Tome of Analysis

Contents

List of Exercises

Works Consulted

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[2] Armstrong, M.A; *Basic Topology.*
[3] Beals, R. *Analysis: An Introduction.*
[4] Bogachev, V.; Smolyanov, O. *Real and Functional Analysis.*
[5] Brown, J.; Churchill, R. *Complex Variables and Applications.*
[6] Erhardt, T. Lectures notes on Functional Analysis
[7] Folland, G. *Real Analysis: Modern Techniques and their Applications.*
[8] Hunter, J. K.; Nachtergaele, B. *Applied Analysis.*
[9] Lang, S. *Complex Analysis.*
[10] Royden, H.L. *Real Analysis.*
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[15] Strichartz, R. S. *The Way of Analysis.*

CHAPTER 1

Winter 2018

1. Problem 1

EXERCISE 1.1 (Perfect sets are uncountable). *Let $A \subseteq \mathbb{R}$ be closed and perfect, and nonempty. Then A is uncountable.*

PROOF. It is first easy to show that A is at least countable. Let $x \in A$ and select $x_1 \in B_1(x)$. For each $n = 1, 2, \ldots$, select

$$x_{n+1} \in (A \cap B_{\epsilon_n}(x)) \setminus \{x\} \quad \epsilon_n = d(x_n, x) \tag{1}$$

This definition implies $d(x_n, x_m) > 0$ for $n \neq m$. Suppose that $d(x_n, x_m) = 0$, then

$$\epsilon_n = d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) = \epsilon_m \tag{2}$$

$$\epsilon_m = d(x_m, x) \leq d(x_m, x_n) + d(x_n, x) = \epsilon_n \tag{3}$$

Therefore, $\epsilon_n = \epsilon_m$, indicating $n = m$, since ϵ strictly decreases. Therefore, A is at least countable.

As a closed subset of $\mathbb{R}$, we know A is a complete metric space, so an application of the Baire Category Theorem reveals that

$$A \neq \bigcup_{n=1}^{\infty} \{y_n\} \tag{4}$$

for any sequence y_n, so that A is uncountable. □

2. Problem 2

EXERCISE 1.2 (Hölder condition for compactness). *Show that the following set is compact*

$$A = \{f \in C(X) \mid \|f\| \leq 1, H_\alpha(f) \leq 1\} \tag{5}$$

where (X, ρ) *is a compact metric space and*

$$H_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} \tag{6}$$

PROOF. To apply Arzelà-Ascoli, lets show A is closed, bounded, and equicontinuous. The first two are obvious, so we focus on the last. Let $\epsilon > 0$ be given. If $\rho(x,y) < \epsilon^{1/\alpha}$, then $\rho^\alpha(x,y) \leq \epsilon$ and

$$H_\alpha(f) \leq 1 \implies \frac{|f(x) - f(y)|}{\rho^\alpha(x,y)} \leq 1 \quad \forall f \in A \implies |f(x) - f(y)| \leq \rho^\alpha(x,y) \leq \epsilon \quad \forall f \in A \tag{7}$$

Therefore, $\delta = \epsilon^{1/\alpha}$ is an appropriate equicontinuity constant. □

3. Problem 3

EXERCISE 1.3 (Exact Egorov's Theorem). *Suppose $f_n \to f$ almost everywhere. Find E^k such $f_n \to f$ uniformly on E^k and*

$$\mu\left(\mathbb{R} \setminus \bigcup_{k=1}^{\infty} E^k\right) = 0. \tag{8}$$

PROOF. For each positive integer k and for each integer m, select

$$E_m^k \subseteq [m, m+1] \tag{9}$$

such that $f_n \to f$ uniformly on E_m^k and $\mu([m,m+1] \setminus E_m^k) < 1/k2^{|m|}$. Then define

$$E^k = \bigcup_{m=-k}^{k} E_m^k \tag{10}$$

To see $f_n \to f$ uniformly on E^k, let m be fixed and $\epsilon > 0$. By the uniform convergence of $f_n \to f$ on E_m^k, select N_m such that $n \geq N_m$ implies $\|f_n - f\|_{E_m^k} < \epsilon$. Let $N = \min\{-N_k, \dots, N_k\}$ to see that the convergence is uniform on the union E^k also.

Break apart the complement

$$\mathbb{R} \setminus E^k = \left(\bigcup_{j=-\infty}^{\infty} [j, j+1]\right) \setminus \bigcup_{m=-k}^{k} E_m^k \tag{11}$$

$$= \bigcup_{j=-\infty}^{\infty} \left([j, j+1] \setminus \bigcup_{m=-k}^{k} E_m^k\right) \tag{12}$$

$$= \left(\bigcup_{j=-k}^{k} [j, j+1] \setminus E_j^k\right) \cup \bigcup_{|j|>k} [j, j+1] \tag{13}$$

The finite union can be measured by hand:

$$\mu\left(\bigcup_{j=-k}^{k} [j, j+1] \setminus E_j^k\right) = \sum_{j=-k}^{k} \mu([j, j+1] \setminus E_j^k) \leq \sum_{j=-\infty}^{\infty} 1/k2^{|j|} = 3/k \tag{14}$$

To study the infinite union, label

$$U_k = \bigcup_{|j|>k} [j, j+1] \tag{15}$$

From the definition, we know that $U_1 \supseteq U_2 \supseteq \cdots$ and since no element in $\mathbb{R}$ is infinite, the intersection $\bigcap U_k$ is empty. Thence,

$$\mathbb{R} \setminus \bigcup_{k=1}^{\infty} E^k = \bigcap_{k=1}^{\infty} \mathbb{R} \setminus E^k \tag{16}$$

$$= \bigcap_{k=1}^{\infty} \left[\left(\bigcup_{j=-k}^{k} [j, j+1] \setminus E_j^k\right) \cup \bigcup_{|j|>k} [j, j+1]\right] \tag{17}$$

$$= \left[\bigcap_{k=1}^{\infty} \left(\bigcup_{j=-k}^{k} [j, j+1] \setminus E_j^k\right)\right] \cup \left[\bigcap_{k=1}^{\infty} U_k\right] \tag{18}$$

Therefore,

$$\mu\left(\mathbb{R} \setminus \bigcup_{k=1}^{\infty} E^k\right) < 3/k \quad \forall k \tag{19}$$

which implies the measure of this set equals zero. □

4. Problem 4

Exercise 1.4 (DCT for convergence in measure). *Let f_n be a sequence of measurable functions converging in measure to f and pointwise bounded by $|f_n(x)| \leq g(x)$ where $g \in L^1$. Then*

$$f \in L^1 \quad \textit{and} \quad \lim_{n\to\infty} \int_X |f_n - f| d\mu = 0 \tag{20}$$

Proof. To proceed, we will prove these in the case that $\mu(X) < \infty$ and then extend the result.

To show f is simply integrable, we show $|f(x)| \leq g(x)$ almost everywhere. For each $\epsilon > 0$, define

$$E_\epsilon^n := \{x \in X \mid |f(x) - f_n(x)| \geq \epsilon\} \tag{21}$$

Define

$$S = \bigcap_{n=1}^{\infty} E_\epsilon^n \tag{22}$$

Then

$$\mu(S) \leq \mu(E_\epsilon^n) \quad \forall n \tag{23}$$

The definition of convergence in measure guarantees $\mu(S) = 0$. Therefore

$$\int_X |f(x)| = \int_{X\setminus S} |f(x)| \tag{24}$$

The integrand is then bounded by comparing

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \tag{25}$$

If $x \in X \setminus S$, then $x \notin E_\epsilon^n$, so f is bounded by ϵ and the dominating function

$$|f_n(x) - f(x)| < \epsilon \implies |f(x)| \leq \epsilon + g(x) \tag{26}$$

Therefore,

$$\int |f(x)| \leq \int \epsilon + g(x) = \epsilon\mu(X) + \int g(x) \tag{27}$$

Letting $\epsilon \to 0$, we find the desired result.

Now that $f \in L^1$, let us show

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| = 0 \tag{28}$$

Let $\epsilon > 0$ be given and define E_ϵ^n as above. Select $\delta > 0$ so that $\mu(B) < \delta$ implies

$$\int_B 2g < \epsilon \tag{29}$$

We are ready for the limit

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| = \lim_{n\to\infty} \left(\int_{X\setminus E_\epsilon^n} |f_n(x) - f(x)| + \int_{E_\epsilon^n} |f_n(x) - f(x)| \right) \tag{30}$$

$$\leq \lim_{n\to\infty} \left(\int_{X\setminus E_\epsilon^n} |f_n(x) - f(x)| + \int_{E_\epsilon^n} 2g \right) \tag{31}$$

Select N such that $n \geq N$ implies $\mu(E_\epsilon^n) < \delta$. This bounds the second integral by ϵ from the integral estimate. The first integral has an easy bound by convergence in measure:

$$\int_{X\setminus E_\epsilon^n} |f_n(x) - f(x)| \leq \int_{X\setminus E_\epsilon^n} \epsilon \leq \epsilon\mu(X \setminus E_\epsilon^n) \leq \epsilon\mu(X) \tag{32}$$

Since $\epsilon > 0$ was arbitrary, we have

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| = 0 \tag{33}$$

Now suppose $\mu(X) = \infty$. Let $\epsilon > 0$ be given. Select $F \subseteq X$ such that $\mu(F) < \infty$ and

$$\int_{X \setminus F} 2g < \epsilon \tag{34}$$

by the integrability of $2g$. Then

$$\int_X |f_n(x) - f(x)| = \int_{X\setminus F} |f_n(x) - f(x)| + \int_F |f_n(x) - f(x)| \tag{35}$$

$$\leq \int_{X\setminus F} 2g + \int_F |f_n(x) - f(x)| \tag{36}$$

$$\leq \epsilon + \int_F |f_n(x) - f(x)| \tag{37}$$

Since F has finite measure, we can apply the previous result to show

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| \leq \epsilon + \lim_{n\to\infty} \int_F |f_n(x) - f(x)| \leq \epsilon \tag{38}$$

The selection of ϵ was arbitrary, so we know that this limit equals zero. □

For an extra goodie, we look at how the convergence in measure metric ρ might have been used to solve this problem. Let $\epsilon > 0$ be given. Bound the integral for a fixed n

$$\int_X |f_n(x) - f(x)| = \int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} [1 + |f_n(x) - f(x)|] \tag{39}$$

$$= \operatorname{ess\,sup} \frac{|f_n - f|}{1 + |f_n - f|} \times \int_X [1 + |f_n(x) - f(x)|] \tag{40}$$

$$\leq \operatorname{ess\,sup} \frac{|f_n - f|}{1 + |f_n - f|} \times (\mu(X) + \|2g\|) \tag{41}$$

This essential supremum exists because the ratio in the integrand is bounded by one, so we can find the ess sup as a limit of L^p norms. Select p such that

$$\operatorname{ess\,sup} \frac{|f_n - f|}{1 + |f_n - f|} = \left(\int \left(\frac{|f_n - f|}{1 + |f_n - f|} \right)^p \right)^{1/p} + O(\epsilon/(\mu(X) + \|2g\|)) \tag{42}$$

$$\leq (\rho(f_n, f))^{1/p} + O(\epsilon/(\mu(X) + \|2g\|)) \tag{43}$$

Combining these estimates shows

$$\int_X |f_n(x) - f(x)| \leq \rho(f_n, f)^{1/p} (\mu(X) + \|2g\|) + O(\epsilon) \tag{44}$$

Select N such that $n \geq N$ implies

$$\rho(f_n, f) = O((\epsilon/(\mu(X) + \|2g\|))^p) \implies \rho(f_n, f)^{1/p} = O(\epsilon/(\mu(X) + \|2g\|)) \tag{45}$$

This result assumes $\mu(X) < \infty$, so we still have to extend the result as in the previous proof.

5. Problem 5

EXERCISE 1.5 (Weak convergence is finite). *Suppose* $\{x_n\} \subseteq X$ *converges weakly to* $x_0 \in X$. *Then* $\|x_0\| \leq \liminf \|x_n\|$.

PROOF. Recall that $\lim \phi(x_n)$ exists for each ϕ, so we may replace the limit with its $\liminf$ and proceed in the double dual

$$\|x_0\| = \|x_0^*\| = \sup_{\|\phi\|=1} \|x_0^*(\phi)\| = \sup_{\phi} |\phi(x_0)| = \sup_{\phi} \lim_{n} |\phi(x_n)| \tag{46}$$

$$= \sup_{\phi} \sup_{n\geq 1} \inf_{m\geq n} |\phi(x_m)| = \sup_{n\geq 1} \sup_{\phi} \inf_{m\geq n} |\phi(x_m)| \tag{47}$$

$$\leq \sup_{n\geq 1} \inf_{m\geq n} \sup_{\|\phi\|=1} |\phi(x_m)| \leq \liminf_{n\to\infty} \|\phi\|\|x_n\| \leq \liminf_{n\to\infty} \|x_n\| \tag{48}$$

□

6. Problem 6

Exercise 1.6 (Bounding linear maps via elements in the dual space). *Suppose $T : E \to F$ is such that $\phi \circ T \in E^*$ is bounded for every $\phi \in F^*$. Then T is bounded.*

Proof. Apply uniform boundedness. Define $T^* : F^* \to E^*$ by sending $\phi \mapsto \phi \circ T$ and define $J : X \to Y^{**}$ by sending $J(x)(\phi) = T^*(\phi)(x) = \phi(Tx)$.

The statement is precisely that

$$\|\phi \circ T\| = \sup_{\|x\|=1} |\phi(Tx)| = \sup_x |J(x)(\phi)| < \infty \quad \forall \phi \in Y^* \tag{49}$$

Uniform boundedness implies

$$\sup_x \|J(x)\| < \infty \tag{50}$$

Indicating $\|J\| < \infty$.

Now we show $\|T^*\| < \infty$.

$$\|T^*\| = \sup_{\|\phi\|=1} \|T^*(\phi)\| = \sup_{\|\phi\|=1} \sup_{\|x\|=1} \|T^*(\phi)(x)\| = \sup_{\|\phi\|=1} \sup_{\|x\|=1} \|J(x)(\phi)\| \tag{51}$$

$$\leq \sup_{\|\phi\|=1} \sup_{\|x\|=1} \|J(x)\| \|(\phi)\| \leq \sup_{\|\phi\|=1} \sup_{\|x\|=1} \|J\| \|x\| \| \|\phi\| \leq \|J\| < \infty \tag{52}$$

Therefore, T is bounded. □

7. Problem 7

EXERCISE 1.7 (Liouville's Theorem and a Characterization of Polynomials).

(a) State Liouville's theorem.

(b) Suppose f is entire and there exists $C > 0$ and $p \in \mathbb{N}$ such that $|f(z)| \leq C|z|^p$ for all $|z| \geq 1$. Then f is a polynomial.

For (a):

THEOREM 1 (Liouville's theorem). *A bounded entire function is constant.*

Now for (b):

PROOF. As in the proof of the Liouville theorem, f is entire so it has a power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{53}$$

where the coefficients are given by Cauchy's differentiation formula

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi^{k+1}} d\xi \tag{54}$$

If $k > p$, then $k + 1 - p > 1$, proving that

$$\left| \int_\gamma \frac{f(\xi)}{\xi^{k+1}} d\xi \right| \leq \int_\gamma \frac{|f(\xi)|}{|\xi|^p} \frac{|\xi|^p}{|\xi|^{k+1}} d\xi \leq C \int_\gamma \frac{|\xi|^p}{|\xi|^{k+1}} d\xi = 0 \tag{55}$$

Therefore, $a_k = 0$ for $k > p$, so f is a polynomial of degree at most p. □

8. Problem 8

Exercise 1.8 (Gaussian integral with a shift).

(a) Prove that

$$\int_{\mathbb{R}} e^{-(x+ia)^2} dx = \int_{\mathbb{R}} e^{-x^2} dx. \tag{56}$$

(b) Use part (a) to prove

$$\int e^{-ix\xi - \frac{x^2}{2\sigma^2}} = e^{-\frac{\xi^2\sigma^2}{2}} \sigma\sqrt{2\pi}. \tag{57}$$

Proof. To attack part (a), for $a = 0$, we already have the result, so suppose $a > 0$. The case $a < 0$ is handled similarly. Integrating around the counterclockwise rectangle $\{-R, R, R+ia, -R+ia\}$ captures the nonexisting poles of the function, so we should have

$$\int_{-R}^{R} e^{-x^2} dx + \int_0^a e^{-(R+iy)^2} dy + \int_R^{-R} e^{-(x+ia)^2} dx + \int_a^0 e^{-(R+iy)^2} dy = 0 \tag{58}$$

Two of these integrals vanish as $R \to \infty$:

$$\left| \int_0^a e^{-(R+iy)^2} dy \right| \leq \int_0^a |e^{-(R+iy)^2}| dy = \int_0^a |e^{-R^2 - 2Ryi + y^2}| dy \leq a e^{a^2 - R^2} \to 0 \tag{59}$$

Similarly

$$\int_a^0 e^{-(R+iy)^2} dy \to 0 \tag{60}$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-x^2} dx + \int_{\infty}^{-\infty} e^{-(x+ia)^2} dx = 0 \tag{61}$$

which indicates

$$\int_{-\infty}^{\infty} e^{-(x+ia)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx \tag{62}$$

For part (b), let $\sigma > 0$ be fixed. Show that

$$\int_{\mathbb{R}} e^{-ix\xi} e^{-\frac{x^2}{2\sigma^2}} dx = \sigma\sqrt{2\pi} e^{-\frac{\sigma^2\xi^2}{2}} \tag{63}$$

To apply the previous part, we complete the square:

$$\frac{x^2}{2\sigma^2} + ix\xi = \frac{1}{2\sigma^2}(x^2 + ix\xi 2\sigma^2) = \frac{1}{2\sigma^2}\left[(x + i\xi\sigma^2)^2 - (i\xi\sigma^2)^2\right] \tag{64}$$

$$= \frac{1}{2\sigma^2}(x + i\xi\sigma^2)^2 - \frac{1}{2\sigma^2}(i\xi\sigma^2)^2 \tag{65}$$

$$= \frac{1}{2\sigma^2}(x + i\xi\sigma^2)^2 + \frac{1}{2\sigma^2}\left[\xi^2\sigma^4\right] \tag{66}$$

$$= \frac{1}{2\sigma^2}(x + i\xi\sigma^2)^2 + \frac{1}{2}\xi^2\sigma^2 \tag{67}$$

Therefore

$$\int e^{-ix\xi - \frac{x^2}{2\sigma^2}} = e^{-\frac{\xi^2\sigma^2}{2}} \int e^{-\frac{1}{2\sigma^2}(x + i\xi\sigma^2)^2} \tag{68}$$

In this integral, substitute $u = x/\sqrt{2\sigma^2}$ and simplify by applying the previous result

$$\int e^{-\frac{1}{2\sigma^2}(x+i\xi\sigma^2)^2}dx = \int e^{-(x/\sqrt{2\sigma^2}+i\xi\sigma^2/\sqrt{2\sigma^2})^2}dx = \sqrt{2\sigma^2}\int e^{-(u+i\xi\sigma/\sqrt{2})^2} = \sqrt{2\sigma^2}\int e^{-u^2}du \tag{69}$$

$$= \sqrt{2\sigma^2}\sqrt{\pi} \tag{70}$$

$$= \sigma\sqrt{2\pi} \tag{71}$$

Therefore,

$$\int e^{-ix\xi-\frac{x^2}{2\sigma^2}} = e^{-\frac{\xi^2\sigma^2}{2}}\sigma\sqrt{2\pi} \tag{72}$$

□

CHAPTER 2

Spring 2018

1. Problem 1

EXERCISE 2.1 (Fundamental Limit Interchange). *Suppose* $f_n \to f$ *uniformly and* $\lim_{x\to x_0} f_n(x)$ *exists for each* f_n. *Then*

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x) \tag{73}$$

PROOF. Let $L_n := \lim_{x\to x_0} f_n(x)$. Then we are going to show

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} L_n \tag{74}$$

Equivalently

$$\lim_{x\to x_0}\lim_{n\to\infty} |f_n(x) - L_n| = 0 \tag{75}$$

Select N so that $\|f_n - f_m\| < \epsilon/3$ for all $n, m \geq N$. Select $\delta > 0$ so that $|f_N(x) - L_N| < \epsilon/3$ for all $0 < |x - x_0| < \delta$. For any $n > N$, another distance can be estimated:

$$|f_n(x) - L_n| = |f_n(x) - f_N(x) + f_N(x) - L_N + L_N - L_n| \tag{76}$$

$$\leq |f_n(x) - f_N(x)| + |f_N(x) - L_N| + |L_N - L_n| \tag{77}$$

The first summand and the second summand are bounded by convergence and continuity, respectively. The final summand is bounded by Cauchiness. Verify:

$$|L_N - L_n| = \lim_{x\to x_0} |f_N(x) - f_n(x)| \leq \|f_N - f_n\| \tag{78}$$

Therefore

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x) \tag{79}$$

□

2. Problem 2

Exercise 2.2 (Arzelà-Ascoli on smooth functions). *Let* $\{f_n : [0,1] \to \mathbb{R}\}$ *be twice differentiable such that* $f_n(0) = f_n'(0) = 0$ *and* $|f_n''(x)| \leq 1$ *uniformly. Then there exists a subsequence which converges uniformly.*

Proof. This is a direct application of Arzela-Ascoli. Include the derivatives in an ambient space

$$\{f_n'\}_{n=1}^{\infty} \subseteq C([0,1]) \tag{80}$$

If we show the sequence is bounded and equicontinuous, then the uniformly converging subsequence will be summoned by Ascoli himself.

For boundedness, compute the sup norm of each derivative

$$\|f_n'\| = \sup_{x\in[0,1]} |f_n'(x)| \leq \sup_{x\in[0,1]} \int_0^x |f_n''(y)|dy \leq 1 \tag{81}$$

For equicontinuity, we can show the sequence of derivatives is Lipschitz. Bound the derivative

$$|f_n'(y) - f_n'(x)| \leq \int_x^y |f_n''(z)|dz \leq y - x \tag{82}$$

Therefore,

$$\sup_{x\neq y} \frac{|f_n'(x) - f_n'(y)|}{|x-y|} \leq 1 \tag{83}$$

Therefore, we may select f_{n_k}' a subsequence converging uniformly to f', which we now prove equals the derivative.

Define a function

$$f(x) = \int_0^x f'(y)dy \tag{84}$$

We will show that $f_{n_k} \to f$ uniformly.

The definition of integration in $\mathbb{R}$ begets

$$f_n(x) = \int_0^x f_n'(y)dy \tag{85}$$

Now for the limit

$$\sup_{x\in[0,1]} |f_{n_k}(x) - f(x)| = \sup_{x\in[0,1]} \left| \int_0^x f_{n_k}'(y) - f'(y)dy \right| \leq \sup_{x\in[0,1]} \int_0^x |f_{n_k}'(y) - f'(y)|dy \tag{86}$$

$$\leq \int_0^1 |f_{n_k}'(y) - f'(y)|dy \leq \|f_{n_k}' - f'\| \tag{87}$$

Therefore, the convergence is uniform. □

3. Problem 3

Exercise 2.3 (Integration and uniform continuity).

(a) Show that if $f \in L^1(\mathbb{R})$ and f is uniformly continuous, then $\lim_{x\to\infty} f(x) = 0$.
(b) Was the assumption of uniform continuity necessary to conclude that f decays?

Proof. Part (a): to show that $\lim_{x\to\infty} f(x) = 0$, suppose otherwise. First select $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Let $x_1 = 0$ and for each $n = 2, 3, \ldots$ select $x_n > x_{n-1} + \delta$ such that $|f(x_n)| > 2\epsilon$. If $x \in B_\delta(x_n)$, then $|f(x) - f(x_n)| < \epsilon$, so that $|f(x)| > \epsilon$.

Therefore, the integral on a single ball is positive:

$$\int_{B_\delta(x_n)} |f(x)| \geq \epsilon\delta \tag{88}$$

There are infinitely many of these balls contained in the real line, so this shows

$$\int |f| \geq \sum_{n=1}^{\infty} \int_{B_\delta(x_n)} |f(x)| = \infty \tag{89}$$

Part (b): The assumption of uniform continuity is necessary to conclude that f decays, as the following function demonstrates:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{90}$$

Certainly, $\int f = 0$, because $\mu(\mathbb{Q}) = 0$ but $\lim_{x\to\infty} \neq 0$ and in fact this limit does not exist. □

4. Problem 4

Exercise 2.4 (DCT on convergence in measure). *See Winter 2018 Exercise 4.*

5. Problem 5

EXERCISE 2.5 (Perturbed compact operators have closed range). *Let $K : X \to X$ be a compact operator and suppose $A = I + K$ has a trivial kernel. Then $A(X)$ is closed.*

PROOF. We can show $A(X)$ is weakly closed. Suppose $\{Ax_n\}_{n=1}^{\infty} \subseteq A(X)$ converges weakly. We will find a subsequence $\{x_{n_k}\}$ such that $Kx_{n_k} \to Kx$.

Let $\phi \in X^*$. By assumption the limit $|\phi(Ax_n)|$ is finite, so that we can deteremine

$$\sup_{n\geq 1} |\phi(x_n)| = \sup_{n\geq 1} |\phi(A^{-1}Ax_n)| \leq \|A^{-1}\| \sup_{n\geq 1} |\phi(Ax_n)| < \infty \tag{91}$$

where $\|A^{-1}\| := \|(A|_{A(X)})^{-1}\| < \infty$ because the open mapping theorem applies once we realize the trivial kernel makes $A|_{A(X)}$ a surjection. This means the sequence is uniformly bounded, so by the compactness of K, select a subsequence $Kx_{n_k} \to Kx$.

Now it remains to prove that

$$\phi(Ax_{n_k}) \to \phi(Ax) \quad \forall \phi \in X^* \tag{92}$$

We will need

$$\lim_{k\to\infty} |\phi(x_{n_k} - x)| = \lim_{k\to\infty} |\phi(A^{-1}Ax_{n_k} - A^{-1}Ax)| \leq \|A^{-1}\| \lim_{k\to\infty} |\phi(Ax_{n_k} - Ax)| \tag{93}$$

$$\leq \|A^{-1}\| \lim_{k\to\infty} |\phi((K+I)x_{n_k} - (K+I)x)| \tag{94}$$

$$\leq \|A^{-1}\| \lim_{k\to\infty} \Big[\phi(Kx_{n_k} - Kx)| + |\phi(x_{n_k} - x)| \Big] \tag{95}$$

The first limit equals zero since $Kx_{n_k} \to Kx$, leaving us with

$$\lim_{k\to\infty} |\phi(x_{n_k} - x)| \leq \|A^{-1}\| \lim_{k\to\infty} |\phi(x_{n_k} - x)| \tag{96}$$

This inequality proves that this limit equals zero since $\|A^{-1}\| < 1$. Therefore,

$$\lim_{k\to\infty} \phi(Ax_{n_k}) = \lim_{k\to\infty} \phi(x_{n_k} + Kx_{n_k}) = \phi(x + Kx) = \phi(Ax) \tag{97}$$

See Lemma 7.3.1 of [**4**] for a proof that does not suppose $\ker A = 0$. □

6. Problem 6

Exercise 2.6 (Weakly converging operators have a bounded limit). *Suppose $A_n : X \to Y$ is a sequence of bounded linear operators converging weakly to A in the sense that for all $\phi \in Y^*$ and $x \in X$ the following limit holds*

$$\lim_{n\to\infty} \phi(A_n x) = \phi(Ax). \tag{98}$$

Then $\sup \|A_n\| < \infty$ *and A is bounded.*

Proof. Define a few linear maps

$$A_n^* : Y^* \to X^* \quad \phi \mapsto \phi \circ A_n \tag{99}$$

$$T_n^x : Y^* \to \mathbb{R} \quad \phi \mapsto \phi(A_n x) \tag{100}$$

$$J_n : X \to Y^{**} \quad x \mapsto T_n^x \tag{101}$$

Fixing $x \in X$, we know

$$\lim_{n\to\infty} \phi(A_n x) = \phi(Ax) \implies \sup_{n\geq 1} |\phi(A_n x)| = \sup_{n\geq 1} |T_n^x(\phi)| < \infty \quad \forall \phi \in Y^* \tag{102}$$

Uniform boundedness implies that $\sup_{n\geq 1} \|T_n^x\| < \infty$. Since x was fixed, this is true for any x, so that uniform boundedness can be applied again on

$$\sup_{n\geq 1} \|T_n^x\| = \sup_{n\geq 1} \|J_n(x)\| < \infty \quad \forall x \in X \tag{103}$$

so that $\sup_{n\geq 1} \|J_n\| < \infty$.

After we show

$$\|J_n\| = \sup_{\|x\|=1} \|J_n(x)\| = \sup_{\|x\|=1} \sup_{\|\phi\|=1} \|J_n(x)(\phi)\| = \sup_{\phi} \sup_{x} \|\phi(A_n x)\| \tag{104}$$

$$= \sup_{\phi} \|\phi \circ A_n\| = \|A_n^*\| = \|A_n\| \tag{105}$$

it is true that $\sup \|A_n\| = \sup \|J_n\| < \infty$.

Now we are ready to show A is bounded, working in the double dual.

$$\|A^*\| = \sup_{\|\phi\|=1} \|A^*(\phi)\| = \sup_{\phi} \sup_{x} |\phi(Ax)| = \sup_{\phi} \sup_{x} \lim_{n\to\infty} |\phi(A_n x)| \tag{106}$$

$$\leq \liminf_{n\to\infty} \sup_{\phi} \sup_{x} |\phi(A_n x)| \leq \liminf_{n\to\infty} \sup_{\phi} \|\phi \circ A_n\| \leq \liminf_{n\to\infty} \sup_{\phi} \|\phi\| \|A_n\| \tag{107}$$

$$\leq \liminf_{n\to\infty} \|A_n\| \leq \sup_{n\geq 1} \|A_n\| < \infty \tag{108}$$

Therefore, $\|A^*\|$ is bounded, proving that $\|A\|$ is bounded. □

7. Problem 7

EXERCISE 2.7 (Complex Fundamental Theorem of Algebra). *State Rouché's theorem and prove the fundamental theorem of algebra.*

THEOREM 2 (Rouché's Theorem). *If $h = f + g$ and*

$$|f| > |g| \tag{109}$$

on the contour C, then h and f have the same number of roots inside C.

THEOREM 3 (Complex Fundamental Theorem of Algebra). *Prove that a polynomial*

$$P(z) = \sum_{k=0}^{n} a_k z^k \tag{110}$$

has exactly n roots and the radius of the disk about zero containing all the roots may be estimated.

PROOF. Reduce the polynomial to a monic

$$p(z) = \sum_{k=0}^{n} c_k z^k \tag{111}$$

where $c_n = 1$ by dividing by a_n. Select

$$R > \sum_{k=0}^{n-1} |c_k| \tag{112}$$

Then for $|z| = R$, we have

$$|p(z) - z^n| = |c_{n-1}z^{n-1} + \cdots + c_1 z + c_0| \tag{113}$$

$$\leq |c_{n-1}||z|^{n-1} + \cdots + |c_1||z| + |c_0| \tag{114}$$

$$= |c_{n-1}|R^{n-1} + \cdots + |c_1|R + |c_0| \tag{115}$$

$$\leq |c_{n-1}|R^{n-1} + \cdots + |c_1|R^{n-1} + |c_0|R^{n-1} \tag{116}$$

$$< R^n = |z|^n \tag{117}$$

Then taking $h = p(z)$, $f = z^n$ and $g = p(z) - z^n$ in Rouché's theorem, we see that $p(z)$ and z^n have the same number of zeroes inside the disk of radius R about the origin.

We scaled the polynomial to be monic, so when we unscale it, we can see all the roots lie in the disk of radius $R|a_n|$ about the origin. □

8. Problem 8

Exercise 2.8 (Complex integral involving a cosh). *Evaluate*

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} dx \tag{118}$$

Proof. This integral can be evaluated by appealing to the residues of the complexified function at $z = \pm i/2$. For any R, enclose these residues in the rectangle with vertices in counter-clockwise order

$$\{R - i, R + i, -R + i, -R - i\} \tag{119}$$

Then the integral over this rectangle is given by either the residue theorem or directly

$$\int_\gamma f(z)dz = I_1 + I_2 + I_3 + I_4 \tag{120}$$

$$= \int_{-R}^{R} f(x-i) + \int_{-1}^{1} f(R+iy)dy + \int_{R}^{-R} f(x+i) + \int_{1}^{-1} f(R+iy)dy \tag{121}$$

The important contributions are given by

$$I_1 = \int_{-R}^{R} f(x-i) = \int_{-R}^{R} \frac{e^{-2\pi i (x-i)\xi}}{\cosh(\pi(x-i))} = -e^{-2\pi\xi} \int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} \tag{122}$$

and similarly

$$I_3 = \int_{R}^{-R} f(x+i) = -\int_{-R}^{R} \frac{e^{-2\pi i (x+i)\xi}}{\cosh(\pi(x+i))} = e^{2\pi\xi} \int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} \tag{123}$$

As $R \to \infty$, they each converge to a constant multiple of the desired integral.

The other integrals vanish as $R \to \infty$, and this can be seen:

$$|I_2| = \left|\int_{-1}^{1} f(R+iy)dy\right| \le \int_{-1}^{1} \frac{|e^{-2\pi i (R+iy)\xi}|}{|\cosh(\pi(R+iy))|} \tag{124}$$

$$\le \int_{-1}^{1} \frac{e^{2\pi y\xi}}{|\cosh(\pi(R+iy))|} \tag{125}$$

Given $\epsilon > 0$, select by uniform continuity $\delta > 0$ such that

$$\left|\frac{1}{|\cosh(\pi(R+iy))|} - \frac{1}{|\cosh(\pi R)|}\right| < \epsilon \tag{126}$$

Also realize that on this interval $e^{2\pi y\xi} \le e^{2\pi\xi}$. Then break the integral into pieces of size δ.

$$\int_{-1}^{1} \frac{e^{2\pi y\xi}}{|\cosh(\pi(R+iy))|} \le \left(\sum_{k=0}^{n} \int_{-1+k\delta}^{-1+(k+1)\delta} + \int_{-1+(n+1)\delta}^{1}\right) e^{2\pi\xi} \left(\frac{1}{|\cosh(\pi R)|} + \epsilon\right) \tag{127}$$

$$\le \int_{-1}^{1} e^{2\pi\xi} \left(\frac{1}{|\cosh(\pi R)|} + \epsilon\right) \tag{128}$$

$$= 2e^{2\pi\xi} \left(\frac{1}{|\cosh(\pi R)|} + \epsilon\right) \tag{129}$$

Send $\epsilon \to 0$ and $R \to 0$ to see the quantity vanish as $R \to \infty$. Similar for the other integral.

The residue theorem says that

$$\lim_{R\to\infty} [I_1 + I_2 + I_3 + I_4] = 2\pi i \operatorname{Res}(f, \pm i/2) = 2(e^{\pi\xi} - e^{-\pi\xi}) \tag{130}$$

But we know I_2 and I_4 vanish so we are left with

$$2(e^{\pi\xi} - e^{-\pi\xi}) = \lim_{R\to\infty} I_1 + I_3 = e^{2\pi\xi} I - e^{-2\pi\xi} I = (e^{\pi\xi} - e^{-\pi\xi})(e^{\pi\xi} + e^{-\pi\xi}) I \tag{131}$$

This implies

$$I(\xi) = \frac{2}{e^{\pi\xi} + e^{-\pi\xi}} = \frac{1}{\cosh(\pi\xi)} \tag{132}$$

Let us compute these residues directly

$$\lim_{z\to i/2}(z - i/2)f(z) = \lim_{z\to i/2}\frac{(z-i/2)e^{-2\pi i z\xi}}{\cosh(\pi z)} = \lim_{z\to i/2}\frac{e^{-2\pi i z\xi} + (z-i/2)\times e^{-2\pi i z\xi}(-2\pi i\xi)}{\pi\sinh(\pi z)} \tag{133}$$

$$= \frac{e^{-2\pi i(i/2)\xi}}{\pi i} = \frac{e^{\pi\xi}}{\pi i} \tag{134}$$

Similarly

$$\lim_{z\to -i/2}(z + i/2)f(z) = \lim_{z\to -i/2}\frac{(z+i/2)e^{-2\pi i z\xi}}{\cosh(\pi z)} = \lim_{z\to i/2}\frac{e^{-2\pi i z\xi} + (z+i/2)\times e^{-2\pi i z\xi}(-2\pi i\xi)}{\pi\sinh(\pi z)} \tag{135}$$

$$= \frac{e^{-2\pi i(-i/2)\xi}}{\pi i} = \frac{e^{-\pi\xi}}{\pi i} \tag{136}$$

□

CHAPTER 3

Winter 2019

1. Problem 1

EXERCISE 3.1 (Prove Arzelà-Ascoli). *Let K be a compact metric space and let A be a subset of $C(K)$. Prove that A is compact if and only if A is closed, bounded, and equicontinuous.*

PROOF. This exercise is asking us to prove the Arzelà-Ascoli Theorem.

First we present a diagonal argument: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\overline{A}$. Construct a diagonal subsequence as follows. Select a countable dense subset $\{x_k\}_{k=1}^{\infty} \subseteq K$ and select nested subsequences

$$f_n = f_{1,n} \supseteq \cdots \supseteq f_{k,n} \supseteq f_{k+1,n} \supseteq \cdots \tag{137}$$

in such a way that $\lim_{n\to\infty} f_{k,n}(x_k)$ exists, by the completeness of $\mathbb{R}$ and the boundedness of $\overline{A}$. It will be proven that $f_{n,n}$ is Cauchy.

Let $\epsilon > 0$ be given. Select $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/3$. Cover K and extract a finite subcover

$$K \subseteq \bigcup_{x\in K} B_\delta(x) \implies K \subseteq \bigcup_{i=1}^{n} B_\delta(x_i) \tag{138}$$

For each x_i, select N_i for which $n, m \geq N_i$ implies $|f_{i,n}(x_i) - f_{i,m}(x_i)| < \epsilon/3$. Set $N = \max\{N_i\}$.

Now we are in a position to prove that if $n, m > N$ we have $\|f_m - f_n\| < \epsilon$. For each $x \in K$, there exists $|x - x_i| < \delta$. Then we may write

$$\|f_m - f_n\| = \sup_{x\in K} |f_m(x) - f_m(x_i) + f_m(x_i) - f_n(x_i) + f_n(x_i) - f_n(x)| \tag{139}$$

$$\leq \sup |f_m(x) - f_m(x_i)| + \sup |f_m(x_i) - f_n(x_i)| + \sup |f_n(x_i) - f_n(x)| \tag{140}$$

The first and last suprema are bounded by $\epsilon/3$ due to the equicontinuity estimate. The middle supremum equals $\max_i |f_m(x_i) - f_n(x_i)|$, and the selection of N guarantees that this quantity is bounded by $\epsilon/3$, completing the proof.

An alternative proof which may be considered more explicit is given as follows, which rests on the equivalence that

$$\text{compact} \iff \text{complete and totally bounded} \tag{141}$$

in a metric space.

Let $\epsilon > 0$ be given. Our goal is to determine $\{f_1, \ldots, f_n\} \subseteq \overline{A}$ such that

$$\overline{A} \subseteq \bigcup_{i=1}^{n} B_\epsilon(f_i) \tag{142}$$

To proceed, we represent each function in $\overline{A}$ by a bounded "step" function, found by exploiting the compact domain K. Since $\overline{A}$ is bounded and equicontinuous, there exists $M > 0$ and $\delta > 0$ such that for all $h \in \overline{A}$ we have $\|h\| < M$ and that $|x - y| < \delta$ implies $|h(x) - h(y)| < \epsilon$.

To exploit the compact domain K, extract a finite subcover as follows

$$K \subseteq \bigcup_{x \in K} B_\delta(x) \tag{143}$$

$$\subseteq \bigcup_{i=1}^{L} B_\delta(x_i) \tag{144}$$

Define a collection of functions

$$G = \left\{ g : \bigcup_{i=1}^{L} B_\delta(x_i) \cap K \to \mathbb{R} \mid g(B_\delta(x_i)) = \epsilon y_i, |\epsilon y_i| < M, y_i \in \mathbb{Z} \right\} \tag{145}$$

For each $f \in \overline{A}$, select $\{y_1, \ldots, y_L\}$ satisfying

$$\epsilon y_i \leq f(x_i) \leq \epsilon(y_i + 1) \tag{146}$$

and $|\epsilon y_i| < M$, so that we may define

$$g_f(B_\delta(x_i)) = \epsilon y_i \tag{147}$$

Then $\|f - g_f\| < \epsilon$. Consider this collection of functions $\overline{G} = \{g_f\}_f = \{g_1, \ldots, g_N\}$. Realize that

$$\overline{A} \subseteq \bigcup_{i=1}^{N} B_\epsilon(g_i) \tag{148}$$

We are now so close, because we just need to invert each $g_i \to f_i$ where f_i is simply *an* element of $\overline{A}$ such that $\|f_i - g_i\| < \epsilon$ as described above, so that an appropriate ϵ-net is

$$\overline{A} \subseteq \bigcup_{i=1}^{N} B_\epsilon(f_i) \tag{149}$$

□

2. Problem 2

EXERCISE 3.2 (Squeeze theorem for Euclidean sets). *Let $K \subset U \subseteq \mathbb{R}^n$ where K is compact. Find V such that $K \subseteq V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact.*

PROOF. Cover K with balls interior to U and extract a finite subcover.

$$K \subseteq \bigcup_{x \in K} B_{\epsilon_x}(x) \cap K \tag{150}$$

$$\subseteq \bigcup_{i=1}^{n} B_{\epsilon_i}(x_i) \cap K \tag{151}$$

Define a family of open sets to help us find an appropriate set V. Set

$$V_\eta = \bigcup_{i=1}^{n} B_{\epsilon_i - \eta}(x_i) \subseteq \bigcup_{i=1}^{n} B_{\epsilon_i}(x_i) \subseteq U \tag{152}$$

Now refine. For each $x \in K$, select $\epsilon > 0$ and x_i such that $|x - x_i| < \epsilon < \epsilon_i$ by density. Then setting $\eta_x < \epsilon_i - \epsilon$, we can realize another open cover

$$K \subseteq \bigcup_{x \in K} V_{\eta_x} \tag{153}$$

$$\subseteq \bigcup_{j=1}^{m} V_{\eta_j} = V \tag{154}$$

Setting V as indicated, we can tell $\overline{V} \subseteq U$, as desired. □

3. Problem 3

EXERCISE 3.3 (Product of absolutely continuous functions). *Let $f, g : [0,1] \to \mathbb{R}$ be absolutely continuous. Then their product is absolutely continuous.*

PROOF. Select $\delta_1, \delta_2 > 0$ via absolute continuity so that

$$\sum_{i=1}^{n} |y_i - x_i| < \delta_1 \implies \sum_{i=1}^{n} |f(y_i) - f(x_i)| < \epsilon/\|g\| \tag{155}$$

$$\sum_{i=1}^{n} |y_i - x_i| < \delta_2 \implies \sum_{i=1}^{n} |g(y_i) - g(x_i)| < \epsilon/\|f\| \tag{156}$$

and set $\delta = \min\{\delta_1, \delta_2\}$. Then we have

$$\sum_{i=1}^{n} |(fg)(y_i) - (fg)(x_i)| = \sum_{i=1}^{n} |f(y_i)g(y_i) - f(x_i)g(x_i)| \tag{157}$$

$$= \sum_{i=1}^{n} |f(y_i)g(y_i) - f(y_i)g(x_i) + f(y_i)g(x_i) - f(x_i)g(x_i)| \tag{158}$$

$$\leq \sum_{i=1}^{n} |f(y_i)||g(y_i) - g(x_i)| + |g(x_i)||f(y_i) - f(x_i)| \tag{159}$$

$$\leq \sum_{i=1}^{n} \|f\||g(y_i) - g(x_i)| + \|g\||f(y_i) - f(x_i)| \tag{160}$$

$$\leq \|f\| \sum_{i=1}^{n} |g(y_i) - g(x_i)| + \|g\| \sum_{i=1}^{n} |f(y_i) - f(x_i)| \tag{161}$$

These sums are both bounded by ϵ if $\sum_i |y_i - x_i| < \delta$, indicating $fg : [0,1] \to \mathbb{R}$ is absolutely continuous. □

4. Problem 4

EXERCISE 3.4 (Radon-Nikodym). *If $\mu(X) < \infty$, $\{E_k\}_{k=1}^n$ are measurable, and $\{c_k\}_{k=1}^n$ are real, define a measure ν by*

$$\nu(E) := \sum_{k=1}^{n} c_k \mu(E \cap E_k) \tag{162}$$

PROOF. Verify that ν is a measure by checking countable additivity. Let $\{A_j\}_{j=1}^\infty$ be a sequence of disjoint measurable sets. Then

$$\nu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{k=1}^{n} c_k \mu\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap E_k\right) \tag{163}$$

$$= \sum_{k=1}^{n} c_k \mu\left(\bigcup_{j=1}^{\infty} (A_j \cap E_k)\right) \tag{164}$$

$$= \sum_{k=1}^{n} c_k \sum_{j=1}^{\infty} \mu(A_j \cap E_k) \tag{165}$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{n} c_k \mu(A_j \cap E_k) \tag{166}$$

$$= \sum_{j=1}^{\infty} \nu(A_j) \tag{167}$$

To see that $\nu \ll \mu$, suppose $\mu(A) = 0$. Then

$$\nu(A) = \sum_{k=1}^{n} \mu(A \cap E_k) \leq \sum_{k=1}^{n} \mu(A) = 0 \tag{168}$$

Now we show the Radon-Nikodym derivative equals $\sum_k c_k \mathbf{1}_{E_k}$. Observe:

$$\int_A \frac{d\nu}{d\mu} d\mu = \int_A \sum_{k=1}^{n} c_k \mathbf{1}_{E_k} \tag{169}$$

$$= \sum_{k=1}^{n} \int_A c_k \mathbf{1}_{E_k} d\mu \tag{170}$$

$$= \sum_{k=1}^{n} \int_{A \cap E_k} c_k d\mu \tag{171}$$

$$= \sum_{k=1}^{n} c_k \mu(A \cap E_k) \tag{172}$$

$$= \nu(A) \tag{173}$$

showing that which was to be shown. □

5. Problem 5

EXERCISE 3.5 (Closed unit ball in the weak topology). *Let X be a Banach space and $B = \{x \in X \mid \|x\| \leq 1\}$. Show that B is closed in the weak topology. Is the unit sphere closed in the weak topology?*

PROOF. Let $x \in X$ be a limit point of B. We may assume $x \neq 0$, so that there exists $\phi : X \to \mathbb{R}$ such that $\phi(x) = \|x\|$ and $\|\phi\| = 1$. Then for each $\epsilon > 0$, select x_ϵ in the neighborhood

$$\{y \in X \mid |\phi(y) - \phi(x)| < \epsilon\} \cap B \tag{174}$$

Then we have $\phi(x) < \phi(x_\epsilon) + \epsilon$. Then

$$\|x\| < \phi(x_\epsilon) + \epsilon \leq \|\phi\||x_\epsilon| + \epsilon \leq 1 + \epsilon \tag{175}$$

Since $\epsilon > 0$ is arbitrary, this means $\|x\| \leq 1$.

To see that the sphere is not necessarily closed in the weak topology, consider the Banach space $B = C([0,1])$ and the sequence of functions $f_n(x) = x^n$. The linear functional

$$\int : C([0,1]) \to \mathbb{R} \tag{176}$$

is bounded, but

$$\int f_n = \frac{1}{n+1} x^{n+1} \Big|_0^1 = \frac{1}{n+1} \to 0 \tag{177}$$

and $\|0\| \neq 1$. □

6. Problem 6

Exercise 3.6 (Spectrum is compact). *Let* $\sigma(A) \subseteq \mathbb{C}$ *be the spectrum of a bounded linear operator* $A : X \to X$*. Then* $\sigma(A)$ *is compact.*

Proof. To prove $\sigma(A)$ is bounded, recall the following sufficient condition for the convergence of a Neumann series which explicitly reconstructs the inverse

$$\|T\| < 1 \implies (I - T)^{-1} = \sum_{k=0}^{\infty} T^k \tag{178}$$

Therefore, if $I - T$ is not invertible, then $\|T\| \geq 1$. Recall the definition of $\sigma(A)$

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not invertible}\} \tag{179}$$

If $A - \lambda I$ is not invertible, certainly $I - A/\lambda$ is not invertible, so that $\|A/\lambda\| \geq 1$. This implies $|\lambda| \leq \|A\|$ so that $\sigma(A)$ is bounded.

For closure, suppose $\lambda_n \to \lambda$ is a sequence satisfying $\lambda_n \in \sigma(A)$ and $\lambda \notin \sigma(A)$. The latter assumption brings into existence a bounded map $B : X \to X$ such that

$$B(A - \lambda I) = I \tag{180}$$

A little Banach algebra reveals

$$B(A - \lambda_n I) = B(A - \lambda I) - B(\lambda_n I - \lambda I) \tag{181}$$

$$= I - B(\lambda_n I - \lambda I) \tag{182}$$

If N is selected so that $n \geq N$ implies $|\lambda_n - \lambda| < \frac{1}{\|B\|}$, then we may realize

$$\|B(\lambda_n I - \lambda I)\| < \|B\|/\|B\| = 1 \tag{183}$$

indicating $I - B(\lambda_n I - \lambda I) = B(A - \lambda_n I)$ is invertible, so that $A - \lambda_n I$ is invertible, contradicting the selection $\lambda_n \in \sigma(A)$. Therefore, $\lambda \in \sigma(A)$, so that $\sigma(A)$ is closed.

Now since $\sigma(A)$ lies in a finite-dimensional space, closed and bounded exactly prove that $\sigma(A)$ is compact. A proof that $\sigma(A)$ is non-empty is saved for Fall 2019 Exercise 5. □

7. Problem 7

EXERCISE 3.7 (Rouché's Theorem on a Geometric Progression).

(a) State Rouché's theorem.

(b) Find the number Z_n of zeroes of p_n as $n \to \infty$ within the closed contour $C = \partial B_1(1/2)$ where

$$p_n(z) = z^2 - 2\left(\frac{z}{3} + \cdots + \frac{z^n}{3^n}\right) \tag{184}$$

THEOREM 4 (Rouché's Theorem). *Suppose $h = f + g$ where f and g are holomorphic on the interior of some closed contour C and moreover that $|f(z)| > |g(z)|$ on the contour C. Then f and h have the same number of zeros in the interior of C.*

Now begin the problem.

PROOF. Note that $p_n(z)$ converges uniformly on the set disk $D = \overline{B_1(1/2)}$ to

$$p(z) = z^2 - 2\frac{z/3}{1 - z/3} = z^2 - \frac{2z}{3 - z}. \tag{185}$$

Solve for the zeroes:

$$0 = z^2 - \frac{2z}{3 - z} = z^2(3 - z) - 2z = z\left[z(3 - z) - 2\right] = z[-z^2 + 3z - 2] = -z(z - 1)(z - 2) \tag{186}$$

Only two of the roots $z = 0$ and $z = 1$ lie inside the contour C. None of the roots lie on the contour, which indicates $m = \min|z^2 - 2z/(3 - z)| > 0$ where the minimum is taken over C.

Now we can apply the Rouché theorem with $h = p$ as defined above, $f = p_n$ and $g = p - p_n$. On the contour C, let us verify the inequality. Let N be such that

$$|g(z)| = |p(z) - p_n(z)| = \left|\sum_{k=n+1}^{\infty} \frac{z^k}{3^k}\right| < m/2 \quad \text{for all } n \geq N \text{ and } z \text{ in the disk} \tag{187}$$

Then

$$|f(z)| = \left|z^2 - 2\left(\frac{z}{3} + \cdots + \frac{z^n}{3^n}\right)\right| \tag{188}$$

$$= \left|z^2 - 2\left(\frac{z}{3 - z} - \sum_{k=n+1}^{\infty} \frac{z^k}{3^k}\right)\right| \tag{189}$$

$$= \left|z^2 - \frac{2z}{3 - z} + \sum_{k=n+1}^{\infty} \frac{2z^k}{3^k}\right| \tag{190}$$

$$\geq ||z^2 - 2z/(3 - z)| - |\Sigma|| \tag{191}$$

$$> m/2 \tag{192}$$

Since $|g(z)| < m/2$ and $|f(z)| > m/2$, this shows that Rouché's theorem applies, indicating p_n and p have the same number of roots, namely 2, inside the contour C. □

8. Problem 8

Exercise 3.8 (A sector-based contour integral). *Evaluate for $p \geq 1$*

$$\int_{\mathbb{R}} \frac{dx}{1+x^{2p}} \tag{193}$$

Proof. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{1+z^{2p}} \tag{194}$$

Capture the pole with least argument $x_0 = \exp(i\pi/2p)$ in the sector $S_R = \{re^{i\theta} \mid 0 \leq r \leq R, 0 \leq \theta \leq \pi/p\}$. Compute the residue

$$\operatorname{Res}(f, x_0) = \lim_{z \to x_0} (z - x_0) f(z) = \lim_{z \to x_0} \frac{z - x_0}{1 + z^{2p}} = \lim_{z \to x_0} \frac{1}{2pz^{2p-1}} = \frac{1}{2px_0^{2p-1}} \tag{195}$$

The residue theorem states

$$\int_{\partial S_R} f(z)dz = \left(\int_0^R + \int_{\text{arc}} + \int_{\text{line}} \right) f(z)dz = 2\pi i \operatorname{Res}(f, z_0) = \frac{\pi i}{px_0^{2p-1}} \tag{196}$$

The line integral can be found by parametrizing $\gamma : [0, R] \to \mathbb{C}$ by $\gamma(t) = (R-t)e^{i\pi/p}$. Then

$$\int_{\text{line}} f(z)dz = \int_0^R \frac{1}{1+(R-t)^{2p}} (-e^{i\pi/p})dt \tag{197}$$

$$= -e^{i\pi/p} \int_0^R f(z)dz \tag{198}$$

$$= -x_0^2 \int_0^R f(z)dz \tag{199}$$

The arc integral vanishes as $R \to \infty$. Set $\rho(t) = Re^{it}$ for $t \in [0, \pi/p]$. Then $\rho'(t) = Rie^{it}$ and the arc integral equals

$$\int_{\text{arc}} f(z)dz = \int_0^{\pi/p} f(\rho(t))\rho'(t)dt = \int_0^{\pi/p} \frac{Rie^{it}}{1+R^{2p}e^{2ipt}} \tag{200}$$

Taking absolute values, we see

$$\left| \int_{\text{arc}} f(z)dz \right| \leq \int_0^{\pi/p} \left| \frac{Rie^{it}}{1+R^{2p}e^{2ipt}} \right| \leq \int_0^{\pi/p} \frac{R}{R^{2p}-1} dt \leq \frac{\pi}{p} \cdot \frac{R}{R^{2p}-1} \to 0 \tag{201}$$

Therefore, we can take limits and rearrange the integral-residue equation to find

$$(1 - x_0^2) \int_0^\infty f(z)dz = \frac{\pi i}{px_0^{2p-1}} = \frac{\pi i}{px_0^{2p}/x_0} = \frac{\pi i}{p(-1)/x_0} = -\frac{\pi i x_0}{p} \tag{202}$$

Therefore,

$$\int_{\mathbb{R}} \frac{dx}{1+x^{2p}} = \frac{2\pi i x_0}{p(x_0^2 - 1)} = \frac{2\pi i}{p(x_0 - x_0^{-1})} = \frac{2\pi i}{p(2i \sin(\pi/2p))} = \frac{\pi}{p \sin(\pi/2p)} \tag{203}$$

□

CHAPTER 4

Spring 2019 TODO 4,6,8

1. Problem 1

EXERCISE 4.1 (Non-contractive mapping). *Define* $T : \mathbb{R} \to \mathbb{R}$ *by*

$$T(x) := \frac{\pi}{2} + x - \arctan(x) \tag{204}$$

Show that $|T(x) - T(y)| \leq |x - y|$ *for all* $x, y \in \mathbb{R}$ *and that* T *has no fixed points in* $\mathbb{R}$. *State the contraction mapping theorem and explain why this example does not contradict the theorem.*

PROOF. For the Lipschitz estimate, we estimate the first derivative of T by formal differentiation rules:

$$T'(w) = 1 - \frac{1}{1 + w^2} \leq 1 \tag{205}$$

The fundamental theorem reveals

$$T(x) = \int_0^x T'(w)dw \tag{206}$$

$$T(y) = \int_0^y T'(w)dw \tag{207}$$

Therefore the difference can be estimated

$$|T(x) - T(y)| = \left| \int_y^x T'(w)dw \right| \leq |x - y| \tag{208}$$

The contraction mapping theorem states that if $|T(x) - T(y)| \leq c|x - y|$ for some $c < 1$, then the map T has a fixed point. In this example we did not select such a $c < 1$, so we are comfortable now proving that actually is no fixed point. Suppose $T(x) = x$. Then $\pi/2 = \arctan(x)$, which is never true. □

2. Problem 2

EXERCISE 4.2 (Squeeze theorem for compact Euclidean sets). *Suppose K is a compact set contained in an open set U. Find an open set V whose closure is compact and*

$$K \subseteq V \subseteq \overline{V} \subseteq U \tag{209}$$

PROOF. See Exercise 3.2. $\square$

3. Problem 3

Exercise 4.3 (Lipschitz functions preserve measure zero sets). *Prove that a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ maps sets of Lebesgue measure zero to sets of Lebesgue measure zero. For which values of n and m does the same statement hold for Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}^m$?*

Proof. If $n < m$, then measure can spring up from nothing, as in the following example. Consider a line segment in $\mathbb{R}^2$, having zero measure. The projection map is Lipschitz, and sends the line segment to a subset of full measure in $[0, 1]$.

If $n = m$, identify a Lipschitz coefficient M and proceed by covering the image of a measure zero set A

$$A \subseteq \bigcup_{k=1}^{\infty} B_{\delta_k}(x_k) \quad \text{and} \quad \sum_{k=1}^{\infty} \mu(B_{\delta_k}) < \epsilon / M^n \tag{210}$$

Then the image is contained also in balls with expanded radii

$$f(A) \subseteq \bigcup_{k=1}^{\infty} B_{M\delta_k}(f(x_k)) \tag{211}$$

A dilation by M introduces a factor M^n, so that

$$\sum_{k=1}^{\infty} \mu(B_{M\delta_k}) = \sum_{k=1}^{\infty} M^n \mu(B_{\delta_k}) = M^n \sum_{k=1}^{\infty} \mu(B_{\delta_k}) < M^n \epsilon / M^n = \epsilon \tag{212}$$

If $n > m$, then zero measure sets remain zero measure by realizing that cubes in $\mathbb{R}^m$ have zero measure in $\mathbb{R}^n$ by the construction of the product measure. From this it follows that balls also have measure zero when included into higher dimensional spaces. □

4. Problem 5

Exercise 4.4 (Inversions and Estimates in Banach spaces). *Let X be a Banach space and $A \in L(X)$ be a bounded linear operator. Show that there exists a bounded linear operator $B \in L(X)$ satsifying $AB = BA = I_X$ if and only if there exists a constant $\gamma > 0$ such that*

$$\|x\| \leq \gamma \|Ax\| \quad \text{and} \quad \|\phi\| \leq \gamma \|A^* \phi\| \quad \text{for all } x \in X \text{ and } \phi \in X^* \tag{213}$$

Proof. Suppose a bounded inverse B exists satisfying $AB = BA = I_X$. Set $\gamma = \|B\|$. Then for any $x \in X$ we have

$$\|x\| = \|I_X x\| = \|B(Ax)\| \leq \gamma \|Ax\| \tag{214}$$

To prove the other estimate we take adjoints: $(I_X)^* = (AB)^* = B^* A^*$. Recall that $\|B^*\| = \|B\| = \gamma$, so for any $\phi \in X^*$ we can estimate directly

$$\|\phi\| = \|I_X^* \phi\| = \|B^* A^* \phi\| \leq \|B^*\| \|A^* \phi\| = \gamma \|A^* \phi\| \tag{215}$$

For the converse, let $\gamma > 0$ entail the above estimates. We can see A is injective because if $Ax = Ay$, then setting $z = x - y$ shows

$$\|x - y\| = \|z\| \leq \gamma \|Az\| = \gamma \|Ax - Ay\| = 0. \tag{216}$$

The second estimate will let us show A is surjective. To apply the open mapping theorem, we verify that $\frac{1}{\gamma} U \subseteq \overline{A(U)}$, where $U = \{x \in X \mid \|x\| < 1\}$. Suppose $y \notin \overline{A(U)}$. The set $\overline{A(U)}$ is closed, balanced, and convex, so there exists a linear functional $\phi : X \to \mathbb{C}$ such that $|\phi(y)| > 1$ and $|\phi(Ax)| \leq 1$ for $\|x\| \leq 1$. Since $\phi(Ax) = A^*(\phi)(x)$, the second estimate shows $\|A^* \phi\| \leq 1$. Putting these all together,

$$\frac{1}{\gamma} < \frac{1}{\gamma} |\phi(y)| \leq \frac{1}{\gamma} \|\phi\| \|y\| \leq \|A^* \phi\| \|y\| \leq \|y\| \tag{217}$$

we see that $\|y\| \geq 1/\gamma$. Therefore, if $\|y\| < 1/\gamma$, then $y \in \overline{A(X)}$, so that $\frac{1}{\gamma} U \subseteq \overline{A(X)}$ which implies A is surjective. Now that A is a continuous bijection, the inverse mapping theorem implies that A^{-1} is a bounded linear operator. □

The interested reader is welcomed to read Proposition 6.8.5 of [**4**], which outlines a more general case of the surjectivity aspect of this exercise, but not the injectivity. Theorem 4.13 of [**11**] does the same.

5. Problem 7

Exercise 4.5 (Two Contour Integrals). *Using complex analysis, evaluate the integrals*

$$I_1 = \int_0^\infty \frac{1-\cos x}{x^2} dx, \quad I_2 = \int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta. \tag{218}$$

Proof. The first integral can be evaluated as in Fall 2019 Exercise 8 by taking

$$\int_0^\infty \frac{1-\cos x}{x^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{1-\cos x}{x^2} = \Re \int_{-\infty}^\infty \frac{1-e^{iz}}{z^2}. \tag{219}$$

The residue of the integrand is found by a series about $z = 0$:

$$\frac{1-e^{iz}}{z^2} = \frac{1-(1+iz-z^2/2+\cdots)}{z^2} = -\frac{i}{z}+\cdots \tag{220}$$

so that $\text{Res}_{z=0} = -i$. For an appropriate contour, take a semicircular arc with a dimple at the origin. The contour integral can be split into a few integrals, most importantly the line segments and dimple

$$\int_{\text{dimple}} + \int_\epsilon^R \frac{1-e^{iz}}{z^2} + \int_{-R}^{-\epsilon} \frac{1-e^{iz}}{z^2} = 2\pi i(-i) = 2\pi \tag{221}$$

Since the dimple 'winds around' the origin one-half times,

$$\int_{\text{dimple}} = 2\pi i/2(-i) = \pi, \tag{222}$$

so that

$$\int_\epsilon^R \frac{1-e^{iz}}{z^2} + \int_{-R}^{-\epsilon} \frac{1-e^{iz}}{z^2} = 2\pi i(-i) = \pi. \tag{223}$$

As $\epsilon \to 0$ and $R \to \infty$, we see

$$\int_{-\infty}^\infty \frac{1-e^{iz}}{z^2} = 2\pi \tag{224}$$

The second integral is handled similarly to Winter 2021 Problem 7. Let $z = e^{i\theta}$ for $\theta \in [0, 2\pi]$ so that

$$-2i\int \frac{1}{z^2+4z+1} = -2i\int \frac{1/z}{z+4+1/z} \tag{225}$$

$$= -2i\int_0^{2\pi} \frac{e^{-i\theta}}{e^{i\theta}+e^{-i\theta}+4} ie^{i\theta} d\theta \tag{226}$$

$$= 2\int_0^{2\pi} \frac{1}{2\cos\theta+4} d\theta \tag{227}$$

$$= \int_0^{2\pi} \frac{1}{\cos\theta+2} d\theta = \tag{228}$$

The first integral can be evaluated with the residue theorem. Identify the poles by solving

$$z^2+4z+1 = 0 \iff z = \frac{-4\pm\sqrt{16-4}}{2} = -2\pm\sqrt{3}. \tag{229}$$

Only the pole at $z = -2+\sqrt{3}$ lies inside the contour of integration, so the residue here is the only one we need to compute, as follows:

$$\text{Res} = \lim_{z\to-2+\sqrt{3}} \frac{z-(-2+\sqrt{3})}{z^2+4z+1} = \lim \frac{1}{2z+4} = \frac{1}{2(-2+\sqrt{3})+4} = \frac{1}{2\sqrt{3}}. \tag{230}$$

Therefore,

$$\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta = -2i\int = -2i(2\pi i/2\sqrt{3}) = \frac{2\pi}{\sqrt{3}}. \tag{231}$$

□

6. Problem 8

EXERCISE 4.6 (Polynomial ideals TODO).

PROOF. The estimate

$$|f(z)| \leq A(1 + |z|^{-s}) \tag{232}$$

implies that

$$|z^s f(z)| \leq A|z|^s + A, \tag{233}$$

which indicates $z^s f(z)$ is a polynomial, say:

$$z^s f(z) = a_0 + \cdots + a_r z^r \tag{234}$$

so that

$$f(z) = \frac{a_0}{z^s} + \cdots + a_r z^{r-s} \tag{235}$$

Conversely, if $f(z)$ is a sum as written above, then take $A = \sum |a_k|$. By the triangle inequality

$$|f(z)| \leq |a_0||z|^{-s} + |a_1||z^{1-s}| + \cdots + |a_r||z^{r-s}| \tag{236}$$

□

CHAPTER 5

Fall 2019

1. Problem 1

EXERCISE 5.1 (A vanishing argument for odd functions). *Let* $f : [-1, 1] \to \mathbb{R}$ *be a continuous odd function and suppose*

$$\int_{-1}^{1} f(x) x^{2k-1} dx = 0 \tag{237}$$

for all $k > 0$*. Then* $f(x) \equiv 0$*.*

PROOF. Include $f \in L^2$ by noticing that f being uniformly continuous implies f^2 is uniformly continuous. Then consider the subspace of odd functions in L^2 which also contains f:

$$F = \{g \in L^2 \mid g(-x) = -g(x) \quad \forall x \in [-1, 1]\} \tag{238}$$

which has a countable dense subset, namely, $\{x, x^3, x^5, \dots\}$ by a similar argument to the Weierstrass approximation theorem. The inner product in this space naturally arises as

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx \tag{239}$$

and we know $\langle f, x^m \rangle = 0$ for any basic element x^m. Therefore, $f \equiv 0$. □

2. Problem 2

Exercise 5.2 (Urysohn in a metric space).

(a) Let X be a locally compact Hausdorff space and $K \subset V \subset X$ where K is compact and V is open. State the Urysohn Lemma in terms of K and V.

(b) Let (X, d) be a metric space. For a non-empty subset $A \subset X$, the function

$$d_A(x) := \inf\{d(x,a) \mid a \in A\} \tag{240}$$

is uniformly continuous.

(c) For disjoint closed sets A and B, define a continuous function $f : X \to [0,1]$ for which $f(A) = 0$ and $f(B) = 1$. Relate this function to the Urysohn Lemma.

Lemma 1 (Urysohn Lemma). *A topological space X is normal if and only if for all $K \subset V$ with K compact and V open, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(K) = 0$ and $f(X \setminus V) = 1$,*

For the rest of the problem:

Proof. To perform part (b), the function $d(\cdot, F)$ can be argued to be uniformly continuous as follows. Let $0 < d(x,y) < \epsilon$. Then

$$|d(x,F) - d(y,F)| = \left|\inf_f d(x,f) - \inf_f d(y,f)\right| = \inf_f |d(x,f) - d(y,f)| \tag{241}$$

For any $f \in F$, we have $\inf \leq |d(x,f) - d(y,f)|$. Then

$$|d(x,F) - d(y,F)| \leq |d(x,f) - d(y,f)| \leq d(x,y) < \epsilon \tag{242}$$

Therefore, $d(\cdot, F)$ is uniformly continuous.

Now to perform part (c), define

$$f(x) := \frac{d_A(x)}{d_A(x) + d_B(x)}. \tag{243}$$

The denominator is never equal to zero, so this function inherits continuity from the functions it is composed of. To see that $d(x,A)+d(x,B) \neq 0$, suppose otherwise. Then $d(x,A) = d(x,B) = 0$ which implies $x \in A \cap B$, contradicting that A and B are disjoint. Therefore, the function is continuous, and we can look at its action on elements in A or in B: if $x \in A$, then $d(x,A) = 0$, so $f(x) = 0$. If $x \in B$, then $d(x,B) = 0$, so $f(x) = d(x,A)/d(x,A) = 1$.

By setting $A = K$ and $B = X \setminus V$, we can prove the Urysohn Lemma in one direction. □

3. Problem 3

EXERCISE 5.3 (A summatory condition for decaying measure). *Prove that*

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty \implies \mu\left(\limsup_{n\to\infty} E_n\right) = 0 \tag{244}$$

PROOF. Recall the definition of lim sup for sets:

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m. \tag{245}$$

It follows that

$$\mu\left(\limsup_{n\to\infty} E_n\right) \leq \mu\left(\bigcup_{m=n}^{\infty} E_m\right) \quad \forall n \geq 1 \tag{246}$$

By countable subadditivity, we know

$$\mu\left(\bigcup_{m=n}^{\infty} E_m\right) \leq \sum_{m=n}^{\infty} \mu(E_m) \quad \forall n \geq 1 \tag{247}$$

But $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ implies

$$\lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(E_m) = 0 \tag{248}$$

Applying this limit to equation 246 then shows $\mu(\limsup E_n) = 0$:

$$\mu\left(\limsup_{n\to\infty} E_n\right) \leq \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(E_m) = 0 \tag{249}$$

□

See Exercise 6.2 for a proof invoking continuity from above.

4. Problem 4

Exercise 5.4 (Absolutely continuous measures). *Let* (X, Σ, μ) *be a finite measure space and suppose* $f, g > 0$. *Define the measures*

$$\nu(E) = \int_E f d\mu \quad \eta(E) = \int_E g d\mu \tag{250}$$

Is $\nu \ll \eta$*? Is* $\eta \ll \nu$*?*

Proof. Compute $R = \operatorname{ess\,sup} g(x)/f(x)$. This is finite because X is a finite measure space and f and g are strictly positive.

If $\nu(E) = 0$, we can show $\eta(E) = 0$. By the definition of the Lebesgue integral, select

$$f \leq \sum_{k=1}^{n} c_k \mathbf{1}_{E_k} \tag{251}$$

satisfying

$$\int_E f \leq \int \sum_{k=1}^{n} c_k \mathbf{1}_{E_k} < \epsilon / R \tag{252}$$

By the selection of R, it can be readily seen that

$$\frac{g(x)}{f(x)} \leq R \implies g(x) \leq R f(x) \tag{253}$$

Apply the monotonicity of the integral to find

$$\int_E g \leq R \int_E f < R\epsilon / R = \epsilon \tag{254}$$

Since $\epsilon > 0$ is arbitrary, this implies $\eta(E) = 0$, so that $\eta \ll \nu$. A similar argument can be made with $R' = 1/R$ to show that $\nu \ll \eta$. □

5. Problem 5

Exercise 5.5 (Spectrum is closed and bounded). *Show that the spectrum of a bounded linear operator on a complex Banach space is a non-empty compact subset of $\mathbb{C}$. Does the same hold for operator on real Banach spaces?*

Proof. Examining Winter 2019 Exercise 6 shows that *this* problem now requires us to show only that the spectrum is non-empty. Let $B : X \to X$ be a bounded linear operator on a Banach space. Suppose the spectrum were empty. That is,

$$\sigma(B) = \{\lambda \in \mathbb{C} \mid \; B - \lambda I \text{ is not invertible}\} = \{\} \tag{255}$$

Then $B - \lambda I$ is invertible for any complex λ, so the resolvent $R_\lambda = (B - \lambda I)^{-1}$ is defined for any λ. By the open mapping, each resolvent is bounded. Let $\phi \in \mathcal{L}(X)^*$ be a non-zero linear functional on the space of bounded opeators. Define a function $F : \mathbb{C} \to \mathbb{C}$

$$F(\lambda) = \phi(R_\lambda) \tag{256}$$

is entire with the sense of operator norm convergence. Taking the modulus we see that $|F(\lambda)| \to 0$ as $|\lambda| \to \infty$, which indicates $F \equiv 0$, so that $X = 0$, a contradiction. Therefore, $B - \lambda I$ is not invertible for some λ.

The compactness does not hold in a real setting. Consider the operator

$$\int : C([0,1]) \to C([0,1]) \tag{257}$$

$$f \mapsto \int_0^x f(t)dt \tag{258}$$

where $C([0,1])$ is the set of continuous real valued functions define over $[0,1]$. Let us follow a familiar derivation of the eigenvalues. Let

$$\int_0^x f(t)dt = \lambda f(x) \tag{259}$$

The very act of writing this indicates that we may differentiate on either side, so that

$$f(x) = \lambda f'(x) \iff f(x) = ke^{\lambda x} \tag{260}$$

But then the spectrum contains the real line, so it must not be compact. ☐

6. Problem 6

Exercise 5.6 (Sequence of Bounded Operators on a Banach space).

(a) Let $\{A_n : X \to X\}_{n=1}^{\infty}$ be a sequence of bounded linear operators on a Banach space X such that $A_n x$ converges for every $x \in X$. Show the following operator on X is bounded:

$$Ax := \lim_{n\to\infty} A_n x. \tag{261}$$

(b) Can the same conclusion be drawn if X is not a Banach space?

Proof. We can argue that $\|A\| < \infty$ using uniform boundedness. The convergence hypothesis implies that

$$\sup_{n\geq 1} \|A_n x\| < \infty \quad \forall x \in X \tag{262}$$

Uniform boundedness says

$$\sup_{n\geq 1} \|A_n\| < \infty \tag{263}$$

We will use this estimate to prove that

$$\|A\| = \sup_{\|x\|=1} \lim_{n\to\infty} \|A_n x\| < \infty \tag{264}$$

The limit within the supremum always exists by the convergence hypothesis, so for any x it is true that

$$\lim_{n\to\infty} \|A_n x\| = \liminf_{n\to\infty} \|A_n x\| = \sup_{n\geq 1} \inf_{m\geq n} \|A_m x\| \tag{265}$$

This is substituted into the equation for $\|A\|$, and we interchange some limits to see

$$\|A\| = \sup_{\|x\|=1} \lim_{n\to\infty} \|A_n x\| = \sup_{\|x\|=1} \sup_{n\geq 1} \inf_{m\geq n} \|A_m x\| \tag{266}$$

$$= \sup_{n\geq 1} \sup_{\|x\|=1} \inf_{m\geq n} \|A_m x\| \tag{267}$$

$$\leq \sup_{n\geq 1} \inf_{m\geq n} \sup_{\|x\|=1} \|A_m x\| \tag{268}$$

$$= \sup_{n\geq 1} \inf_{m\geq n} \|A_m\| \tag{269}$$

$$\leq \sup_{n\geq 1} \|A_n\| \tag{270}$$

$$< \infty \tag{271}$$

Consider the sequence of bounded operators

$$T_n : C(\mathbb{R}) \to \mathbb{R} \tag{272}$$

$$f \mapsto \int_{-n}^{n} f(x)dx \tag{273}$$

Each integration T_n is over a compact domain, so the operators are bounded. But the limit operator is integration over the whole real line, which is unbounded, for example in the case of constant functions. □

7. Problem 7

EXERCISE 5.7 (Maximum modulus principle and its sibling). *Let $\Omega \subseteq \mathbb{C}$ be a connected domain and let $f(z)$ be holomorphic on Ω. Show that neither $\Re[f(z)]$ nor $|f(z)|$ attain a maximum on Ω unless f is constant.*

PROOF. This is the maximum modulus principle, which the question is asking us to prove. Suppose $|f(z_0)| \geq |f(z)|$ over Ω. Find a power series

$$f(z) = a_0 + a_1(z - z_0) + \cdots \tag{274}$$

in a region about z_0. If f is constant, we are done, so suppose $a_1 \neq 0$. In this case, f is a locally an open mapping, so select $r > 0$ such that $f(B_r(z_0))$ is open. Note that a_0 lies in this set, so select $\delta > 0$ such that

$$B_\delta(a_0) \subseteq f(B_r(z_0)) \tag{275}$$

Let $a_0 = a + bi$. If $a > 0$, find $a_0 + \delta/2 = f(z_0 + w)$ where $|w| < r$. Then

$$|f(z_0 + w)| = |(a + \delta/2) + bi| = \sqrt{(a + \delta/2)^2 + b^2} > \sqrt{a^2 + b^2} = |a_0| = |f(z_0)| \tag{276}$$

violates that $|f(z_0)|$ is the maximum modulus. Similarly if $a < 0$, subtract $\delta/2$ to find the same contradiction.

To prove the maximum real part principle, suppose $\Re[f(z_0)] \geq \Re[f(z)]$ and pass f to the exponential function. We have:

$$e^{f(z)} = e^{\Re f(z) + \Im f(z)} \tag{277}$$

which implies

$$|e^{f(z)}| = e^{\Re f(z)} \tag{278}$$

The previous result shows that if this function has a maximum, then the function is constant, which indicates $\Re f(z)$ is constant by the monotonicity of the exponential function, completing the proof. ☐

8. Problem 8

EXERCISE 5.8 (Sinc Integral!). *Integrate*

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \tag{279}$$

PROOF. Consider the complexified function $f(z) = e^{iz}/z$. Then

$$\int_{-\infty}^{\infty} \frac{\sin z}{z} = \Im \int_{-\infty}^{\infty} \frac{e^{iz}}{z} \tag{280}$$

By writing e^{iz} as a Taylor series, we can divide to argue that the residue at zero equals one:

$$\frac{e^{iz}}{z} = \frac{1 + iz - z^2/2 + \cdots}{z} = \frac{1}{z} + i + \cdots \tag{281}$$

Integrate f around a semicircular arc with radius R and a dimple in the lower half-plane centered at the origin with radius ϵ. Then the residue theorem says this path integral captures the pole, so that

$$\int_{\text{dimple}} + \int_{\epsilon}^{R} + \int_{\text{arc}} + \int_{-R}^{-\epsilon} = 2\pi i \tag{282}$$

The arc integral vanishes as $R \to \infty$ in the upper half-plane. Set $z = Re^{i\theta}$. Then

$$e^{iz} = e^{iRe^{i\theta}} = e^{iR(\cos\theta + i\sin\theta)} = e^{iR\cos\theta - R\sin\theta} \implies |e^{iz}| = e^{-R\sin\theta} \tag{283}$$

Simplify the following integral

$$\int_{\text{arc}} = \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} Rie^{i\theta} d\theta = i \int_0^{\pi} e^{iRe^{i\theta}} d\theta \tag{284}$$

Then we can apply the previous estimate

$$\left| \int_{\text{arc}} \right| \leq \int_0^{\pi} e^{-R\sin\theta} d\theta \tag{285}$$

The integrand converges to 0 because $\sin\theta$ is nonnegative on this interval. Moreover, $e^{-R\sin\theta}$ is continuous and $[0, \pi]$ is compact, so we may exchange limits. Therefore, the arc integral vanishes.

The dimple integral is handled by letting $\epsilon \to 0$. Set $z = \epsilon e^{i\theta}$. We have

$$\int_{\text{dimple}} = i \int_{\pi}^{2\pi} e^{i\epsilon e^{i\theta}} d\theta \tag{286}$$

Again, we can apply integral interchange because of smoothness, so the dimple integral is sent to πi as $\epsilon \to 0$.

Therefore, as $R \to \infty$ and $\epsilon \to 0$, we see

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} = \pi i \tag{287}$$

Taking, the imaginary part, we see:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi \tag{288}$$

□

CHAPTER 6

Spring 2020 TODO

1. Problem 2

Exercise 6.1 (Metric for Closed Sets). *Suppose (X, d) is a bounded metric space. Let*

$$d(x, A) := \inf\{d(x, a) \mid a \in A\}. \tag{289}$$

And define

$$d_H(A, B) := \inf\{\epsilon > 0 \mid A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\} \tag{290}$$

where

$$N_\epsilon(A) := \{x \in X \mid d(x, A) < \epsilon\}. \tag{291}$$

Show that d_H is a distance function on the space of all closed subsets in X.

Proof. To see that zero distance implies equality, suppose $d_H(A, B) = 0$. Select $\epsilon > 0$ such that $A \subseteq N_\epsilon(B)$ and $B \subseteq N_\epsilon(A)$. Sending $\epsilon \to 0$, $N_\epsilon(B) \to B$ by closure and similarly $N_\epsilon(A) \to A$. The inclusions then indicate that $A \subseteq B$ and $B \subseteq A$, so that $A = B$.

For symmetry, statements around a logical 'and' may be commuted, so that

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\} \tag{292}$$

$$= \inf\{\epsilon > 0 \mid B \subseteq N_\epsilon(A) \text{ and } A \subseteq N_\epsilon(B)\} = d_H(B, A) \tag{293}$$

For the triangle inequality, we will prove the estimate

$$d_H(A, C) \leq d_H(A, B) + d_H(B, C). \tag{294}$$

Let $\epsilon > 0$ satisfy

$$A \subseteq N_\epsilon(B) \tag{295}$$

$$B \subseteq N_\epsilon(A) \tag{296}$$

and $\epsilon' > 0$ satisfy

$$B \subseteq N_{\epsilon'}(C) \tag{297}$$

$$C \subseteq N_{\epsilon'}(B) \tag{298}$$

The N operator acts convexly, so

$$A \subseteq N_\epsilon(B) \subseteq N_\epsilon(N_{\epsilon'}(C)) \subseteq N_{\epsilon+\epsilon'}(C). \tag{299}$$

Therefore, $A \subseteq N_{\epsilon+\epsilon'}(C)$. For the reverse inclusion, note that

$$C \subseteq N_{\epsilon'}(B) \subseteq N_{\epsilon'}(N_\epsilon(A)) \subseteq N_{\epsilon'+\epsilon}(A) \tag{300}$$

so that $C \subseteq N_{\epsilon'+\epsilon}(A)$. Therefore,

$$d_H(A, C) \leq \epsilon + \epsilon'. \tag{301}$$

Taking the infimum over the indicated ϵ and ϵ' yields

$$d_H(A, C) \leq d_H(A, B) + d_H(B, C). \tag{302}$$

□

2. Problem 3

Exercise 6.2 (Borel-Cantelli). *Consider the measure space* $(X, \mathcal{M}, \mu)$ *with* μ *a positive measure. Let* $\{E_k\}$ *be a countable family of measurable sets satisfying*

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty. \tag{303}$$

Define

$$E := \{x \in \mathbb{R} \mid x \in E_k \textit{ for infinitely many } k\}. \tag{304}$$

Prove the following:

(a) E *is measurable*

(b) $\mu(E) = 0$.

Proof. For part (a), we can construct E from σ-algebra operations:

$$E = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k \tag{305}$$

For part (b), the definition of a converging series from basic real analysis tells us that

$$\lim_{m\to\infty} \sum_{k=m}^{\infty} \mu(E_k) = 0. \tag{306}$$

Intersections are decreasing and the converging series guarantees the first set has finite measure from the following estimate

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k) < \infty \tag{307}$$

Now apply continuity from above:

$$\mu(E) = \lim_{m\to\infty} \mu\left(\bigcup_{k=m}^{\infty} E_k\right) \leq \sum_{k=m}^{\infty} \mu(E_k) \tag{308}$$

where the inequality follows from the countable subadditivity of measure. Applying the limit $m \to \infty$ on either side yields the desired result. □

See Exercise 5.3 for a proof invoking monotonicity of measure.

3. Problem 4

EXERCISE 6.3 (Indicator function limit). *Let $f : [0,1] \to \mathbb{R}$ be continuous and $g : [0,1] \to [0,1]$ measurable. Compute the limit*

$$\lim_{n\to\infty} \int_0^1 f(g(x)^n)dx \tag{309}$$

PROOF. The function f has domain a compact set, so f is bounded. The domain of integration has finite measure and the function $f \circ g^n$ is measurable, so the bounded convergence theorem allows the limit interchange

$$\lim_{n\to\infty} \int_0^1 f(g(x)^n)dx = \int_0^1 \lim_{n\to\infty} f(g(x)^n)dx \tag{310}$$

As $n \to \infty$ the function g^n becomes an indicator function

$$\lim_{n\to\infty} g(x)^n = \begin{cases} 1 & x \in E \\ 0 & x \in [0,1] \setminus E \end{cases} \tag{311}$$

where $E = g^{-1}(\{1\})$. The continuity of f then reveals $f \circ g^n$ becomes an indicator function

$$\lim_{n\to\infty} f(g(x)^n) = \begin{cases} f(1) & x \in E \\ f(0) & x \in [0,1] \setminus E \end{cases} \tag{312}$$

Therefore,

$$\lim_{n\to\infty} \int_0^1 f(g(x)^n) = \int_0^1 \begin{cases} f(1) & x \in E \\ f(0) & x \in [0,1] \setminus E \end{cases} dx \tag{313}$$

$$= \int_E f(1)dx + \int_{[0,1]\setminus E} f(0)dx \tag{314}$$

$$= \mu(E)f(1) + \mu([0,1] \setminus E)f(0) \tag{315}$$

□

CHAPTER 7

Fall 2020

1. Problem 1

Exercise 7.1 (Continuous bijections, compactness, Hausdorff, and gluing).

(a) Let $f : X \to Y$ be a continuous bijection where X is compact and Y is Hausdorff. Then f is a homeomorphism.

(b) Let $X = A \cup B$ where A and B are closed subsets of X. Suppose $f : X \to Y$ is a map such that $f|_A$ and $f|_B$ are continuous. Then f is continuous.

Proof. For part (a): we show $(f^{-1})^{-1}(S)$ is closed for any closed $S \subseteq X$. Let $S \subseteq X$ be closed. The inverses simplify to $f(S)$ because f is a bijection. Since X is compact and S is closed, it follows that S is compact. Continuous images of compact maps are compact, so we know $f(S)$ is compact. Since Y is Hausdorff, this implies $f(S)$ is closed.

For part (b), let us glue by hand. We know $f(X) = f(A) \cup f(B)$, so if $U \subseteq f(X)$ is open, $U = U \cap f(A) \cup U \cap f(B)$. By writing this we see

$$f^{-1}(U) = f^{-1}(U \cap f(A) \cup U \cap f(B)) = f^{-1}(U \cap f(A)) \cup f^{-1}(U \cap f(B)) \tag{316}$$

$$= (f|_A)^{-1}(U) \cup (f|_B)^{-1}(U) \tag{317}$$

Since these restrictions are continuous, $f^{-1}(U)$ is a union of open sets, therefore indicating that f is continuous. □

2. Problem 2

[TODO]

EXERCISE 7.2 (An equivalence relation with closure). *Let $\sim$ be an equivalence relation on a topological space X. Assume each equivalence class is a closed set in X. Then a set of finitely many points in $X/\sim$ is closed in the quotient topology.*

PROOF. Let S be a set of finitely many points in $X/\sim$:

$$S = \{[x_1], \ldots, [x_n]\} \tag{318}$$

To show S is closed, we show its inverse image (under the quotient map) is closed:

$$q^{-1}(S) = \{x \in X \mid q(x) \in S\} = \bigcup_{i=1}^{n} \{x \in X \mid x \sim x_i\} \tag{319}$$

We assumed each equivalence class is a closed set in X, so this is a finite union of closed sets, which is closed.

For an example of X a Hausdorff where its quotient $X/\sim$ is not, consider $X = \mathbb{R}$ under the relation $a \sim b \iff a - b \in \mathbb{Q}$. Then the quotient space $X/\sim$ is not Hausdorff.

This can be seen by supposing distinct equivalence classes $[x]$ and $[y]$ lie in disjoint open sets U and V. Select representatives $x < y$. Select an interior neighborhood $B_\delta(x) \subseteq q^{-1}(U)$ for some $\delta > 0$. Approximate

$$|y - x - r| < \delta \tag{320}$$

for some rational r. Then $y - r \in B_\delta(x)$ implies $[y - r]$ in U. But the rationality of r implies $[y] = [y - r]$. Therefore, U and V are not disjoint. $\square$

3. Problem 3

EXERCISE 7.3 (Slicing the range of an integral). *Let X be a finite measure space. Let $f : X \to \mathbb{R}$ be a measurable function and define for each $k = 1, 2, \dots$*

$$E_k = \{x \in X \mid k \le |f(x)| < k+1\} \tag{321}$$

Then $f \in L^1(X)$ if and only if

$$\sum_{k=1}^{\infty} k\mu(E_k) < \infty \tag{322}$$

PROOF. In the forward direction, suppose $f \in L^1(X)$. The set

$$X = \bigcup_{k=1}^{\infty} E_k \tag{323}$$

is always a disjoint union and we have

$$\sum_{k=1}^{\infty} k\mu(E_k) \le \sum_{k=1}^{\infty} \int_{E_k} |f| = \int_X |f| < \infty \tag{324}$$

In the reverse direction, suppose the sum is finite. Then add $\mu(X) < \infty$ to the sum

$$\mu(X) + \sum_{k=1}^{\infty} k\mu(E_k) < \infty \tag{325}$$

We are free to measure the set X as follows

$$\mu(X) = \sum_{k=1}^{\infty} \mu(E_k) \tag{326}$$

Combining the sums shows

$$\sum_{k=1}^{\infty} (k+1)\mu(E_k) < \infty \tag{327}$$

Compare this to the integral

$$\int_X |f| = \sum_{k=1}^{\infty} \int_{E_k} |f| \le \sum_{k=1}^{\infty} \int_{E_k} (k+1) = \sum_{k=1}^{\infty} (k+1)\mu(E_k) < \infty \tag{328}$$

To see that the finite measure hypothesis is necessary, consider the function $f : \mathbb{R} \to \mathbb{R}$ sending $1 \le x \mapsto 1/x$ and $1 > x \mapsto 0$. $E_k = \emptyset$ for all $k \ge 1$. Then

$$\sum_{k=1}^{\infty} k\mu(E_k) = 0 < \infty \tag{329}$$

But

$$\int_{-\infty}^{\infty} f = \int_1^{\infty} \frac{1}{x} = \infty \tag{330}$$

□

4. Problem 4

EXERCISE 7.4 (Logarithmic Fubini FIXME). *Let* $f : [0,1] \to \mathbb{R}$ *be integrable and set*

$$g(x) := \int_x^1 \frac{f(t)}{t} dt \tag{331}$$

PROOF. Then g is integrable and

$$\int_0^1 g(x)dx = \int_0^1 \int_x^1 f(t)dtdx = \int_0^1 \int_0^1 \frac{f(t)}{t} \cdot 1_{x \le t \le 1}(t)dtdx \tag{332}$$

$$= \int_0^1 \int_0^1 \frac{f(t)}{t} \cdot 1_{0 \le x \le t}(x)dxdt = \int_0^1 \int_0^t \frac{f(t)}{t} dxdt = \int_0^1 f(t)dt \tag{333}$$

□

5. Problem 5

EXERCISE 7.5 (Weak convergence is unique in a reflexive space).

(a) If X is reflexive, show that a weakly converging sequence converges to a point.
(b) Show that the conclusion need not be true if X is not reflexive.

PROOF. For part (a), suppose $x \in X$ satisfies the property that $\lim \phi(x_n)$ exists for each $\phi \in X^*$. Define a linear functional on the dual space

$$x^* : X^* \to \mathbb{R} \tag{334}$$

$$\phi \mapsto \lim_{n\to\infty} x_n^*(\phi) \tag{335}$$

Then $x = J^{-1}(x^*)$ is the *unique* candidate limit because X is reflexive, completing the proof.

For part (b), consider the space $B = C([0,1])$ and the functions $f_n(x) = x^n$ with norm $\|f_n\| = 1$. The dual space is given by the measures on $[0,1]$, so that any linear functional equals

$$\phi(f) = \int f d\nu \tag{336}$$

for some measure ν. Then $\phi(f_n) \to \phi(0)$. But this limit is not unique because $\phi(0) = \phi(1_E)$ for any set E of measure zero, say $E = \mathbb{Q} \cap [0,1]$. □

6. Problem 6

EXERCISE 7.6 (One-stop Banach space decomposition). *Let X_0 be a one-dimensional subspace of a Banach space X. Summon a closed subspace X_1 such that $X = X_0 \dot{+} X_1$.*

PROOF. Select a non-zero $z \in X_0$ and $\phi : X \to \mathbb{R}$ such that $\phi(z) = 1$ by extending the linear functional

$$\text{span}\{z\} = X_0 \to \mathbb{R} \tag{337}$$

$$\lambda z \mapsto \lambda|z| \tag{338}$$

to a continuous linear functional via the Hahn-Banach theorem. Let $X_1 = \ker \phi$ which is a closed subspace because ϕ is a continuous linear functional. We will prove that

$$X = X_0 \dot{+} X_1 \tag{339}$$

A sufficient condition is that X_0 and the kernel are complemented: $X_0 \cap X_1 = \{0\}$ and $X_0 + X_1 = X$.

To show $X_0 \cap X_1 = \{0\}$, suppose $x \in X_0$ and $\phi(x) = 0$. Then $\phi(x) = \phi(\lambda z) = \lambda = 0$, implying $x = 0$. To show $X_0 + X_1 = X$, let $x \in X$. We will break x into an X_0 summand and a kernel summand:

$$x = \phi(x)z + (x - \phi(x)z) \tag{340}$$

Because $z \in X_0$ and $\phi(x)$ is a scalar, it is certain that $\phi(x)z \in X_0$. To verify that the second term lies in the kernel, simply evaluate

$$\phi(x - \phi(x)z) = \phi(x) - \phi(x)\phi(z) = \phi(x) - \phi(x) = 0 \tag{341}$$

The decomposition is defined for any $x \in X$, so we are done. $\square$

7. Problem 7

Exercise 7.7 (Coercive estimate on entire functions). *If $f(z)$ is an entire function such that $|f(z)| \to \infty$ as $|z| \to \infty$, then find constants $c > 0$ and $R > 0$ such that $|f(z)| > c|z|$ for all $|z| > R$.*

Proof. First we see that f is a polynomial, because f is entire and $|f(z)| \to \infty$ as $|z| \to \infty$. Set

$$f(z) := a_0 + a_1 z + \cdots + a_n z^n \tag{342}$$

The following reverse triangle inequality needs no absolute value

$$|f(z)| = |a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0| \tag{343}$$

$$\geq ||a_n z^n| - |a_{n-1} z^{n-1} + \cdots + a_0|| \tag{344}$$

when z satisfies

$$|a_n z| > R > |a_{n-1}| + |a_{n-2}| + \cdots + |a_0| \tag{345}$$

because

$$|a_{n-1} z^{n-1} + \cdots + a_0| \leq \sum_{k=n-1}^{0} |a_k z^k| \leq \sum_{k=n-1}^{0} |a_k z^{n-1}| \tag{346}$$

$$= |z^{n-1}| \sum_{k=0}^{n-1} |a_k| \tag{347}$$

$$\leq |z^{n-1}||a_n z| \tag{348}$$

$$= |a_n z^n| \tag{349}$$

Therefore,

$$|f(z)| \geq |a_n z^n| - |a_{n-1} z^{n-1} + \cdots + a_0| \tag{350}$$

$$\geq |a_n z^n| - \sum_{k=0}^{n-1} |a_k z^k| \tag{351}$$

$$= \sum_{k=0}^{n-1} \frac{|a_n z^n|}{n} - |a_k z^k| \tag{352}$$

$$= \sum_{k=0}^{n-1} |z^n| \left(\frac{|a_n|}{n} - \frac{|a_k|}{|z^{n-k}|} \right) \tag{353}$$

For each $k = 0, \ldots, n-1$, select $R_k > R$ such that $|z| > R$ implies

$$\frac{|a_n|}{n} - \frac{|a_k|}{|z^{n-k}|} > \frac{|a_n|}{2n} \tag{354}$$

Then if $|z| > \max\{R_k\}$, we know

$$|f(z)| > \sum_{k=0}^{n-1} |z^n| \frac{|a_n|}{2n} > |z| \frac{|a_n|}{2} \tag{355}$$

□

8. Problem 8

EXERCISE 7.8 (Semi-circular contour integral). *Evaluate*

$$\int_0^\infty \frac{1+x^2}{1+x^4} \tag{356}$$

PROOF. Consider the zeroes of the denominator lying in the top-half plane:

$$\theta_1 = \frac{i+1}{\sqrt{2}} \quad \theta_2 = \frac{i-1}{\sqrt{2}} \tag{357}$$

Enclose them in a semicircular path of radius R and apply the residue theorem

$$\int_{-R}^{R} f(z) + \int_\gamma f(z) = 2\pi i \operatorname{Res}(z = \theta_1, \theta_2) \tag{358}$$

Setting $\gamma(t) = Re^{it}$ for $t \in [0, \pi]$, we can see the arc integral vanishes as $R \to \infty$:

$$\left| \int_0^\pi f(\gamma(t))\gamma'(t)dt \right| = \left| \int_0^\pi \frac{1+(Re^{it})^2}{1+(Re^{it})^4} Rie^{it} dt \right| \tag{359}$$

$$\leq \int_0^\pi \frac{1+R^2}{R^4-1} Rdt \tag{360}$$

$$= \pi \frac{1+R^2}{R^4-1} R \to 0 \tag{361}$$

Let us compute the residues now

$$\operatorname{Res}(f, \theta_1) = \lim_{z \to \theta_1} \frac{(z-\theta_1)(1+z^2)}{1+z^4} = \lim \frac{1+z^2}{4z^3} \tag{362}$$

$$= \frac{1+i}{4i\frac{1+i}{\sqrt{2}}} = \frac{\sqrt{2}}{4i} \tag{363}$$

$$\operatorname{Res}(f, \theta_2) = \frac{1-i}{4i\frac{1-i}{\sqrt{2}}} = \frac{\sqrt{2}}{4i} \tag{364}$$

By taking limits we can see

$$\int_{-\infty}^{\infty} f(z) = 2\pi i \left[\frac{\sqrt{2}}{2i} \right] = \pi\sqrt{2} \tag{365}$$

The integral in question is half this because the integrand is even:

$$\int_0^\infty \frac{1+x^2}{1+x^4} = \frac{\pi\sqrt{2}}{2} \tag{366}$$

□

CHAPTER 8

Winter 2021

1. Problem 1

EXERCISE 8.1 (Types of compactness). *Give the definitions of compactness and limit point compactness of a topological space. Show that every compact space is limit point compact. Give an example that the converse is not true.*

DEFINITION 1 (Compactness). *A topological space X is called compact if for every open cover of X, there exists a finite subcollection of that cover which also covers X.*

DEFINITION 2 (Limit point compactness). *A topological space X is called limit point compact every infinite subset $S \subseteq X$ has a limit point.*

Now for the real workout:

PROOF. If X is compact, then X is limit point compact. Suppose not, then for any $x \in X$, select an open set $U \ni x$ such that $S \cap U \subseteq \{x\}$. Cover

$$X \subseteq \bigcup_{x \in X} U \tag{367}$$

$$\subseteq \bigcup_{i=1}^{n} U_i \tag{368}$$

The infinite set S can be included

$$S = \bigcup_{i=1}^{n} U_i \cap S \subseteq \bigcup_{i=1}^{n} \{x\} \tag{369}$$

which contradicts that S is an infinite subset. Therefore, S is limit point compact.

To see a limit point compact space which is not compact, consider $\mathbb{Z} \times \{0, 1\}$ where $\mathbb{Z}$ has the standard topology and the topology of $\{0, 1\}$ is $\mathcal{T} = \{\{\}, \{0, 1\}\}$. Any point is a limit point, so any infinite subset contains a limit point. □

2. Problem 2

EXERCISE 8.2 (Continuous maps preserve connectedness). *If X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.*

PROOF. Suppose $f(X) = A \cup B$ is a separation. Then X has a separation:

$$X = f^{-1}(f(X)) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B). \tag{370}$$

This contradicts that X is connected, so we must instead have that $f(X)$ is connected. □

3. Problem 3

Exercise 8.3 (Countable complement measure space). *Let X be uncountable. Define the countable complement measure space*

$$\mathcal{M} = \{E \subseteq X \mid X \setminus E \textit{ is at most countable or } E \textit{ is at most countable}\} \tag{371}$$

$$\mu : \mathcal{M} \to [0, \infty] \quad E \mapsto \begin{cases} \mu(E) = 0 & E \textit{ at most countable} \\ \mu(E) = 1 & X \setminus E \textit{ at most countable} \end{cases} \tag{372}$$

(a) Prove that $\mathcal{M}$ is a σ-algebra and that μ is a measure on $\mathcal{M}$.
(b) Prove that $\mathcal{M}$ is the *σ-algebra generated by $\mathcal{E} = \{\{x\} : x \in X\}$.*

Proof. To see that $\mathcal{M}$ forms a σ-algebra, let $\{E_n\}_{n=1}^{\infty}$ lie in $\mathcal{M}$. Then to verify the union

$$E = \bigcup_{n=1}^{\infty} E_n \tag{373}$$

lies in $\mathcal{M}$, we show that E is either at most countable or $X \setminus E$ is at most countable. If each E_n is at most countable, then the union is certainly at most countable, so suppose $X \setminus E_k$ is at most countable. Then

$$X \setminus E = \bigcap_{n=1}^{\infty} X \setminus E_n \subseteq X \setminus E_k \tag{374}$$

Therefore, $X \setminus E$ is at most countable. Unions are included, and complements are included by the definition, so $\mathcal{M}$ forms a σ-algebra.

Now we show μ is a measure. Let $\{E_n\}_{n=1}^{\infty}$ be disjoint sets in $\mathcal{M}$. If each E_n is at most countable, then their union is at most countable and $\mu(E_n) = 0$, so we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 0 = \sum_{n=1}^{\infty} 0 = \sum_{n=1}^{\infty} \mu(E_n) \tag{375}$$

Otherwise, at least one $X \setminus E_k$ is at most countable, so $\mu(E_k) = 1$ and disjointness implies $\mu(E_n) = 0$ for $n \neq k$ and that the union is at most countable. Then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 1 = \mu(E_k) = \sum_{n=1}^{\infty} \mu(E_n) \tag{376}$$

Therefore, μ respects countable additivity, so is a measure.

To show that $\mathcal{M}$ is the σ-algebra generated by the singletons $\mathcal{E} = \{\{x\} \mid x \in X\}$, let $\mathcal{E}'$ be a σ-algebra containing $\mathcal{E}$. Note that $\mathcal{E}'$ contains all countable unions, countable intersections, and complements of singletons. If $E \in \mathcal{M}$, then $E = \{x_1, \dots\}$ or $X \setminus E = \{x_1, \dots\}$. Both of these lie in $\mathcal{E}'$, so that $E \in \mathcal{E}'$, therefore, $\mathcal{M} \subseteq \mathcal{E}'$, indicating $\mathcal{M}$ is the σ-algebra generated by singletons. $\square$

4. Problem 4

Exercise 8.4 (Convergence in measure metric). *Let $f_n : E \to \mathbb{R}$ be a sequence of measurable functions where $\mu(E) < \infty$. Then $f_n \to 0$ in measure if and only if*

$$\lim_{n\to\infty} \int \frac{|f_n|}{1+|f_n|} = 0 \tag{377}$$

Proof. Suppose the limit is zero. Let $\epsilon > 0$ be given. Select N such that

$$n \geq N \implies \int \frac{|f_n|}{1+|f_n|} < \epsilon^2 \tag{378}$$

We are going to prove that the following set has small measure for such n.

$$F = \{x \in E \mid |f_n(x)| > \epsilon\} \tag{379}$$

Measure by integrating. If $x \in F$, then $1 < |f_n(x)|/\epsilon$, so we have

$$\mu(F) = \int_F d\mu < \int_F \frac{|f_n(x)|}{\epsilon} \tag{380}$$

$$= \frac{1}{\epsilon} \int_F |f_n| \tag{381}$$

$$\leq \frac{1}{\epsilon} \int_F \frac{|f_n|}{1+|f_n|} \tag{382}$$

$$\leq \frac{1}{\epsilon} \int \frac{|f_n|}{1+|f_n|} \tag{383}$$

$$\leq \frac{1}{\epsilon}\epsilon^2 \tag{384}$$

$$\leq \epsilon \tag{385}$$

Suppose convergence in measure holds. Let $\epsilon > 0$ be given and define $\epsilon' = \epsilon/(2\mu(E))$ and define

$$F = \{x \in E \mid |f_n(x)| > \epsilon'\} \tag{386}$$

Break up the norm integral:

$$\int \frac{|f_n|}{1+|f_n|} = \int_{E\setminus F} \frac{|f_n|}{1+|f_n|} + \int_F \frac{|f_n|}{1+|f_n|} \tag{387}$$

If $x \in E \setminus F$, then

$$|f_n(x)| \leq \epsilon' \implies \frac{|f_n(x)|}{1+|f_n(x)|} < \epsilon' \tag{388}$$

Also, $|f_n|/(1+|f_n|) < 1$, so each integral can be bounded

$$\int \frac{|f_n|}{1+|f_n|} < \mu(E \setminus F)\epsilon' + \mu(F) \tag{389}$$

By convergence in measure, select N such that $n \geq N$ implies $\mu(F) < \epsilon/2$. Then we are done. □

5. Problem 5

EXERCISE 8.5 (Projection operator and closed subspaces). *Let $X = X_1 \dot{+} X_2$ and define $P(x_1+x_2) = x_1$. Then P is a linear operator satisfying $P^2 = P$ and moreover, P is bounded if and only if both X_1 and X_2 are closed.*

PROOF. To see P is a linear operator, let $x, y \in X$ decompose uniquely into $x = x_1 + x_2$ and $y = y_1 + y_2$. Then $x + y = (x_1 + y_1) + (x_2 + y_2)$ uniquely, so we see

$$P(x+y) = x_1 + y_1 = Px_1 + Py_1 = Px + Py \tag{390}$$

Moreover, $P^2 = P$ because if $x \in X$, then $x = x_1 + x_2$ uniquely. Projecting, we know $Px = x_1 + 0$ uniquely, so that $P(Px) = x_1 = Px$. Therefore, $P^2 = P$.

Now to see P is bounded implies X_1 and X_2 are closed, we just write them as follows

$$X_1 = P(X) \tag{391}$$

$$X_2 = \ker P \tag{392}$$

The closed graph theorem applies, so that

$$\Gamma = \{(x_1 + x_2, x_1) \mid x_1 \in X_1, x_2 \in X_2\} = X \times X_1 \tag{393}$$

is a closed subspace of $X \times X$, indicating X_1 is a closed subspace. Kernels of bounded operators are closed, so X_2 is closed.

Conversely, if X_1 and X_2 are closed subspaces, they are themselves Banach spaces, so we may define the direct sum of $Y = X_1 \oplus X_2$ under the norm

$$\|(x_1, x_2)\| = \|x_1\|_{X_1} + \|x_2\|_{X_2} \tag{394}$$

Completeness is inherited from the completeness of X_1 and X_2. For any $x = x_1 + x_2 \in X$, we know $\|x\|_X \leq \|(x_1, x_2)\|_Y$, so this shows $X \cong Y$ by mapping $x_1 + x_2 \mapsto (x_1, x_2)$.

Note that P acts on the space Y by $P(x_1, x_2) = (x_1, 0)$, and this means

$$\|P\|_{Y \to Y} = \sup_{y \in Y} \frac{\|Py\|}{\|y\|} = \sup_{(x_1, x_2) \in Y} \frac{\|x_1\|_{X_1}}{\|x_1\|_{X_1} + \|x_2\|_{X_2}} \leq 1 \tag{395}$$

so P is bounded as an operator on Y. By the isomorphism, we know P is bounded as an operator on X. $\square$

6. Problem 6

EXERCISE 8.6 (Closed subspaces are reflexive). *Show that a closed subspace of a reflexive Banach space is reflexive.*

PROOF. Let $S \subseteq X$ be a closed subspace of a reflexive space X. The theorem of Kakutani states that S is reflexive if and only if the closed unit ball B_S in S is compact in the weak topology $\sigma(S, S^*)$. This topology is induced by the weak topology $\sigma(X, X^*)$. The theorem of Banach and Alaoglu states that B_X is compact in the weak-* topology. Since X is reflexive, the weak topology and weak-* topology coincide $\sigma(X, X^*) = \sigma(X^*, X)$. By compactness in the weak-* topology, $B_S \subseteq B_X$ is compact also in the weak topology because it is a weakly closed subset of a compact space. Therefore the theorem of Kakutani implies S is reflexive. $\square$

7. Problem 7

Exercise 8.7 (Funky sine integral). *Evaluate*

$$\int_0^\pi \frac{d\theta}{2+\sin(2\theta)} \tag{396}$$

Proof. Substitute $z = e^{i\theta}$ into the integral and find $d\theta = dz/2iz$, so that the integral may be interpreted as a contour integral of a rational function about the unit circle

$$\int_0^\pi \frac{d\theta}{2+\sin(2\theta)} = \int_0^\pi \frac{d\theta}{2+\frac{z-1/z}{2i}} = 2i\int_0^\pi \frac{d\theta}{4i+z-1/z} = \int_\gamma \frac{dz}{z(4i+z-1/z)} = \int_\gamma \frac{dz}{4iz+z^2-1} \tag{397}$$

The poles are found by the quadratic formula

$$4iz + z^2 - 1 = 0 \implies z = -2i \pm \sqrt{3}i \tag{398}$$

But only one of them lies within the unit disk, namely, $z_0 = -2i + \sqrt{3}i$. To compute the residue, we evaluate a limit with l'Hôpital's rule:

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} \frac{z-z_0}{4iz+z^2-1} = \lim_{z\to z_0} \frac{1}{4i+2z} = \frac{1}{2\sqrt{3}i} \tag{399}$$

Then we can find by the Residue Theorem that

$$\int_0^\pi \frac{d\theta}{2+\sin(2\theta)} = \int_\gamma \frac{dz}{4iz+z^2-1} = 2\pi i\operatorname{Res}(f, z_0) = 2\pi i\frac{1}{2\sqrt{3}i} = \frac{\pi}{\sqrt{3}} \tag{400}$$

□

8. Problem 8

EXERCISE 8.8 (Entire functions, singularities, and injectivity). *Let $f : \mathbb{C} \to \mathbb{C}$ be entire with*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{401}$$

(a) Show that f has an essential singularity at infinity if $a_n \neq 0$ for infinitely many n.
(b) Show that if f is injective, then $f(z) = a_0 + a_1 z$.

PROOF. For (a), suppose f has no essential singularity at infinity. Then one of the limits

$$\lim_{z\to\infty} f(z) \qquad \lim_{z\to\infty} 1/f(z) \tag{402}$$

exists. If the first one exists, then f is bounded, indicating f is constant, so suppose the first one does not exist and the second one does. Suppose $1/f(z) \to a \neq 0$. Then the limit $f(z) \to 1/a$ exists, a contradiction. Therefore, $1/f(z) \to 0$, and $|f(z)| \to \infty$, which indicates f is a polynomial.

For (b), if $f(z)$ is not a polymomial, then f has an essential singularity at infinity, which means f is not injective. Therefore suppose $f(z) = a_0 + \cdots + a_n z^n$. Injectivity means that $f(z) = a_n(z - r)^n$ for some unique root r. Substitute

$$f(e^{2\pi i/n} + r) = a_n \tag{403}$$

$$f(1 + r) = a_n \tag{404}$$

and apply the injectivity of f to find $1 = e^{2\pi i/n}$, which means $n = 1$. Therefore, $f(z) = a_0 + a_1 z$. □

CHAPTER 9

Spring 2021

1. Problem 1

EXERCISE 9.1 (Product and box topologies). *Let*

$$X = \prod_{n=1}^{\infty} [0,1] \tag{405}$$

$$S = \{(x_n) \in X \mid \exists N : n \geq N \implies x_n = 0\} \tag{406}$$

$$= \bigcup_{N=1}^{\infty} \{(x_n) \mid n \geq N \implies x_n = 0\} \tag{407}$$

(a) Show that if X is considered with the product topology, then the closure of S is X.
(b) Show that if X is considered with the box topology, then S is closed in X.

PROOF. For part (a), Let $x = (x_n) \in X \setminus S$. We will show x is a limit point of S. Define a sequence of elements in S:

$$s^1 := (x_1, 0, \dots) \tag{408}$$

$$s^2 := (x_2, x_2, 0, \dots) \tag{409}$$

$$\vdots \tag{410}$$

$$s^n := (x_1, x_2, \dots, x_n, 0, 0, \dots) \tag{411}$$

If U is a given neighborhood of x, we argue that $s^n \in U$ for some n. From the definition of the product topology, select a basic element $B \subseteq U$ containing x and realize B as a product of finitely many not necessarily trivial sets:

$$B = \left(\prod_{n=1}^{N} U_n\right) \times \prod_{n=N+1}^{\infty} [0,1] \tag{412}$$

Then $s^N \in B \subseteq U$.

For part (b), suppose $x \in X \setminus S$ is a limit point of S. Select

$$s \in \prod_{n=1}^{\infty} \begin{cases} B_{|x_n|}(x_n) & x_n \neq 0 \\ B_1(0) & x_n = 0 \end{cases} \tag{413}$$

Since $x \notin S$, there exists a subsequence x_{n_k} such that $x_{n_k} \neq 0$, so we may write the product as

$$\left(\prod_{k=1}^{\infty} B_{|x_{n_k}|}(x_{n_k})\right) \times \prod_{x_n=0} B_1(0) \tag{414}$$

We can see that if s lies in the first product, then s does not lie in S, because $0 \notin B_{|x_{n_k}|}(x_{n_k})$, so there is no N after which all the elements are zero. This is a contradiction, so we know $x \in S$. Therefore, S is closed. □

2. Problem 2

Exercise 9.2 (Continuity and connectedness in discrete topologies). *Let X be a topological space and $\mathbb{Z}$ in the standard topology. Consider the property*

$$P(X) := \text{“every continuous function } f : X \to \mathbb{Z} \text{ is constant”} \tag{415}$$

(a) With $\mathbb{R}$ in the standard topology, show that $P(\mathbb{R})$ is true.
(b) For an arbitrary topological space X, find and prove a characterization of $P(X)$ in terms of X.

Proof. With $\mathbb{R}$ in the standard topology, we $P(\mathbb{R})$ is true. This can be seen by making a metric space argument. Let $R > 0$ be fixed and a select $\delta > 0$ so that $|f(x) - f(y)| < 1/2$ for any $x, y \in [-R, R]$. Each integer is isolated, so this inequality implies $f(x) = f(y)$, so that f is constant on expanding intervals, indicating f is constant on $\mathbb{R}$.

Now we show $P(X)$ is true if and only if X is connected.

Suppose X is connected. The continuous image of a connected space is connected, so $f(X)$ is connected. The connected subsets of $\mathbb{Z}$ are precisely the singletons, so we know $f(X) = \{n\}$, indicating f is constant.

Conversely, suppose X is not connected. It is possible to define a continuous function which is non-constant by separating $X = A \cup B$ and defining $f(A) = 1$ and $f(B) = 0$. The open sets in $\mathbb{Z}$ are precisely the singletons, so any preimage equals A, B, or is trivial, so that f is continuous. Therefore if X is connected, every continuous function $f : X \to \mathbb{Z}$ is constant. □

3. Problem 3

Exercise 9.3 (Measuring with an expanding ruler). *Let $A \subseteq \mathbb{R}$ be a set of positive finite measure. Define a function*

$$\varphi : \mathbb{R} \to [0, \infty) \tag{416}$$

$$x \mapsto \mu(A \cap (-\infty, x]) \tag{417}$$

(a) Show that φ is continuous.
(b) Find $x \in \mathbb{R}$ such that $\mu(A \cap (-\infty, x)) = \mu(A \cap (x, \infty))$.

Proof. Let us show that φ is continuous. Let $\epsilon > 0$ be given and $x \in \mathbb{R}$. Let $0 < |x - y| < \epsilon$. We show $|\varphi(x) - \varphi(y)| < \epsilon$ also in two similar cases. Let $y > x$. Then $(-\infty, x] \subseteq (-\infty, y]$, so we know that

$$\varphi(y) - \varphi(x) = \mu(A \cap (-\infty, y]) - \mu(A \cap (-\infty, x]) = \mu(A \cap (-\infty, y] \setminus A \cap (-\infty, x]) = \mu(A \cap (x, y]) \tag{418}$$

$$\leq \mu((x, y]) \leq \epsilon \tag{419}$$

For $y < x$, make a similar argument.

To prove the existence of $x \in \mathbb{R}$ such that $\mu(A \cap (-\infty, x)) = \mu(A \cap (x, \infty))$, examine the difference

$$d(x) = \mu(A \cap (-\infty, x)) - \mu(A \cap (x, \infty)) \tag{420}$$

By the continuity of measure and φ, this can be written

$$d(x) = \varphi(x) - (\mu(A) - \varphi(x)) = 2\varphi(x) - \mu(A) \tag{421}$$

One can plainly see the limits

$$\lim_{x \to \infty} \varphi(x) = \mu(A) \tag{422}$$

$$\lim_{x \to -\infty} \varphi(x) = 0 \tag{423}$$

Therefore, $d(-\infty) = -\mu(A)$ and $d(\infty) = \mu(A)$, so the Intermediate Value Theorem summons $x \in (-\infty, \infty)$ such that $d(x) = 0$ and $\mu(A \cap (-\infty, x)) = \mu(A \cap (x, \infty))$. $\square$

4. Problem 4

Exercise 9.4 (Slicing the domain of an integral). *Let* $f_1, f_2, \ldots$ *and* g *be functions in* $L^1(\mathbb{R})$ *and* $E_n := \{x \in \mathbb{R} \mid |f_n(x)| > |g(x)|\}$. *Suppose* $f_n \to g$ *pointwise almost everywhere and*

$$\lim_{n\to\infty} \int_{E_n} |f_n| = 0 \tag{424}$$

Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - g| = 0 \tag{425}$$

Proof. Break up the domain of the integral

$$\int_{\mathbb{R}} |f_n - g| = \int_{\mathbb{R}\setminus E_n} |f_n - g| + \int_{E_n} |f_n - g| \tag{426}$$

On the set E_n, we have $|g| < |f_n|$, so

$$\lim_{n\to\infty} \int_{E_n} |f_n - g| \le \lim_{n\to\infty} \int_{E_n} 2|f_n| = 0 \tag{427}$$

On the complement, we have $|f_n| < |g|$, so that a dominating function exists, and DCT may be applied:

$$\lim_{n\to\infty} \int_{\mathbb{R}\setminus E_n} |f_n - g| = \lim_{n\to\infty} \int_{\mathbb{R}} |f_n - g| \cdot 1_{\mathbb{R}\setminus E_n} \tag{428}$$

$$= \int_{\mathbb{R}} \lim_{n\to\infty} |f_n - g| \cdot 1_{\mathbb{R}\setminus E_n} \tag{429}$$

$$\le \int_{\mathbb{R}} \lim_{n\to\infty} |f_n - g| \tag{430}$$

$$= 0 \tag{431}$$

Therefore

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - g| = 0 \tag{432}$$

□

5. Problem 5

EXERCISE 9.5 (Weakly converging operators). *Let X and Y be Banach spaces. A sequence $A_n \in L(X,Y)$ is said to converge weakly to $A \in L(X,Y)$ if for all $x \in X$ and all $\phi \in Y^*$, the sequence $\phi(A_n x)$ converges to $\phi(Ax)$. Assuming that A_n converges weakly to A, show that $\sup_{n \geq 1} \|A_n\| < \infty$ and that the operator A is bounded.*

PROOF. See Winter 2018 Exercise 6. $\square$

6. Problem 6

EXERCISE 9.6 (Compute a few functional norms). *Let X be the set of continuously differentiable functions $f : [-1,1] \to \mathbb{R}$ under the norm*

$$\|f\| = \sup_{x\in[-1,1]} |f(x)| \tag{433}$$

Determine the boundedness and norms for the following functionals

$$\phi_1(f) = f(0) \quad \phi_2(f) = \int_{-1}^{1} \operatorname{sign}(x) f(x) \quad \phi_3(f) = f'(0) \quad \phi_4(f) = \sum_{n=1}^{\infty} \frac{f(1/n)}{2^n} \tag{434}$$

PROOF. For ϕ_1, the norm is bounded:

$$\|\phi_1\| = \sup_{\|f\|=1} \|\phi_1(f)\| \le \sup_{\|f\|=1} |f(0)| \le \|f\| = 1 \tag{435}$$

Equality is achieved for any f satisfying $|f(0)| = \|f\|$.

For ϕ_2, first simplify

$$\phi_2(f) = \int_{-1}^{0} (-1) f(x) + \int_{0}^{1} f(x) = \int_{0}^{1} f(x) - f(-x) \tag{436}$$

$$\|\phi_2\| = \sup_{\|f\|=1} \|\phi_2(f)\| \le \sup_{\|f\|=1} \int_{0}^{1} |f(x) - f(-x)| \le \|2f\| = 2 \tag{437}$$

For equality, consider a Fourier series converging to the function

$$f(x) = \begin{cases} -1 & x \in [-1,0] \\ 1 & x \in [0,1] \end{cases} \tag{438}$$

Since the functional can be evaluated and equals 2 in the limit, this means the norm equals 2.

For ϕ_3, we can see the functional is unbounded by considering the sequence of functions $f_n(x) = e^{-nx^2}$. In this case, $\|f_n\| = 1$ and each f_n is continuously differentiable. The derivative is $f_n'(x) = -ne^{-nx^2}$ and so

$$\sup_n |\phi_3(f_n)| = \sup_n |n| = \infty \tag{439}$$

For ϕ_4,

$$\|\phi_4\| = \sup_{\|f\|=1} \|\phi_4(f)\| \le \sup_{\|f\|=1} \sum_{n=1}^{\infty} \frac{\|f\|}{2^n} = 1 \tag{440}$$

If f is constant, then

$$\|\phi_4(f)\| = \left\| \sum_{n=1}^{\infty} \frac{f}{2^n} \right\| = \|f\| \tag{441}$$

so the norm equals one. □

7. Problem 7

Exercise 9.7 (Compact convergence in the plane). *Let $\Omega \subseteq \mathbb{C}$ be open and suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions over Ω converging uniformly to $f : \Omega \to \mathbb{C}$. For any $\delta > 0$, the $f_n' \to f'$ uniformly on the set*

$$K_\delta := \{z \in \Omega \mid \overline{B_\delta(z)} \subseteq \Omega\} \tag{442}$$

Proof. Note that if $z \in K_\delta$, then $\overline{B_{\delta/2}(z)} \subseteq K_{\delta/2}$, because if $x \in \overline{B_{\delta/2}(z)}$, then $\overline{B_{\delta/2}(x)} \subseteq \overline{B_\delta(z)} \subseteq \Omega$, so $x \in K_{\delta/2}$.

For any $z \in K_\delta$, the Cauchy Integral Formula can be applied on a circle of radius $r = \delta/2$ around z to find the derivative sequence in terms of the original sequence. For any f_n, we have

$$f_n'(z) = \frac{1}{2\pi i} \int_{C_r(z)} \frac{f_n'(\zeta)}{\zeta - z} \tag{443}$$

It can be shown that

$$\int_{C_r(z)} \frac{f_n'(\zeta)}{\zeta - z} = \int_{C_r(z)} \frac{f_n(\zeta)}{(\zeta - z)^2} \tag{444}$$

Then

$$\begin{aligned}
|f_n'(z) - f'(z)| &= \frac{1}{2\pi} \left| \int_{C_r(z)} \frac{f_n(\zeta)}{(\zeta - z)^2} - \int_{C_r(z)} \frac{f(\zeta)}{(\zeta - z)^2} \right| && (445)\\
&\le \frac{1}{2\pi} \int_{C_r(z)} \frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|^2} && (446)\\
&\le \frac{1}{2\pi r^2} \int_{C_r(z)} |f_n(\zeta) - f(\zeta)| && (447)\\
&\le \frac{1}{2\pi r^2} \|f_n - f\|_{C_r(z)} \mu(C_r(z)) && (448)\\
&\le \frac{1}{r} \|f_n - f\|_{K_{\delta/2}} && (449)\\
&\le \frac{1}{r} \|f_n - f\|_{\Omega} && (450)
\end{aligned}$$

Letting $n \to \infty$ shows $f_n' \to f'$ uniformly on K_δ. □

8. Problem 8

EXERCISE 9.8 (Sector-based contour integral). *Evaluate*

$$\int_0^\infty \frac{x^{1/3}}{1+x^2} \tag{451}$$

PROOF. Substitute $z = x^{1/3}$ to transform the integral into

$$3\int_0^R \frac{z^3}{1+z^6} \tag{452}$$

Find the pole at $e^{i\pi/6}$ and integrate around the sector of radius R and $0 \le \theta \le \pi/3$. The integral can be written as

$$\int_0^R f(z) + \int_{\text{arc}} f(z) + \int_R^0 f(re^{i\pi/3})e^{i\pi/3} = 2\pi i \operatorname{Res}(f, e^{i\pi/6}) \tag{453}$$

This equation is analyzed in three stages: find the residue, express the backwards integral in terms of the forwards integral, and show the middle integral vanishes as $R \to \infty$.

For the residue:

$$\operatorname{Res}(f, e^{i\pi/6}) = \lim_{z\to e^{i\pi/6}} \frac{(z-e^{i\pi/6})z^3}{1+z^6} = \lim_{z\to e^{i\pi/6}} \frac{z^3}{6z^5} = \frac{1}{6}e^{-\pi i/3} \tag{454}$$

For the backwards integral

$$e^{i\pi/3}\int_R^0 f(re^{i\pi/3}) = e^{i\pi/3}\int_R^0 \frac{r^3e^{i\pi}}{1+r^6} = e^{i\pi/3}\int_0^R \frac{r^3}{1+r^6} = e^{i\pi/3}\int_0^R f(z) \tag{455}$$

For the middle integral

$$\int_{\text{arc}} f(z) \le \int_{|z|=R} f(z) \le \int \frac{|z|^3}{|z|^6-1} = 2\pi R\frac{R^3}{R^6-1} \to 0 \tag{456}$$

Letting $R \to \infty$ shows that

$$(1+e^{i\pi/3})\int_0^\infty \frac{z^3}{1+z^6} = 2\pi i\frac{1}{6}e^{-\pi i/3} \tag{457}$$

Therefore,

$$\int_0^\infty \frac{z^3}{1+z^6} = \frac{2\pi i}{6}\frac{e^{-\pi i/3}}{1+e^{i\pi/3}} = \frac{2\pi}{6}\frac{e^{i\pi/2}e^{-\pi i/3}}{1+e^{i\pi/3}} = \frac{2\pi}{6}\frac{e^{i\pi/6}}{1+e^{i\pi/3}} = \frac{2\pi}{6}\frac{1}{e^{-i\pi/6}+e^{i\pi/6}} = \frac{2\pi}{6}\frac{1}{2\cos(\pi/6)} \tag{458}$$

$$= \frac{\pi}{3\sqrt{3}} \tag{459}$$

This means the original integral equals

$$\int_0^\infty \frac{x^{1/3}}{1+x^2} = \frac{\pi}{\sqrt{3}} \tag{460}$$

□

CHAPTER 10

Fall 2021 TODO 1,3,6

1. Problem 1

EXERCISE 10.1 (Connectedness: the plane & a lexicographic order topology TODO).

(a) Show that open, connected subsets of the plane are path-connected.

(b) Show that $X = [0,1] \times [0,1]$ in the lexicographic order topology is not path-connected.

PROOF. For part (a), let $x \in U$ be a fixed base point. Define $x \sim y$ if and only if there exists a continuous function $\gamma : [0,1] \to U$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Consider the equivalence class of the base point.

$$V = \{y \in U \mid x \sim y\} \tag{461}$$

We show V is open by constructing a neighborhood for any point. Let $y \in V$. Select a neighborhood $B_\delta(y) \cap U \subseteq U$. Then this same ball is also interior to V, for if $y' \in B_\delta(y) \cap V$, the convexity of the ball implies $y' \sim y$. The selection of y precisely states $y \sim x$, so the transitivity of the relation implies $y' \sim x$. Therefore, $y' \in V$, so that V is an open set. Writing

$$U = V \cup (U \setminus V) \tag{462}$$

and applying the connectedness of U implies $V = U$, so that U is path-connected.

The above proof could have also been performed in a constructive way as follows. Select a countable collection of rectangles such that

$$U = \bigcup_{i=1}^{\infty} R_i \tag{463}$$

We can explicitly construct *the* path component of U in a tree based fashion, walking through the path-connected neighbors (sufficiently induced by their non-trivial intersection) starting at a base rectangle. Let $R_\ell \sim R_k$ if and only if $R_\ell \cap R_k$. Define the base level of a tree by

$$T_1 = \{R_1\} \tag{464}$$

and proceed in an inductive fashion, walking the rectangles $R_i \sim R_j$ for some $R_j \in T_n$ which have not already been walked. Formally, this can be written

$$T_{n+1} = \{R_i \mid \text{there exists } R_j \in T_n \text{ such that } R_i \cap R_j \text{ is non-empty}\} \setminus \bigcup_{m=1}^{n} T_m. \tag{465}$$

To complete the procedure, consider the entire tree we just generated

$$T = \bigcup_{n=1}^{\infty} T_n. \tag{466}$$

Now we wish to argue that $T = \{R_1, R_2, \dots\}$. Consider the complement $S = \{R_1, R_2, \dots\} \setminus T = \{R^1, R^2, \dots\}$, which we now argue is empty. Rewriting U

$$U = \left(\bigcup_{k=1}^{\infty} R^k\right) \cup \left(\bigcup_{R_i \in T}^{\infty} R_i\right), \tag{467}$$

connectedness implies some $R^k \cap R_i$ is non-empty. By convexity of rectangles, this means $R^k \sim R_i$. Realizing $R_i \in T_m$ implies $R^k \in T_{m-1}$ or $R^k \in T_{m+1}$, so that $R^k \in T$, showing that S is empty.

Therefore,

$$U = \bigcup_{U_i \in T} U_i \tag{468}$$

From the construction of T, there is an explicit path between any two points, namely from a point to the base to the other point.

TODO For part (b), recall that the lexicographic order is induced by relation $(a, b) < (c, d)$ if and only if $a < c$ or $a = c$ and $b < d$. Explicitly, the basic sets are

$$\mathcal{B} = \{\{x \in [0, 1] \times [0, 1] \mid a < x < b\} \mid a, b \in [0, 1] \times [0, 1]\} \tag{469}$$

□

2. Problem 2

EXERCISE 10.2 (Separating functional in a metric space). *Let X be a metric space and let A and B be disjoint closed subsets of X. There exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

PROOF. For any closed set F, define a distance function $d(\cdot, F) : X \to \mathbb{R}_{\geq 0}$.

$$d(x, F) = \inf\{d(x, y) \mid y \in F\} \tag{470}$$

The function $d(\cdot, F)$ can be argued to be continuous in a sequential fashion. Suppose $x_n \to x$. Then

$$\lim_{n\to\infty} d(x_n, F) = \lim_{n\to\infty} \inf_{f\in F} d(x_n, f) = \inf_{n\geq 1} \sup_{m\geq n} \inf_{f\in F} d(x_m, f) \leq \inf_{n\geq 1} \inf_{f\in F} \sup_{m\geq n} d(x_m, f) \tag{471}$$

$$\leq \inf_{f\in F} \inf_{n\geq 1} \sup_{m\geq n} d(x_m, f) = \inf_{f\in F} \lim_{n\to\infty} d(x_n, f) = \inf_{f\in F} d(x, f) = d(x, F) \tag{472}$$

To finish, the definition of infimum says $d(x, F) \leq d(x, f)$ for all $f \in F$, so $\lim d(x_n, F) = d(x, F)$, indicating $d(\cdot, F)$ is continuous.

Since A and B are disjoint closed sets and X is a metric space, the following distances are nonzero for any $a \in A$ and $b \in B$.

$$d(a, B) \quad d(b, A) \tag{473}$$

This implies $d(x, A) + d(x, B) \neq 0$ for if the sum were zero, then both the summands would equal zero, which indicates $x \in A$ and $x \in B$, contradicting that A and B are disjoint.

Now we are free to define a continuous function

$$f(x) := \frac{d(x, A)}{d(x, A) + d(x, B)} \tag{474}$$

If $x \in A$, then $d(x, A) = 0$, so $f(x) = 0$. If $x \in B$, then $d(x, B) = 0$, so $f(x) = d(x, A)/d(x, A) = 1$. □

3. Problem 3

EXERCISE 10.3 (Compactess and continuity from above TODO). *Let* $E \subseteq \mathbb{R}$ *and define* $\mathcal{O}_n = \{x \in \mathbb{R} \mid d(x, E) < 1/n\}$

(a) Show that if E *is compact, then* $\lim \mu(\mathcal{O}_n) = \mu(E)$.

(b) Show that the conclusion may be false if E *is closed and unbounded or if* E *is open and bounded.*

PROOF. If E is compact, then each $\mathcal{O}_n$ is bounded, because E is bounded. Moreover, we have the following inclusions by the monotonicity of $1/n$:

$$\mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \cdots \supseteq \mathcal{O}_n \supseteq \cdots \tag{475}$$

To apply the Lebesgue measure's continuity from above, note that

$$\bigcap_{n=1}^{\infty} \mathcal{O}_n = E \tag{476}$$

because E is closed. Therefore we are free to determine that

$$\lim_{n\to\infty} \mu(\mathcal{O}_n) = \mu\left(\bigcap_{n=1}^{\infty} \mathcal{O}_n\right) = \mu(E) \tag{477}$$

For a first counterexample, consider the closed and unbounded sequence $E_j = [j, \infty) \cap \mathbb{N}$ with the counting measure. Each set is countable, so the intersection is empty, so the result from continuity does not apply, because

$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = 0 \neq \mu(E_1) - \lim_{j\to\infty} \mu(E_j) = \infty \tag{478}$$

In a similar way, we can exploit σ-finiteness by setting $E_j = [j, \infty)$ with the Lebesgue measure to observe an identical inequality.

□

4. Problem 4

EXERCISE 10.4 (DCT in two ways). *Evaluate the limit*

$$\lim_{n\to\infty}\int_0^1 \frac{ndx}{(1+nx)^2(1+x+x^2)} \tag{479}$$

PROOF. Substitute $u = nx$ in the integral

$$\int_0^1 \frac{ndx}{(1+nx)^2(1+x+x^2)} = \int_0^n \frac{du}{(1+u)^2(1+u/n+(u/n)^2)} \tag{480}$$

$$= \int_{\mathbb{R}} \frac{1_{[0,n]}(u)du}{(1+u)^2(1+u/n+(u/n)^2)} \tag{481}$$

We have the inequality

$$u \in [0,n] \implies 1/(1+u/n+(u/n)^2) < 1 \tag{482}$$

Therefore, the integrand is bounded by $1/(1+u)^2 \cdot 1_{[0,\infty)}$, which is integrable, so we may apply DCT.

$$\lim_{n\to\infty}\int_{\mathbb{R}} \frac{1_{[0,n]}(u)du}{(1+u)^2(1+u/n+(u/n)^2)} = \int_0^\infty \frac{1}{(1+u)^2}du = 1 \tag{483}$$

This could also have been proven with integration by parts as follows. Separate with some parentheses

$$I_n = \int_0^1 \left(\frac{n}{(1+nx)^2}\right)\frac{1}{1+x+x^2}dx \tag{484}$$

to reveal that the integrand can be rewritten for an integration by parts as follows:

$$I_n = \int_0^1 \left(-\frac{1}{1+nx}\right)' \frac{1}{1+x+x^2}dx \tag{485}$$

$$= \left(-\frac{1}{1+nx}\right)\frac{1}{1+x+x^2}\Bigg|_0^1 - \int_0^1 \left(-\frac{1}{1+nx}\right)\left(-\frac{1+2x}{(1+x+x^2)^2}\right) \tag{486}$$

$$= \left(-\frac{1}{1+n}\right)\frac{1}{1+1+1^2} + 1 - \int_0^1 \frac{1+2x}{(1+nx)(1+x+x^2)^2} \tag{487}$$

The integrand decays in n which is not tied to $1/x^2$, so that limit/integral interchange applies, proving that

$$\lim_{n\to\infty} I_n = 1 \tag{488}$$

If one desires a concrete dominating function to supply to the DCT, they can take 2. □

5. Problem 5

EXERCISE 10.5 (Injectivity and a coercive estimate). *Let X and Y be Banach spaces and $A : X \to Y$ be a bounded linear operator. Show that the following are equivalent:*

(a) A is injective and the range of A is closed

(b) There exists a constant $M > 0$ such that

$$\|x\| \leq M\|Ax\| \quad \forall x \in X \tag{489}$$

PROOF. If A is injective and the range of A is closed, consider the operator

$$A|_{A(X)} : A(X) \to A(X) \tag{490}$$

which is known to have a closed range. This means the domain is also closed, from which we can decide that the graph is closed, so that the operator and its inverse are continuous. Therefore, we take

$$M = \|(A|_{A(X)})^{-1}\|_{B(A(X))} \tag{491}$$

Now lets show M is satisfactory. Let $x \in X$ be given, then $Ax \in A(X)$, and we can determine that

$$\|(A|_{A(X)})^{-1}(Ax)\| \leq M\|Ax\| \tag{492}$$

It remains to simply contract $(A|_{A(X)})^{-1}(Ax) = x$ to find the desired result.

Now suppose there exists such an $M > 0$. To show that A is injective, suppose $Ax = Ay$. By linearity we have the inequality

$$\|x - y\| \leq M\|Ax - Ay\| = 0 \tag{493}$$

which implies $x = y$ by squeezing the difference down. We are left to show that $A(X)$ is closed. Suppose $Ax_n \to y$. By the estimate involving, we can determine that the sequence $\{x_n\}$ is Cauchy, from which we select its limit x. Then to show $y = Ax$, let us utilize the triangle inequality

$$\|Ax - y\| \leq \|Ax - Ax_n\| + \|Ax_n - y\| \tag{494}$$

The left summand vanishes because A is bounded and the right summand vanishes by the assumption $Ax_n \to y$. Therefore, $A(X)$ is closed.

The above shows (a) $\iff$ (b). □

6. Problem 6

Exercise 10.6 (Vanishing Condition on a Hilbert Space). *Let H be a Hilbert space and let A_n be a sequence of bounded linear operators on H. Assume for every $x, y \in H$ that* $\lim\langle y, A_n x\rangle = 0$*.*

(a) Does it follow that $\lim \|A_n\| = 0$*?*

(b) Does it follow that $\sup \|A_n\| < \infty$*?*

Provide counterexamples or proofs.

Proof. For part (b), we provide a proof. Let n be fixed. For any $x \in H$, set $y = A_n x$. The vanishing assumption indicates that $\langle A_n x, A_n x\rangle = 0$. □

7. Problem 7

EXERCISE 10.7 (Rouché's theorem for a half-plane). *Let $a > 1$. Show that the equation*

$$a - z - e^{-z} = 0 \tag{495}$$

has exactly one solution in the right half-plane.

PROOF. This can be shown with Rouché's theorem, whose setup we now perform. Let $f = a - z$ and $g = -e^{-z}$ and consider the semicircular contour with side $[Ri, -Ri]$ and arc $Re^{i\theta}$ for $\theta \in [-\pi/2, \pi/2]$ where $R > 0$ is arbitrary.

For the side, de Moivre's theorem implies $|g| = 1$. A quick computation can exactly determine the modulus of $f = a - z$, where $z \in [Ri, -Ri]$

$$|f| = |a - z| = \sqrt{a^2 + |z|^2} \geq a > 1 = |g| \tag{496}$$

The inequalities follow by the monotonicity of the square root and the given information about a. Now for the arc, we shall have $z = Re^{i\theta}$. This means

$$|g| = |e^{-Re^{i\theta}}| = |e^{-R\cos(\theta) - iR\sin(\theta)}| \tag{497}$$

$$= |e^{-R\cos(\theta)}e^{-iR\sin(\theta)}| \tag{498}$$

$$= e^{-R\cos(\theta)} \tag{499}$$

The domain $\theta \in [-\pi/2, \pi/2]$ indicates $-R\cos(\theta) > 0$, so that as g vanishes as $R \to \infty$, showing trivially that $|f| > |g|$.

Therefore all along the semicircular contour, we have that $|f| > |g|$, which indicates $a - z - e^{-z}$ has the same number of zeros as f, a linear function with only zero. Sending $R \to \infty$, we can decide that the equation has only one root in the right half-plane. □

8. Problem 8

Exercise 10.8 (Another semicircular contour). *Integrate*

(500) $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx$$

Proof. To evaluate this, we integrate the complexified function

(501) $$f(z) = \frac{e^{iz}}{z^2 + a^2}$$

along a semicircular contour which captures the pole $z = ai$. Letting $R \to \infty$ will capture the desired integral in the real part. The residue theorem implies

(502) $$\int_{-R}^{R} \frac{e^{ix}}{x^2 + a^2} dx + \int_{\text{arc}} f(z) dz = 2\pi i \operatorname{Res}(f, ai)$$

The arc integral can be shown to vanish

(503) $$\left| \int_{\text{arc}} f(z) dz \right| \leq \int \frac{|e^{iz}|}{|z|^2 - a^2} dz$$

When $z = a + bi$, we know $|e^{iz}| = |e^{i(a+bi)}| = e^{-b}$, so in the upper half-plane, we know $\sup |e^{iz}| = 1$. Therefore,

(504) $$\int \frac{|e^{iz}|}{|z|^2 - a^2} dz \leq \int \frac{1}{R^2 - a^2} dz = \frac{\pi R}{R^2 - a^2} \to 0$$

As $R \to \infty$ we are left with the residue

(505) $$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} dx = 2\pi i \operatorname{Res}(f, ai)$$

(506) $$= 2\pi i \lim_{z \to ai} \frac{(z - ai)e^{iz}}{(z - ai)(z + ai)}$$

(507) $$= 2\pi i \lim_{z \to ai} \frac{e^{iz}}{z + ai}$$

(508) $$= 2\pi i e^{-a}/2ai$$

(509) $$= \frac{\pi e^{-a}}{a}$$

Taking the real part shows

(510) $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{\pi e^{-a}}{a}$$

□

CHAPTER 11

Winter 2022

1. Problem 1

EXERCISE 11.1 (Dini's Theorem). *Suppose that* $\{f_n\}$ *is a sequence of continuous functions from* $[0,1]$*, where each* $f_n(x)$ *is monotone increasing. And suppose that* $f_n(x)$ *converges to a continuous function* $f(x)$ *pointwisely on* $[0,1]$*.*

(a) Show that $f_n(x)$ *in fact uniformly converges to* $f(x)$ *on* $[0,1]$*.*

(b) Give an example where the uniform convergence fails if the limit function $f(x)$ *is not continuous.*

PROOF. For (a), let $f_n \to f$ as described. Define

$$E_n = \{x \in [0,1] \mid f(x) < f_n(x) + \epsilon\}. \tag{511}$$

Since $f_1(x) \leq f_2(x) \leq \cdots \leq f(x)$, we have

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq [0,1] \tag{512}$$

and by pointwise convergence it follows that

$$[0,1] = \bigcup_{n=1}^{\infty} E_n. \tag{513}$$

Extracting a finite subcover, we find

$$[0,1] = \bigcup_{n=1}^{N} E_n = E_N \tag{514}$$

Now show the convergence is uniform. Let $n > N$. Then $x \in E_N$ and

$$\|f_n - f\| = \sup_{x\in[0,1]} |f_n(x) - f(x)| \tag{515}$$

$$= \sup f(x) - f_n(x) \tag{516}$$

$$\leq \epsilon \tag{517}$$

For (b), consider $f_n(x) = 1 - x^n$, converging to $1 - 1_{\{1\}}$. A uniformly converging sequence of continuous functions converges to a continuous function, so we can see that the convergence must not be uniform by contradiction. □

2. Problem 2

Exercise 11.2 (Urysohn's Lemma in a Metric Space). *Suppose that X is a metric space with the distance function $d(\cdot,\cdot)$. For a point $x \in X$ and a subset A, let*

$$d(x, A) := \inf\{d(x, y) \mid y \in A\}. \tag{518}$$

(a) Let A and B be two disjoint closed subsets in X. Show that

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)} : X \to [0, 1] \tag{519}$$

is continuous.

(b) Use (a) to verify that, for a closed subsets A and an open subset U such that $A \subseteq U$, there always exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Proof. The verification that f is continuous and $f|_A = 0$ and $f|_B = 1$ is left for Exercise Fall 2019 Exercise 5.2 or Fall 2021 Exercise 2.

Now for part (b), set $B = U^c$ and $V = f^{-1}([0, 1/2))$. We show that $V \subseteq U$ and $\overline{V} \subseteq U$. Let $x \in V$. Then $f(x) \in [0, 1/2)$. Since $f|_B = 1$, this means $x \notin B = U^c$, so that $x \in U$. Let $\{x_n\} \subseteq V$ be a sequence converging to x. The continuity of f yields $f(x) = \lim f(x_n) \leq 1/2$, so that $x \notin B$ again indicating $x \in U$.

Finally, $A \subseteq V$ because $A = f^{-1}(0)$, so that

$$A \subseteq V \subseteq \overline{V} \subseteq U. \tag{520}$$

□

Note that Winter 2019 Exercise 2 provides an alternative construction of V.

3. Problem 3

EXERCISE 11.3 (Radon-Nikodym). *Repeat of Exercise 3.4.*

4. Problem 4

EXERCISE 11.4 (Absolute continuity).

(a) Let f and g be absolutely continuous functions on $[0,1]$. Show that their product is also absolutely continuous.

(b) Give an example of a function on $[0,1]$ which is uniformly continuous but not absolutely continuous.

PROOF. Part (a) is proven in Exercise 3.3.

For part (b), consider the Devil's staircase. It is a continuous function on a compact set and thus uniformly continuous. But an absolutely continuous function has bounded variation, which the Devil's staircase does not, so we know the function is not absolutely continuous by contradiction. □

5. Problem 5

EXERCISE 11.5 (Scaling Mean Operator). *Let $X = C([0,1])$ be the Banach space of all continuous complex-valued functions on $[0,1]$ with the maximum norm. Consider the linear operator $A : X \to X$ defined by*

$$(Af)(x) = x\int_0^1 f(y)dy, x \in [0,1]. \tag{521}$$

(a) Show that A is bounded and determine its operator norm.
(b) Determine the spectrum of the operator A.

PROOF. To show A is bounded, estimate

$$\begin{aligned}
\|A\| &= \sup_{\|f\|_\infty=1} \|Af\|_\infty && (522)\\
&= \sup_{\|f\|_\infty=1} \sup_{x\in[0,1]} \left|x\int_0^1 f(y)dy\right| && (523)\\
&\le \sup_{\|f\|_\infty=1} \sup_{x\in[0,1]} |x|\int_0^1 |f(y)|dy && (524)\\
&\le \sup_{\|f\|_\infty=1} \int_0^1 |f(y)|dy && (525)\\
&\le 1. && (526)
\end{aligned}$$

To compute the operator norm, consider $A : L^2([0,1]) \to L^2([0,1])$, extended by density. By applying Fubini's theorem

$$\begin{aligned}
\langle Af, g\rangle &= \int_0^1 \left(x\int_0^1 f(y)dy\right)\overline{g(x)}dx && (527)\\
&= \int_0^1\int_0^1 xf(y)dy\overline{g(x)}dx && (528)\\
&= \int_0^1\int_0^1 xf(y)\overline{g(x)}dxdy && (529)\\
&= \int_0^1 f(y)\int_0^1 x\overline{g(x)}dxdy && (530)\\
&= \int_0^1 f(y)\overline{\int_0^1 \overline{x}g(x)dx}dy && (531)\\
&= \int_0^1 f(y)\overline{\int_0^1 xg(x)dx}dy && (532)\\
&= \langle f, A^*g\rangle && (533)
\end{aligned}$$

we can find the adjoint of A is equal to

$$A^*f = \int_0^1 yf(y)dy. \tag{534}$$

which we can also verify is bounded. To apply the formula $\|A\| = \sqrt{\|A^*A\|}$, determine the composed operator

$$A^*Af = \int_0^1 y\left(y\int_0^1 f(x)dx\right)dy \tag{535}$$

$$= \int_0^1 y^2dy\int_0^1 f(x)dx \tag{536}$$

$$= \frac{1}{3}\int_0^1 f(x)dx \tag{537}$$

Taking norms shows

$$\|A^*Af\| \leq \frac{\|f\|_1}{3} \tag{538}$$

with equality on constant functions, so that $\|A^*A\| = 1/3$ and $\|A\| = 1/\sqrt{3}$.

To find the spectrum, we first prove that A is a compact operator. Let $\{f_n\}$ be a sequence in the unit ball. Then $A(\{f_n\})$ is bounded because A is bounded. To prove the sequence is equicontinuous, let $\epsilon > 0$ be given. If $|x-y| < \epsilon$, then the estimate

$$|Af_n(x) - Af_n(y)| = \left|x\int_0^1 f(t)dt - y\int_0^1 f(t)dt\right| \tag{539}$$

$$\leq |x-y|\int_0^1 |f(t)|dt \tag{540}$$

$$\leq |x-y| \tag{541}$$

$$< \epsilon \tag{542}$$

is true for any f_n. Therefore the hypotheses of Arzelà-Ascoli apply, so that a converging subsequence exists, proving that A is compact. Therefore it suffices to find the eigenvalues of A^*, which we do now. Let

$$A^*f = \lambda f \tag{543}$$

Because A^*f is constant, this means $f(y) \equiv f$ is constant, so we in fact have

$$A^*f = \int_0^1 yfdy = f\int_0^1 ydy = \frac{f}{2} \tag{544}$$

so that $\lambda = 1/2$. Therefore, $\sigma(A) = \{0, 1/2\}$. □

6. Problem 6

Exercise 11.6 (Banach Space Decomposition). *Repeat of Exercise 7.6.*

7. Problem 7

Exercise 11.7 (Product of real and imaginary part Liouville). *Let $f = u+iv$ be an entire function such that $|u||v|$ is bounded. Prove that f must be a constant function.*

Proof. Expand

$$-if^2 = -i(u^2 + 2iuv - v^2) = 2uv + i(v^2 - u^2) \tag{545}$$

If $\sup |u||v| = M$, then the entire function e^{-if^2} has a bound:

$$|e^{-if^2}| = |e^{2uv+i(v^2-u^2)}| = e^{2uv} \leq e^{2|uv|} = e^{2|u||v|} \leq e^{2M}. \tag{546}$$

Therefore e^{-if^2} is constant, so that its absolute value e^{2uv} is also constant, which indicates $uv \equiv \pm M$. If we set $u = \pm M/v$, then the Cauchy-Riemann equations for f may be applied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{547}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{548}$$

In particular, the equations yield

$$\frac{\partial u}{\partial x} = \mp\frac{M}{v^2}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \tag{549}$$

$$\frac{\partial u}{\partial y} = \mp\frac{M}{v^2}\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} \tag{550}$$

Substituting the right half of the first equation into the right half of the second equation shows

$$\frac{M^2}{v^2}\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x} \iff \left(\frac{M^2}{v^2} + 1\right)\frac{\partial v}{\partial x} = 0 \tag{551}$$

Since $M^2/v^2 + 1 > 0$, this implies the partial derivative equals zero. The symmetry of the equations dictates that each partial derivative $\{u_x, u_y, v_x, v_y\}$ vanishes, so that u and v are both constant, proving that f is constant. □

8. Problem 8

EXERCISE 11.8 (Series for a Complex Integral). *Prove that*

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta} d\theta = \sum_{n=0}^{\infty} \frac{1}{(n!2^n)^2}. \tag{552}$$

PROOF. Substitute $z = e^{i\theta}$, so that

$$\cos\theta = \frac{1}{2}(z + 1/z) \quad d\theta = \frac{dz}{iz} \tag{553}$$

Then the above integral can be interpreted as a winding number about the unit circle.

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta} d\theta = \frac{1}{2\pi}\int_\gamma e^{1/2(z+1/z)}\frac{dz}{iz} = \frac{1}{2\pi i}\int_\gamma \frac{e^{1/2(z+1/z)}}{z} dz \tag{554}$$

To summon the appropriate poles to integrate over, expand

$$e^{1/2z} = \sum_{k=0}^{\infty} \frac{1}{k!2^k z^k} \tag{555}$$

Then

$$\frac{e^{1/2(z+1/z)}}{z} = \sum_{k=0}^{\infty} \frac{1}{k!2^k z^k}\frac{e^{z/2}}{z} = \sum_{k=0}^{\infty} \frac{1}{k!2^k z^k}\frac{e^{z/2}}{z} = \sum_{k=0}^{\infty} \frac{1}{k!2^k z^{k+1}}\left(1 + z/2 + (z/2)^2/2! + \cdots\right) \tag{556}$$

Integrating ignores all the terms which are not a multiple of $1/z$, so we note that the coefficient of $1/z$ equals

$$\frac{(1/2)^k/k!}{k!2^k} = \frac{1}{(k!2^k)^2} \tag{557}$$

by finding the numerator from the expansion of $e^{z/2}$ and the denominator from the denominator in the expansion involving $e^{1/2z}$. Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta} d\theta = \sum_{k=0}^{\infty} \frac{1}{(k!2^k)^2}. \tag{558}$$

□

For a version of this problem with an arbitrary scaling factor, see Problem 5.10.6 of [**13**].

CHAPTER 12

Virtuoso Section

1. Fall 2000 Problem 7

EXERCISE 12.1 (The Volterra Operator). *Write*

$$Tf(x) = \int_0^x f(s)ds. \tag{559}$$

(a) Show that T defines a bounded linear operator on the Banach space $C([0,1])$, endowed with its usual norm.
(b) Show that this operator on $C([0,1])$ is compact.

PROOF. For part (a), linearity follows by the linearity of the integral. For boundedness, determine an upper bound as follows:

$$\|Tf\|_\infty = \sup_{x\in[0,1]} \left|\int_0^x f(s)ds\right| \leq \sup_{x\in[0,1]} \int_0^x |f(s)|ds = \int_0^1 |f(s)|ds \leq \|f\|_\infty \tag{560}$$

Therefore, $\|T\| \leq 1$, indicating T is bounded.

For part (b), let $\{f_n\} \subseteq C([0,1])$ be a sequence in the unit ball. Then we verify $T(\{f_n\})$ is precompact. Because T is bounded, so is its image. All we have to show is equicontinuity. Let $\epsilon > 0$ be given. If $|x-y| < \epsilon$, then

$$|Tf_n(x) - Tf_n(y)| = \left|\int_0^x f_n(s)ds - \int_0^y f_n(s)ds\right| = \left|\int_y^x f_n(s)ds\right| \leq \int_y^x |f_n(s)|ds \tag{561}$$

$$\leq |x-y|\|f\|_\infty \tag{562}$$

$$\leq |x-y| \tag{563}$$

$$< \epsilon \tag{564}$$

Therefore the hypotheses of Arzelà-Ascoli apply, indicating a converging subsequence exists. □

2. Fall 2001 Problem 1

EXERCISE 12.2 (Integrals are continuous in mean). *Show that if* $f \in L^1(\mathbb{R})$ *then* $\int_{-\infty}^{\infty} |f(x+h) - f(x)|dx \to 0$ *as* $h \to 0$.

PROOF. Let $h_n \to 0$. Set $f_n(x) = f(x + h_n)$. Select $F \subseteq \mathbb{R}$ such that $\mu(F) < \infty$ and

$$\int_{\mathbb{R}\setminus F} 2|f| < \epsilon. \tag{565}$$

From this arises the estimate

$$\int_{\mathbb{R}} |f_n - f| = \int_{\mathbb{R}\setminus F} |f_n - f| + \int_F |f_n - f| \tag{566}$$

$$\leq \int_{\mathbb{R}\setminus F} 2|f| + \int_F |f_n - f| \tag{567}$$

$$\leq \epsilon + \int_F |f_n - f|. \tag{568}$$

Now select $\delta > 0$ so that if $\mu(B) < \delta$, then

$$\int_B 2|f| < \epsilon. \tag{569}$$

By Egorov's theorem there exists $E \subseteq F$ such that $\mu(F \setminus E) < \delta$ and $f_n \to f$ uniformly on E. Then

$$\int_F |f_n - f| = \int_{F\setminus E} |f_n - f| + \int_E |f_n - f| \tag{570}$$

$$\leq \int_{F\setminus E} 2|f| + \mu(E)\|f_n - f\|_E \tag{571}$$

$$\leq \epsilon + \mu(E)\|f_n - f\|_E. \tag{572}$$

Sending $n \to \infty$ and $\epsilon \to 0$ shows

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f| = 0. \tag{573}$$

□

3. Fall 2001 Problem 2

EXERCISE 12.3 (Basel problem with Fourier analysis).

(a) *Find the Fourier coefficients* $\widehat{f}(k)$ *for the function* $f(x) = x$ *with respect to the exponential system* $e^{2\pi ikx}$ $(k \in \mathbb{Z})$ *on* $[-\frac{1}{2}, \frac{1}{2}]$.

(b) *Use the result of part (a) to compute*

$$\sum_{k=1}^{\infty} \frac{1}{k^2}. \tag{574}$$

PROOF. Introduce the inner product on $L^1 \cap L^2$

$$\langle f, g \rangle = \int_{-\frac{1}{2}} \frac{1}{2} f(x)\overline{g(x)}dx. \tag{575}$$

Then the functions $\{e^{2\pi ikx}\}$ form an orthonormal system and

$$f(x) = \sum_{k=-\infty}^{\infty} \langle f, e^{2\pi ikx} \rangle e^{2\pi ikx} \quad \text{in } L^2 \tag{576}$$

Let us compute these inner products for the given $f(x) = x$.

$$\langle x, e^{2\pi ikx} \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} xe^{-2\pi ikx}dx \tag{577}$$

$$= \int x \left(e^{-2\pi ikx}/(-2\pi ik)\right)' dx \tag{578}$$

$$= \left.\frac{xe^{-2\pi ikx}}{-2\pi ik}\right|_{-\frac{1}{2}}^{\frac{1}{2}} - \int (e^{-2\pi ikx}/(-2\pi ik))dx \tag{579}$$

$$= \frac{1}{-2\pi ik}\left[\frac{e^{-\pi ik}}{2} + \frac{e^{\pi ik}}{2}\right] - \int (e^{-2\pi ikx}/(-2\pi ik))dx \tag{580}$$

(581)

For the part:

$$\int (e^{-2\pi ikx}/(-2\pi ik))dx = \int (e^{-2\pi ikx}/(-2\pi ik))'/(-2\pi ik)dx \tag{582}$$

$$= -\left.\frac{e^{-2\pi ikx}}{4\pi^2k^2}\right|_{-\frac{1}{2}}^{\frac{1}{2}} \tag{583}$$

$$= -\frac{1}{4\pi^2k^2}\left[e^{-\pi ik} - e^{\pi ik}\right] \tag{584}$$

Combining these shows

$$\langle x, e^{2\pi ikx} \rangle = \frac{1}{-2\pi ik}\left[\frac{e^{-\pi ik}}{2} + \frac{e^{\pi ik}}{2}\right] + \frac{1}{4\pi^2k^2}\left[e^{-\pi ik} - e^{\pi ik}\right] \tag{585}$$

$$= \frac{i\cos(\pi k)}{2\pi k} - \frac{i\sin(\pi k)}{2\pi^2k^2} \tag{586}$$

If $k = 0$,

$$\langle x, 1 \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} xdx = 0 \tag{587}$$

so

$$x = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{2\pi i k x} \tag{588}$$

$$= \sum_{k \neq 0} \left[\frac{i\cos(\pi k)}{2\pi k} - \frac{i\sin(\pi k)}{2\pi^2 k^2} \right] e^{2\pi i k x} \tag{589}$$

$$= \sum_{k \neq 0} \frac{i(-1)^k}{2\pi k} e^{2\pi i k x} \tag{590}$$

Apply the Parseval identity for our $f(x) = x$

$$\|f\|_2 = \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2 \tag{591}$$

$$= \sum_{k \neq 0} \frac{1}{4\pi^2 k^2} \tag{592}$$

$$= 2 \sum_{k=1}^{\infty} \frac{1}{4\pi^2 k^2} \tag{593}$$

Then

$$\|f\|_2 = \int x^2 dx = \left. \frac{x^3}{3} \right|_{-1/2}^{1/2} = \frac{1/8}{3} + \frac{1/8}{3} = \frac{1}{12} \tag{594}$$

so that

$$\frac{1}{12} = \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \tag{595}$$

and finally

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}. \tag{596}$$

□

4. Fall 2001 Problem 6

EXERCISE 12.4 (Bounded only on the irrationals). *Show that there does not exist a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{C}$ such that the sequence $\{f_n(x)\}$ is bounded if and only if x is irrational. (Hint: Show that the set $\{x \mid \{f_n(x)\}$ is bounded$\}$ is an F_σ.)*

PROOF. Per the hint, we write

$$\{x \mid \{f_n(x)\} \text{ is bounded}\} = \bigcup_{M=1}^{\infty} \bigcap_{n=1}^{\infty} \{x \mid |f_n(x)| \leq M\} \tag{597}$$

The intersection is over closed sets, so it is closed, indicating this is indeed an F_σ set. At this point we are done because $\mathbb{R} \setminus \mathbb{Q}$ is definitely not an F_σ. In fact, $\mathbb{Q}$ is an F_σ set, because it is a countable union of points, so that the irrationals form a G_δ set. □

5. Fall 2008 Problem 2

EXERCISE 12.5 (Countable product of the interval). *Consider the space $X = [0,1] \times [0,1] \times \cdots$ (the countably-infinite product of $[0,1]$ with the product topology). An element of X may be thought of as a sequence $\{x_n\}_{n=1}^{\infty}$ with each $x_n \in [0,1]$. Show that the function from X to $\mathbb{R}$ defined by*

$$\{x_n\} \mapsto \sum_{n=1}^{\infty} 2^{-n} x_n \tag{598}$$

is continuous.

PROOF. Let $U \subseteq \mathbb{R}$ be an open set. Select $f(y) \in B_\epsilon(p) \subseteq U$. Let $\epsilon' = \epsilon - |f(y) - p|$. By a metric space argument, $B_{\epsilon'}(f(y)) \subseteq B_\epsilon(p) \subseteq U$. Select N such that

$$\sum_{n=N}^{\infty} \frac{2}{2^n} < \epsilon'/2. \tag{599}$$

Define a basic open set in X

$$V = \prod_{n=1}^{N-1} B_{\epsilon'/2}(y_n) \times \prod_{n=N}^{\infty} [0,1]. \tag{600}$$

Certainly $y \in V$, and we show that V is interior to the inverse image. Let $\widetilde{y} \in V$. Then

$$|f(y) - f(\widetilde{y})| = \left| \sum_{n=1}^{\infty} \frac{y_n}{2^n} - \frac{\widetilde{y_n}}{2^n} \right| \tag{601}$$

$$\leq \sum_{n=1}^{N-1} \frac{|y_n - \widetilde{y_n}|}{2^n} + \sum_{n=N}^{\infty} \frac{|y_n - \widetilde{y_n}|}{2^n} \tag{602}$$

$$\leq \sum_{n=1}^{N-1} \frac{\epsilon'/2}{2^n} + \sum_{n=N}^{\infty} \frac{2}{2^n} \tag{603}$$

$$\leq \epsilon'/2 + \epsilon'/2 \tag{604}$$

$$\leq \epsilon' \tag{605}$$

Therefore, $f(\widetilde{y}) \in B_{\epsilon'}(f(y)) \subseteq U$, indicating $\widetilde{y} \in f^{-1}(U)$. □