed-time stability:** Next, we review a sufficient condition for fixed-time stability of the origin for the closed-loop trajectories of (1).

**Lemma 2 ([15]).** Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable, positive definite, proper function, satisfying

$$\dot{V}(x(t)) \leq -c_1(V(x(t)))^{\alpha_1} - c_2(V(x(t)))^{\alpha_2} + c_3V(x(t)), \quad (4)$$

with $c_1, c_2 > 0$, $c_3 \in \mathbb{R}$, $\alpha_1 = 1 + \frac{1}{\mu}$, $\alpha_2 = 1 - \frac{1}{\mu}$ for some $\mu > 1$, along the closed-loop trajectories of (1) under a continuous control input $u(x)$. Then, there exists a neighborhood $D$ of the origin such that for all $x(0) \in D$, the trajectories of (1) reach the origin in a fixed time $T$ where $T, D$ are known functions of $\mu$ and $\frac{1}{\sqrt{\mu - 1}}$.

In this work, without loss of generality, it is assumed that the functions $h_S, h_T, h_G$ are relative-degree one functions. For higher-relative degree functions, higher-order CLF and CBF conditions can be used. For example, if the function $h_T$ is of relative degree 2 (as in the case study presented in Section IV), then following the results in [16], it can be shown that satisfaction of the inequality

$$L_f^2h_S + L_fL_fh_Su + 2L_fh_S + h_S \leq \alpha(-L_fh_S - h_S), \quad (5)$$

for some $\alpha \in \mathcal{K}$ implies that the set $\tilde{S}_S = \{x \mid L_fh_S(x) + h_S(x) \leq 0\} \subset S_S$ is forward-invariant. In this case, one can define $\tilde{h}_S = L_fh_S + h_S$ so that (5) reads $L_f\tilde{h}_S + L_fh_S \leq \alpha(-\tilde{h}_S)$, which is same as (3), thus guaranteeing forward invariance of the set $S_S$. Interested reader is referred to [17] for more details on higher-order CBF conditions.

**III. MAIN RESULTS**

**Robust CBF and CLF:** First, we present conditions for robust CBFs so that the safety requirements (ii) and (iii) in Problem 1 can be satisfied in the presence of the disturbance $d$ and error $\epsilon$. We make the following assumption.

**Assumption 3.** There exist $l_S, l_G, l_T > 0$ such that

$$\left\| \frac{\partial h_S}{\partial x}(x) \right\| \leq l_S, \left\| \frac{\partial h_G}{\partial x}(x) \right\| \leq l_G, \left\| \frac{\partial h_T}{\partial x}(t, x) \right\| \leq l_T, \text{for all } x \in D \subset \mathbb{R}^n, \text{and all } t \geq 0.$$

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Since the functions $h_S, h_T$ are continuously differentiable, Assumption 3 can be easily satisfied in any compact domain $D$. Corresponding to the set $S(t) = \{ x \mid h(t, x) \leq 0 \}$ for some continuously differentiable $h : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$, define $\hat{S}(t) = \{ \hat{x} \mid h(t, \hat{x}) \leq -l \}$, where $l = \sup \| \frac{\partial h(t, x)}{\partial x} \|$ is the Lipschitz constant of the function $h$. We define the notion of a robust CBF as follows.

**Definition 1 (Robust CBF-S).** A continuously differentiable function $h : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called a robust CBF for the set $S(t) = \{ x \mid h(t, x) \leq 0 \}$ for (1) w.r.t. a disturbance $d$ satisfying Assumption 1 if the following condition holds

$$\inf_{u \in U} \left\{ L_f h(t, x(t)) + L_g h(t, x(t))u + \frac{\partial h}{\partial t} (t, x(t)) \right\} \leq \alpha(-h(t, x(t))) - l \gamma,$$

for some locally Lipschitz class-K function $\alpha$ and for all $x(t) \in S(t), t \geq 0$.

Note that we use the worst-case bound of $\| \frac{\partial h}{\partial x} d \| = l \gamma$ to define the robust CBF. This condition can be relaxed if more information than just the upper bound of the disturbance is known. We can now state the following lemma that relates the robust CBF condition with forward invariance of the set $S(t)$ in the presence of the disturbance $d$. For any function $\phi : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ with Lipschitz constant $l_\phi$, define

$$\hat{\phi}(t, \cdot) = \phi(t, \cdot) + l_\phi \epsilon.$$

**Lemma 3.** Let the solution $x(t)$ of (1) exist and be unique in forward time, and $h$ be a robust CBF-S for (2). Then, there exists a control input $u \in U$ such that the set $S(t)$ is forward invariant for the trajectories of (1) for all $\hat{x}(0) \in \hat{S}(0)$.

**Proof.** Using the mean value theorem, we have that there exists $z \in \mathbb{R}^n$ such that

$$h(t, x) = h(t, \hat{x} + (x - \hat{x})) = h(t, \hat{x}) + \frac{\partial h}{\partial x} (t, z)(x - \hat{x}),$$

$$h(t, x) \leq h(t, \hat{x}) + \| \frac{\partial h}{\partial x} (t, z) \| \| (x - \hat{x}) \| \leq h(t, \hat{x}) + l \epsilon.$$

Thus, $h(t, \hat{x}) \leq -l \epsilon$ implies that $h(t, x) \leq 0$. Note that the time derivative of $h$ along the trajectories of (2) reads

$$\dot{\hat{h}}(t, \hat{x}) = L_f h(t, \hat{x}) + L_g h(t, \hat{x})u + \frac{\partial h}{\partial t} (t, \hat{x}),$$

$$\leq L_f h(t, \hat{x}) + L_g h(t, \hat{x})u + \left\| \frac{\partial h}{\partial x} (t, \hat{x}) \right\| \| d(t, \hat{x}) \| + \frac{\partial h}{\partial t} (t, \hat{x}),$$

$$\leq L_f h(t, \hat{x}) + L_g h(t, \hat{x})u + \frac{\partial h}{\partial t} (t, \hat{x}) + l \gamma$$

$$\leq \alpha(-h(t, x(t))) - l \gamma.$$

Thus, using Lemma 1, we have that $\hat{h}(t, \hat{x}(t)) \leq 0$ (or $h(t, \hat{x}(t)) \leq -l$) for all $t \geq 0$, i.e., the set $\hat{S}(t)$ is forward invariant for $\hat{x}(t)$ for all $\hat{x}(0) \in \hat{S}(0)$. Thus, we have $h(t, x) \leq 0$ for all $t \geq 0$, implying forward invariance of the set $S(t)$ for all $\hat{x}(0) \in \hat{S}(0)$.

**Remark 1.** Note that for the robust CBF condition, if the set $\hat{S}(0)$ is empty, then there exists no initial condition for which forward invariance of the set $S$ can be guaranteed based on Lemma 3.

Next, we present a robust CLF condition to guarantee FxTS of the closed-loop trajectories to the goal set. Consider a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ with Lipschitz constant $l_V$. Using the mean-value theorem, we obtain

$$V(x) \leq V(\hat{x}) + l_V \epsilon, \forall x, \hat{x} \in \mathbb{R}^n,$$

from which we obtain that if $V(\hat{x}) \leq -l_V \epsilon$, then $V(x) \leq 0$. Using this and inspired from [10, Definition 2], we define the notion of robust fixed-time CLF (FxT-CLF).

**Definition 2 (Robust FxT-CLF-SG).** A continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a robust FxT-CLF-SG for a set $S_G$ for (1) w.r.t. disturbance $d$ satisfying Assumption 1 if $V(x) < 0$ for $x \in \text{int}(S_G)$, and there exists $\alpha \in K_{\infty}$ such that $V(x) \geq \alpha(\|x\|_{S_G})$ for all $x \notin S_G$, satisfying

$$\inf_{\epsilon \in U} \left\{ L_f V(x) + L_g V(\hat{x})u \right\} \leq -\alpha_1(V(x))^{\gamma_1} - \alpha_2(V(x))^{\gamma_2}$$

$$+ \delta_1 V(x) - l_V \gamma,$$

for all $x \notin S_G$ with $\alpha_1, \alpha_2 > 0$, $\delta_1 \in \mathbb{R}$, $\gamma_1 = 1 + \frac{1}{\mu}$, $\gamma_2 = 1 - \frac{1}{\mu}$ for some $\mu > 1$.

Based on this, we can state the following result.

**Lemma 4.** Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $\hat{V}$ is a robust FxT-CLF-SG for (2). Then, there exists $u \in U$, and a neighborhood $D$ of the set $S_G$ such that for all $\hat{x}(0) \in D$, the closed-loop trajectories of (1) reach the goal set $S_G$ in a fixed time $T$.

**Proof.** Define $\hat{V}(\hat{x})$ per (7) so that $\dot{\hat{V}}(\hat{x}) = \hat{V}(\hat{x})$. Note that (9) implies that there exists $u \in U$ such that

$$\dot{\hat{V}}(\hat{x}) = L_f V(\hat{x}) + L_g V(\hat{x})u + L_d V(\hat{x})$$

$$\leq -\alpha_1 \hat{V}(\hat{x})^{\gamma_1} - \alpha_2 \hat{V}(\hat{x})^{\gamma_2} + \delta_1 \hat{V}(\hat{x}).$$

Thus, from [15, Theorem 1], we obtain that there exists a domain $D$ and fixed time $0 < T < \infty$ (that are functions of $\hat{x}(0)$) such that $\hat{V}(\hat{x}(T)) = 0$ for all $\hat{x}(0) \in D$. Thus, we obtain that $V(\hat{x}(T)) \leq -l_V \epsilon$, which, in light of (8), implies $V(x(T)) \leq 0$ for all $\hat{x}(0) \in D$. Since $V(x) \geq \alpha(\|x\|_{S_G})$, $V(x(T)) \leq 0$ implies that $x(T) \in S_G$, which completes the proof.

The robust FxT-CLF condition guarantees that if the estimated state $\hat{x}$ reaches a certain level set in the interior of the set $S_G$, quantitatively given as $\{ x \mid V(x) \leq -l_V \epsilon \}$, then the actual state $x$ reach the zero sub-level set of $V$, and thus, reach the set $S_G$.

**Remark 2.** For Lemma 4, it is required that the set $\{ x \mid h_G(x) \leq -l_G \epsilon \} \neq \emptyset$. Otherwise, if the minimum value of the function $h_G$ exceeds $-l_G \epsilon$, i.e., $h_{G, \min} \triangleq \min_{x \in S_G} h_G(x) > -l_G \epsilon$, so that $\{ x \mid h_G(x) \leq -l_G \epsilon \} = \emptyset$, it is not possible for $\hat{h}_G(\hat{x})$ to go to zero. In such cases, (9) implies that the closed-loop trajectories only reach the...
set \( \{ x \mid h_G(x) \leq h_G, min + l_G \epsilon \} \), leading to input-to-state stability. In this work, we assume that \( \{ \hat{x} \mid h_G(\hat{x}) \leq \bar{S}_G \} \) for \( i \). The QP formulation is cast as linear inequality constraints in a min-norm control problem.

**Proof.** Part (i): Since \( \hat{x}(t) \in \left( \bar{S}_G \cap \int(\hat{S}_T(t)) \right) \), we have that \( h_G(\hat{x}), h_T(t, \hat{x}), h_G(\hat{x}) \neq 0 \) for all \( t \geq 0 \). Choose any \( \bar{v} \in \mathcal{U} \) and define \( \tilde{\delta}_1 = \frac{L_I h_G + L_T h_T + \epsilon_h}{h_G} \) which is well-defined for all \( \hat{x} \notin \bar{S}_G \), so that (10c) is satisfied with equality. Thus, there exists \( \hat{\tau} \in \left[ \bar{v}^T \tilde{\delta}_1 \tilde{\delta}_2 \tilde{\delta}_3 \right]^T \) such that all the constraints of (10) are satisfied.

Part (ii): The condition (10c) implies that the function \( h_G \) is a robust FxT-CLF-SG for (2). Thus, using Lemma 4 and [15, Theorem 1], we obtain that \( h_G(\hat{x}(t)) \leq 0 \) for \( t \geq \bar{T} \), which implies that \( h_G(\hat{x}(t)) \leq -L_G \gamma \), in which implies \( h_G(x(t)) \leq 0 \) for \( t \geq \bar{T} \) for all \( \hat{x}(0) \in D \cap \int(\hat{S}_G) \cap \int(\hat{S}_T(0)) \). Furthermore, conditions (10d) and (10e) imply that the functions \( h_G \) and \( h_T \) are robust CBFs for (2), and thus, the set \( S_G \) is forward-invariant for the closed-loop trajectories of (1). Thus, the control input \( u = v^* \) solves Problem 1 for all \( \hat{x}(0) \in D \cap \int(\hat{S}_G) \cap \int(\hat{S}_T(0)) \).

It is worth noting that the constraints in the QP (10) are a function of the estimated state \( \hat{x} \), and not the actual state \( x \), which is unknown. Thus, the resulting control input \( u = v^*(\hat{x}) \) is realizable. Before presenting the case study, we provide some discussion on the main result.

**Remark 3.** Theorem 1 guarantees that starting from the intersection of the interiors of the safe sets, the closed-loop trajectories remain inside the interior of these sets. The case when the initial conditions lie on the intersection of the boundaries of the safe sets requires strong viability assumptions such as the existence of \( u \) such that (3) holds for both \( h_G \) and \( h_T \) for all \( x \in \partial S_G \cap \partial D_G \).

**Remark 4.** We impose continuity requirements on the solution of the QP (10) to use the traditional Nagumo’s viability theorem to guarantee forward invariance of a set. Prior work e.g., [2], [8], [10] discusses conditions under which the solution of parametric QPs such as (10) is continuous, or even Lipschitz continuous. More recently, utilizing the concept of strong invariance and tools from non-smooth analysis, forward invariance of a set requiring that the control input is only measurable and locally bounded is discussed in [18].

**Remark 5.** Note that the result in part (iii) of Theorem 1 requires \( \delta_i < 0 \) so that the control input \( u \) solves the convergence requirement of Problem 1. When this condition does not hold, the closed-loop trajectories satisfy the safety requirements, but may not converge to the goal set from any arbitrary initial condition \( x(0) \notin S_G \) (see [10]).
of each agent. We also consider measurement uncertainties in the state estimates as stated in Assumption 2. The system dynamics is under-actuated since there is no control input in the sway degree of freedom (y-axis). The multi-task problem considered for the case study is as follows (see Figure 1):

**Problem.** Compute \( \tau_i \in \mathcal{U} = [-\tau_{u,m}, \tau_{u,m}] \times [-\tau_{r,m}, \tau_{r,m}] \), \( \tau_{u,m}, \tau_{r,m} > 0 \), such that each agent

(i) Reaches an assigned goal region around a point \( g_i \in \mathbb{R}^2 \) within a user-defined time \( T \);
(ii) Keeps their respective point-of-interest \( p_i \in \mathbb{R}^2 \) in their field of view (given as a sector of radius \( R > 0 \) and angle \( \alpha > 0 \));
(iii) Maintains a safe distance \( d_s \) w.r.t. other agents;

where \( \angle(\cdot) \) is the angle of the vector (\( \cdot \)) with respect to the x-axis of the global frame.

Note that (ii) requires safety with respect to a static safe set, while requires (iii) safety with respect to a time-varying safe set. The parameters used in the case study are given in Table I. First, we construct CLF and CBFs to guarantee convergence to the desired location, and invariance of the required safe sets, respectively. Consider the function

\[
 h_{ij} = d_i^2 - \left\| [x_i(t) \ y_j(t)]^T - [x_j(t) \ y_i(t)]^T \right\|^2,
\]

defined for \( i \neq j \), so that \( h_{ij} \leq 0 \) implies that the agents maintain the safe distance \( d_s \). Since the function \( h_{ij} \) is relative degree two function with respect to the dynamics (11), we use the second order safety condition discussed in [9]. Similarly, for keeping the point-of-interest in the field of view, we use two separate CBFs, defined as

\[
 h_{\phi} = \angle \left( p_i - [x_i \ y_i]^T \right) - \phi_i > 0,
\]

\[
 h_{R} = \left\| [x_i - y_i]^T - p_i \right\|^2 - R^2,
\]

so that \( h_{\phi}(z_i) \leq 0, h_{R}(z_i) \leq 0 \) implies that \( z_i \in \mathcal{F} \). For \( h_{\phi}, h_{R} \), we use the relative degree 2 condition (5). Finally, we define the CLF as

\[
 V = \frac{1}{2}(X_i - X_d)^T(X_i - X_d),
\]

where \( X_i \in \mathbb{R}^6 \) is the state vector of the \( i \)-th agent, and \( X_d \in \mathbb{R}^6 \) its desired state, defined as \( X_d = \max_i \{ V_i(t) \} \) with time for the three cases.

Figures 5 and 6 show the control inputs \( \tau_u \) and \( \tau_r \), respectively, for the 4 agents, and it can be seen that the control input constraints are satisfied at all times.

Figures 5 and 6 show the control inputs \( \tau_u \) and \( \tau_r \), respectively, for the 4 agents, and it can be seen that the control input constraints are satisfied at all times.

| Table I: Dynamic parameters as taken from [19]. |
|-----------------|-----------------|-----------------|
| \( m_{11} \) | 5.5404 | \( X_u \) | -2.3015 |
| \( m_{21} \) | 9.6000 | \( Y_u \) | -3.0140 |
| \( m_{22} \) | 5.4000 | \( X_P \) | -2.6809 |
| \( m_{22} \) | 5.4000 | \( X_r \) | -0.0048 |

\( X_u \) and \( X_P \) are given in Table I. First, we construct CLF and CBFs to guarantee convergence to the desired location, and invariance of the required safe sets, respectively. Consider the function

\[
 h_{ij} = d_i^2 - \left\| [x_i(t) \ y_j(t)]^T - [x_j(t) \ y_i(t)]^T \right\|^2,
\]

defined for \( i \neq j \), so that \( h_{ij} \leq 0 \) implies that the agents maintain the safe distance \( d_s \). Since the function \( h_{ij} \) is relative degree two function with respect to the dynamics (11), we use the second order safety condition discussed in [9]. Similarly, for keeping the point-of-interest in the field of view, we use two separate CBFs, defined as

\[
 h_{\phi} = \angle \left( p_i - [x_i \ y_i]^T \right) - \phi_i > 0,
\]

\[
 h_{R} = \left\| [x_i - y_i]^T - p_i \right\|^2 - R^2,
\]

so that \( h_{\phi}(z_i) \leq 0, h_{R}(z_i) \leq 0 \) implies that \( z_i \in \mathcal{F} \). For \( h_{\phi}, h_{R} \), we use the relative degree 2 condition (5). Finally, we define the CLF as

\[
 V = \frac{1}{2}(X_i - X_d)^T(X_i - X_d),
\]

where \( X_i \in \mathbb{R}^6 \) is the state vector of the \( i \)-th agent, and \( X_d \in \mathbb{R}^6 \) its desired state, defined as \( X_d = \max_i \{ V_i(t) \} \) with time for the three cases.

A video of the simulation is available at: https://tinyurl.com/y32oa4p4.
gencness constraints using slack variables so that its feasibility is guaranteed. We showed that under certain conditions, control input defined as the solution of the proposed QP solves the multi-task problem, even in the presence of the considered disturbances and input constraints.

One of the drawbacks of the presented method is conservatism due to the absence of knowledge of the structure of the disturbance. In the future, we would like to study online learning-based methods to learn estimates of the disturbances, so that the formulation can be made less conservative.

REFERENCES


V. CONCLUSION

We considered a multi-task control synthesis problem for a class of nonlinear, control-affine systems under input constraints, where the objectives include remaining in a static safe set and a time-varying safe set and reaching a goal set within a fixed time. We also considered additive disturbances in the system dynamics and bounded state-estimation errors. We utilized robust CBFs to guarantee safety, and robust FXT-CLF to guarantee fixed-time reachability to given goal sets. Finally, we formulated a QP incorporating safety and conver-