“I Went by Twos, He Went by One”:
Multiple Interpretations of Inscriptions as Resources for Mathematical Discussions

Judit N. Moschkovich
Education Department
University of California, Santa Cruz

This article examines a classroom discussion of multiple interpretations of the scales on two distance versus time graphs. The analysis describes how two students and a teacher used multiple meanings for phrases of the form “I went by” and coordinated these meanings with different views of the scales. Students’ ambiguous and shifting meanings did not prove to be obstacles to this discussion. Instead, this teacher used student interpretations as resources, built on them, and connected them to canonical mathematical concepts—in particular by highlighting (Goodwin, 1994) a “unitized” (Lamon, 1994, 1996, 2007) view of the scales. Research in mathematics education describes teaching that promotes conceptual development as having two central features: One is that teachers and students attend explicitly to concepts, and the other is that students wrestle with important mathematics (Hiebert & Grouws, 2007). Not only does this classroom discussion provide an example that it is possible to balance these two features, but the analysis provides the details of how instruction can simultaneously provide explicit attention to concepts while allowing students to wrestle with these concepts.

Graphs, tables, and equations are ubiquitous in mathematics classrooms. Student interpretations of these inscriptions and the connections among these three representations are taken as evidence of conceptual understanding. We should not, however, assume that students and teachers interpret these inscriptions in the same ways. Researchers have described how students interpret algebraic symbols such
as the equal sign (Herscovics, 1989; Kieran, 1992, 2007) in different ways than a teacher might. We know that students interpret graphs in both canonical and noncanonical ways (Bell & Janvier, 1981; Curcio, 1987; Friel, Curcio, & Bright, 2001; Kaput, 1998; Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, 1996, 1998; Moschkovich, Schoenfeld, & Arcavi, 1993; Nemirovsky & Monk, 2000). We also know that different aspects of an inscription, such as the grid marks on a graph, can impact how a student interprets a graph (Stevens & Hall, 1998). Therefore, when students and teachers look at a graph together, we can expect that they may not be looking at, talking about, referring to, or imagining the same things. We should expect that students will interpret inscriptions in both canonical and noncanonical ways and that many aspects of inscriptions will impact these student interpretations.

How do students and teachers manage discussions of multiple interpretations? This article presents an instance of a classroom discussion in which a teacher and two students managed such multiple interpretations. During this classroom discussion, the participants used multiple and shifting meanings for phrases of the form “I went by” to describe the scales on two graphs, which is surprising given the simplicity of the form. The analysis shows the sources of ambiguity and makes visible how participants coordinated these multiple meanings for the phrase “I went by” with different views of the scales. These ambiguous and shifting meanings did not prove to be obstacles to this mathematical discussion. Instead, this teacher used students’ multiple interpretations as resources, building on and connecting them to important mathematical concepts.

Current policy documents in mathematics education reflect a converging agreement that both procedures and concepts are important, mutually supportive and should be developed simultaneously (Kilpatrick, Swafford, & Findell, 2001; National Mathematics Advisory Panel, 2008). However, these documents do not provide guidance or examples for carrying out instruction that balances an emphasis on both. According to a recent review of the research literature (Hiebert & Grouws, 2007), teaching that makes a difference in student achievement and promotes conceptual development has two features: (a) Teachers and students attend explicitly to concepts, and (b) students wrestle with important mathematics. The present classroom discussion provides an example of how one teacher enacted such instruction through a discussion that was grounded in inscriptions, questions, and multiple interpretations that students themselves had generated while also attending to conceptual knowledge.

Mathematics classrooms in the United States show a striking absence of the very two features identified as facilitating conceptual understanding (Hiebert & Grouws, 2007). For example, the TIMSS Video Study (Hiebert, Stigler, & Manaster, 1999) showed that in algebra classrooms in the United States little, if any, time is spent on mathematical reasoning. The study reported that “mathematical reasoning was evident in 20% of the German lessons, 53% of the Japanese les-
sons, and 0% of the U.S. lessons” and that overall, “Japanese teachers emphasized thinking, German and U.S teachers emphasized skills” (p. 198). The TIMSS Video Study described two ways for instruction to address mathematical concepts and procedures: either simply stated by the teacher or developed through examples, demonstrations, and discussions. The data showed that concepts and procedures were usually “developed” in German and Japanese classrooms but just stated in U.S. lessons. The present classroom discussion stands in stark contrast to these findings, providing an instance of a discussion that simultaneously addressed both skills and reasoning and developed an important mathematical concept, unitizing.

The first part of the analysis addresses the following questions: What were the multiple situated meanings of statements of the form “I went by”? How did participants coordinate these meanings with views of the scales? The second part of the analysis examines the teacher’s role during this discussion. In particular, how did the teacher respond to students’ multiple interpretations of the scales? How did the teacher connect student interpretations to canonical mathematical practices and concepts? Analyses of conversations in other settings have shown that meanings for utterances are situated in activity and coordinated with particular ways of viewing and focusing attention on inscriptions (Gee, 1996; Goodwin, 1994; Hindmarsh & Heath, 2000; Stevens & Hall, 1998). Ways of viewing inscriptions have been described as connected to professions (Goodwin, 1994) and disciplines (Stevens & Hall, 1998). This classroom mathematical discussion, like other conversations, involved situated meanings for utterances (rather than static or individual meanings for words) that were coordinated with multiple views of the scales.

Student descriptions of the scales revealed three different situated meanings for phrases of the form “I went by.” One meaning referred to the value of the interval between tick marks on a scale, the second to the number of segments between tick marks, and the third to the value of each segment between tick marks. The same utterance was used with different meanings depending on the view of the inscriptions. The analysis shows how participants coordinated multiple meanings with different views of the scale, grid segments, tick marks, and number labels on the scale.

The phrase “I went by twos” can refer to the action used to construct a scale by making tick marks every two segments and mean “I made tick marks every two segments” (see Figure 1). This description names how many segments are marked by the tick mark that has a number label but need not refer to units or the quantity represented by each segment marked. The same utterance, “I went by twos,” can also refer to the number of units represented by the chunk created between two tick marks, as in “I made tick marks at every segment and each segment represents two units” (see Figure 2). This meaning refers not only to the marks and the number of segments marked but also to the number of units represented in any marked segment. Using this meaning for the phrase “I went by,” one would describe the scale
in Figure 2 by saying “I went by twos” but describe the scale in Figure 1 by saying “I went by half (or point five).”

A discussion centered on how two students labeled the axes on their graphs might seem both simple and procedural. However, the analysis shows that the deceivingly simple actions of labeling, describing, and comparing labels on axes are not so simple for students and, in fact, involve important conceptual understanding. The first step in constructing a graph is to partition each axis into equal segments that will serve as markers for graphing an ordered pair. Labeling an axis with tick marks and numbers divides the number line into segments of equal length and involves an important mathematical concept, unitizing (Lamon, 1994, 1996, 2007). As one makes tick marks or writes number labels on the axis, each segment labeled or marked is assigned a unit. Following Goodwin (1994), the analysis describes how the teacher highlighted a “unitized” view of the scales,
anchoring her descriptions of the scales on a canonical mathematical practice, unitizing.

How is this unitized view of the scales mathematically important? For one thing, scaling is a central aspect of understanding graphs as inscriptions and can be a stumbling block for students (Leinhardt et al., 1990), especially when they are learning about the concept of slope. For example, because the slope of a line is defined as the change in the vertical direction divided by the change in the horizontal direction, how the slope of a line appears on a graph depends on how each axis is unitized. To compare the slopes of two lines, it is essential to first consider the scales on the axes for each graph. If two graphs have different scales on the y axis, the same linear equation will appear to have different slopes on each graph. For example, if the y axis on the first graph “goes by ones” (each space represents one unit) and that on the second graph “goes by twos” (each space represents two units), and one graphs the same function (e.g., \( y = 2x + 4 \)) on both graphs, the slope will look steeper on the first graph than on the second graph. Because the slope is a ratio, the impact of changing the unitizing of the x axis so that the second scale on the x axis “goes by twos” would be to make the same line appear steeper on the second graph. Scaling and unitizing can thus complicate discussions of slope.

Beyond the impact of unitizing on scaling and the centrality of scaling to understanding the concept of slope, the connection between slope and rate is fundamental. Rate is a central mathematical concept that not only spans students’ classroom experiences from middle school to calculus but also can be useful in life outside of the classroom, for example in calculating mileage per gallon of gasoline or comparing prices per ounce at the supermarket. The Curriculum Focal Points (National Council of Teachers of Mathematics, 2006) identify rate as one of the focal points for middle school. Research has shown that the concept of speed involves much more than knowing the procedure “speed equals distance over time” and that understanding the concepts of rate and speed can be challenging for students (Thompson & Thompson, 1996; Thompson, 1994; Thompson & Thompson, 1994).

How can this analysis of how one teacher led a conceptual discussion contribute to teaching practice more generally? This analysis first provides an example of how a teacher balanced explicit attention to concepts while allowing students to wrestle with important mathematics. The teacher accomplished this not by setting a problem but by transforming student questions into a problem that makes connections. Hiebert and Grouws (2007) suggested that one way to ensure that students wrestle with important mathematics is for teachers to choose and set problems that make connections. This classroom discussion provides an example of how teachers can also ensure that students wrestle with important mathematics by transforming student-generated questions into problems that make connections. Hiebert and Grouws gave an example of a problem that makes connections by asking students to graph several linear equations and examine the role played by the
numbers in the equation in determining the position and the slope of the associated lines. This problem closely parallels the topic of this classroom discussion, examining the role played by the numbers on the y-axis scale in determining the slope of line segments on the graph. However, Hiebert and Grouws cautioned against stopping at the analysis of the kind of problems teachers present to students. They emphasized that describing teaching involves close inspection of how students and teachers interact about content “with considerable detail and precision” (p. 393). Although the classroom discussion itself provides an example of “what is possible” for classroom discussions, the analysis provides a description of how the teacher and students attended explicitly to content and reveals the details of the interaction.

Expecting students to generate their own interpretations of graphs and eliciting these interpretations during classroom discussions leads to an important pedagogical question: What pedagogical principles can teachers use to respond to students’ multiple interpretations? In particular, how can teachers build on student interpretations and at the same time also connect student ideas to canonical practices and concepts? The second part of the analysis addresses these questions by examining how this teacher’s role in the discussion, in particular how the teacher wove together multiple perspectives, shared many characteristics with instructional conversations (Goldenberg, 1991; Tharp & Gallimore, 1988, 1991).

Too often we impose “either/or” lenses on classroom interactions (see Warren, Ogonowski, & Pothier, 2005, for a critique of one such dichotomy) and imagine that mathematical discussions draw on only one or another set of dichotomized practices: Content is either procedural or conceptual, meanings are either everyday or academic, and instruction is either “explicit teaching” or “student exploration.” The analysis of this classroom discussion moves away from such dichotomized views. A topic that might at first glance seem procedural provided opportunities for a conceptual discussion, an everyday phrase that might not seem mathematical communicated meaning connected to important mathematical concepts, and an explanation by a teacher simultaneously allowed students to wrestle with important mathematics and paid explicit attention to mathematical concepts.

THEORETICAL FRAMEWORK

The situated and sociocultural theoretical framework for this analysis draws on a situated perspective of learning mathematics (Greeno, 1994). I assume that learning mathematics is a discursive activity that involves participating in a community of practice (Forman, 1996; Lave & Wenger, 1991; Nasir, 2002); developing classroom sociomathematical norms (Cobb, Wood, & Yackel, 1993); and using multiple material, linguistic, and social resources (Greeno, 1994). The analysis presented here also builds on previous work on discussions in mathematics and
Mathematical discussions were initially defined as “purposeful talk on a mathematical subject in which there are genuine pupil contributions and interactions” (Pirie, 1991, p. 143). More recent work has expanded on this definition by also considering mathematical practices (Cobb, Stephan, McClain, & Gravemeijer, 2001) and inscriptions (Latour, 1987; Latour & Woolgar, 1986; Sfard & McClain, 2002). Some analyses have described how students develop new meanings for words as they participate in classroom mathematical discussions (for an example, see Radford, 2001).

In analyzing the situated meanings of phrases the participants used to describe the scales, I use situated in several senses. The multiple meanings of the phrase “I went by” were situated locally in the ecology of this classroom and in the history and interactions that preceded this discussion, and, thus, they may or may not arise in other classrooms. In this particular classroom, these meanings were also situated in time, in the situation, and with respect to artifacts; meanings may shift among participants and for an individual participant at different times, in different situations, and with respect to different artifacts. I assume that meanings for utterances are situated in practices, in particular where and how participants focus joint attention (Rogoff, 1990) and view inscriptions (Goodwin, 1994; Stevens & Hall, 1998). The multiple meanings for this phrase reflect how each participant “highlighted” (Goodwin, 1994) the inscriptions from different perspectives.

I take the notions of “focus of attention” and “perspective” from Rogoff’s (1990) definition of appropriation. In this definition, the central features of appropriation are that the process involves achieving a joint focus of attention and developing shared meanings (as well as transforming what is appropriated). Rogoff distinguished between what she called “skills” and “shifts in perspective.” She defined skills as “the integration and organization of information and component acts into plans for action under relevant circumstances” and shifts of perspective as involving “giving up an understanding of a phenomenon to take another view contrasting with the original perspective” (p. 142). Following these definitions, I examine multiple meanings for utterances, multiple perspectives of the scales, and how participants coordinated these multiple perspectives and meanings.

I selected this particular classroom discussion for several reasons. First, graphing is an especially fruitful mathematical topic on which to focus: It is well researched, well understood, and also seen as key to the development of conceptual understanding (Bell & Janvier, 1981; Curiel, 1987; Friel et al., 2001; Kaput, 1998; Leinhardt et al., 1990; Moschkovich et al., 1993; Nemirovsky & Monk, 2000). Second, a discussion of the labels on the scales might seem procedural when in fact comparing two scales and explaining how a scale impacts the shape of a curve involve important conceptual understanding. Lastly, the teacher’s role in this discus-
sion exemplifies some of the central principles for instructional conversations: eliciting student ideas, building on these ideas, and at the same time connecting to canonical mathematical practices. This particular discussion shows how a teacher used an everyday phrase based on students’ own ideas and ways of talking, one that may appear to have little mathematical meaning and that is not a technical term, and connected this phrase to conceptual content and a canonical mathematical practice.

This discussion illustrates that mathematical discussions with conceptual content need not involve formal or technical language. The phrase “I went by” does not seem, at first glance, particularly mathematical and might be labeled an “everyday” phrase. The phrase certainly is not a technical term, like tangent or function. Nevertheless, both the teacher and the students used this phrase repeatedly, their use of this phrase grounded the discussion on the inscriptions, the students used this phrase to communicate mathematical ideas, and the teacher connected the meaning of this phrase to unitizing. The argument is not that the phrase “I went by” in itself is important, but rather that this phrase is an instance of an everyday phrase that the students and teacher used as a resource for generating a mathematical discussion.¹ This discussion is mathematical not because it involved technical mathematical terms (or a mathematical inscription), but because it involved mathematical concepts and practices.

Instructional conversation (Goldenberg, 1991; Tharp & Gallimore, 1988, 1991) is a pedagogical approach to classroom discussions based on Vygotskian accounts of learning and teaching. These events are described as both instructional in intent and conversational in quality. Instructional conversations are described as times when “teachers and students are responsive to what others say, so that each statement or contribution builds upon, challenges, or extends a previous one” (Goldenberg, 1991, p. 3). Important characteristics of instructional conversations are the teacher’s efforts to focus on student understanding, elicit student ideas, listen carefully, make guesses about the meaning of student contributions, and weave together multiple perspectives. Instructional conversations have been contrasted to direct instruction. Although instructional conversations can include teacher responses such as clarifying, instructing, and other contributions that look like direct instruction, these responses are provided only when necessary and are always tailored to student needs. This classroom discussion is an example of an instructional conversation about a mathematical situation. The second part of the “Analysis” section examines how this teacher responded to students’ multiple meanings for the phrase “I went by” by building on student interpretations and weaving student perspectives with an explanation anchored in unitizing.

¹Data from other lessons also shows that the teacher, these two students, and other students in this classroom used this phrase (or similar phrases) during whole class and small group discussions.
Unitizing as a Mathematical Practice

Cobb et al. (2001) defined mathematical practices as the “taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas” (p. 126). In contrast to social norms and sociomathematical norms, mathematical practices are specific to particular mathematical ideas. Using the term practice serves to shift from purely cognitive accounts of mathematical activity to accounts that assume the social, cultural, and discursive nature of mathematical activity.²

I use the phrase mathematical Discourse practices (Moschkovich, 2007) to mark a view of Discourse as more than utterances or text (Gee, 1996, 1999), to highlight that Discourse is embedded in practices, and to emphasize the plurality of these practices. There is no single set of mathematical Discourse practices or one mathematical community (for a discussion of multiple mathematical Discourses, see Moschkovich, 2002). Mathematical discussions can involve different communities (mathematicians, teachers, or students) and different genres (explanations, proofs, or presentations). Practices vary across communities of research mathematicians, traditional classrooms, and reformed classrooms. However, across these various communities and genres there are common practices that can be labeled as canonical.³

Lamon (1994, 1996, 2007) described partitioning and unitizing as central features of proportional reasoning. Partitioning is the action taken to divide an object or set of objects into equal parts. Lamon (1996) defined unitizing as “the cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity” (p. 170). The example used to illustrate unitizing is how a six-pack of cans of a beverage can be viewed and described as 6 cans (6 one-units) or 1 six-pack (1 six-unit). Similarly, a case of cans can be viewed as 24 cans (24 one-units), 2 twelve-packs (2 twelve-

³I use the terms practice and practices in the sense used by Scribner (1984) for a practice account of literacy to “highlight the culturally organized nature of significant literacy activities and their conceptual kinship to other culturally organized activities involving different technologies and symbol systems” (p. 13).

³There are many ways to mark the assumption that practices are connected to professions, as in professional vision (Goodwin, 1994); to disciplines, as in disciplined perception (Stevens & Hall, 1998); to expertise, as in expert knowledge; or to accepted canons, as in canonical practices. I choose to use the phrase canonical mathematical practice rather than professional vision or disciplined perception for several reasons. First, students in mathematics classrooms are not in training to become professional mathematicians, and thus the phrase professional vision does not seem appropriate. Second, students and teachers in a middle school mathematics classroom are not engaged in the practices of any one academic discipline, as constructing and interpreting graphs of motion can be a part of the practices of several academic disciplines such as mathematics, engineering, and physics. Lastly, this choice marks a connection to Chèche Konnen work on science discourse practices (Rosebery et al., 1992; Warren et al., 2005), which uses the term canonical.
units), or 4 six-packs (4 six-units). Labeling or making tick marks on a number line or axis is another example of unitizing. As I make a tick mark or write a number label, I am assigning a unit to the segment I tick or label and deciding the size of the chunks I am marking.

Although there were several canonical mathematical practices evident in this classroom discussion, the analysis focuses on unitizing. In contrast to views of unitizing as purely cognitive, I assume that unitizing is a practice that is simultaneously cognitive, social, and discursive. As such, unitizing involves not only thinking about a situation in a particular way but also using particular meanings for utterances and particular views of that situation. In the example of a case of cans, one needs to both view the situation while imagining different units (a six-pack or a can) and also use words to describe the unit as “a six-pack” or “a can.”

**SETTING, PARTICIPANTS, AND DATA**

The transcript comes from a larger set of data collected in an eighth-grade bilingual mathematics classroom. Classroom observations and videotaping were conducted during two curriculum units from *Connected Mathematics* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), “Variables and Patterns” and “Moving Straight Ahead.” This discussion occurred during the unit “Moving Straight Ahead.” Data collected included videotapes of whole-class discussions and of one student group for every lesson, as well as videotaped problem-solving sessions in pairs. The classroom was part of a two-way Spanish–English bilingual program, but classes were not conducted as dual-immersion classes after the sixth grade. Some students spoke mainly English, some used both languages, and some spoke mainly Spanish. Instruction was conducted principally in English, with some discussions and explanations in Spanish among students when working in small groups as well as between the teacher and some students. The two students in this discussion, Carlos and David, were native Spanish speakers who were considered to be proficient in both languages and thus bilingual.

This small-group discussion occurred toward the beginning of a classroom period. The teacher usually started the 90-min class with a brief whole-class discussion about a mathematics problem (the problem for that day, a problem from a previous lesson, or a homework problem). Students then worked in groups of two to four, discussing the problem at their tables. The teacher moved from small group to small group, asking and answering questions in each group. Toward the end of the class period there were usually reports or presentations by each group as well as whole-class discussions led by the teacher. On the day of this discussion, students expected that each group would at some point be asked to go to the front of the classroom to explain their graphs and would be expected to describe how and why
they solved a problem as they had, as well as be prepared to answer questions from other students and the teacher.

ANALYSIS

The analysis is presented in two parts. The first part examines the multiple meanings students used for statements of the form “I went by.” I describe how multiple meanings and competing claims arose and how the students coordinated meanings with views of the scales. The second part of the analysis examines the teacher’s role during this discussion. I describe how the teacher responded to these multiple student interpretations and how she connected student interpretations to canonical mathematical practices.

Table 1 shows a summary of the discussion (see Appendix A for full Transcript 1). I first provide some background to frame the subsequent closer analysis of student interpretations and use excerpts of the transcript to describe how the students coordinated multiple meanings for the phrase “I went by” with different views of the scales. I then examine the teacher’s explanations, describing how she highlighted a unitized view of the scales and connected student interpretations to this canonical practice. As an epilogue, I examine a whole-class discussion that occurred several months later in which one of the students, Carlos, explained the impact of a scale on the shape of a graph (see Appendix B for full Transcript 2). Although I do not argue that there was a cause-and-effect relationship between these two events, this second transcript provides some evidence that Carlos changed how he talked about and viewed the scales on a graph.

Background

We join Carlos and David as they were working together in a small group. They were reviewing the graphs each had created independently for a homework problem in the Connected Mathematics unit “Moving Straight Ahead” (Lappan et al., 1998). This problem was framed by a story about a 5-day bicycle trip. In this story, while some riders rode bicycles, others rode in a van and recorded the total dis-

<table>
<thead>
<tr>
<th>TABLE 1</th>
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<tbody>
<tr>
<td><strong>Overview of Transcript 1</strong></td>
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<tr>
<td>Lines 22–28: David and Carlos compare their graphs.</td>
</tr>
<tr>
<td>Lines 29–63: The teacher joins David and Carlos. They discuss the shape of the two curves and the impact of changing a scale on the shape of a curve. They discuss the scales for the two graphs by referring to both the x and the y axes.</td>
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<tr>
<td>Lines 64–74: The teacher, David, and Carlos discuss the y axes. They describe the scale on the y axis of each graph and compare the scales on the two y axes.</td>
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<tr>
<td>Lines 75–87: The teacher responds to Carlos’s claim that he “went by fives.”</td>
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tance from the starting point traveled every half hour. The problem here refers to the second day of the bicycle tour (see Figure 3).

Each student first read his written answer out loud. Their written answers were similar in some ways and different in other ways. They both wrote that the bikers had traveled a total of 45 miles in 5 hours, and they also agreed that the half-hour time interval when the bikers had made the least progress was the interval between 2.0 and 2.5 hours. However, they had written different answers when identifying the half of the trip and the half-hour interval when the bikers had made the most progress. As the students read their answers, Carlos remarked that the written answers were different. David then took his graph out and compared it to Carlos’s graph (see Figures 4 and 5). As they looked at the graphs, they noticed that the two graphs looked different.

On the second day of their bicycle trip, the group left Atlantic City and rode 5 hours South to Cape May, New Jersey. This time, Sidney and Sarah rode in the van. From Cape May, they took a ferry across the Delaware Bay to Lewes, Delaware. Sarah recorded the following data about the distance traveled until they reached the ferry.

<table>
<thead>
<tr>
<th>Time (hr)</th>
<th>Distance (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>8</td>
</tr>
<tr>
<td>1.0</td>
<td>15</td>
</tr>
<tr>
<td>1.5</td>
<td>19</td>
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<tr>
<td>2.0</td>
<td>25</td>
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<tr>
<td>2.5</td>
<td>27</td>
</tr>
<tr>
<td>3.0</td>
<td>34</td>
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<tr>
<td>3.5</td>
<td>40</td>
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<td>4.0</td>
<td>40</td>
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<tr>
<td>4.5</td>
<td>40</td>
</tr>
<tr>
<td>5.0</td>
<td>45</td>
</tr>
</tbody>
</table>

1. Make a coordinate graph of the (time, distance) data given in the table.
2. Sidney wants to write a report describing Day 2 of the tour. Using information from the table and the graph, what would she write about the days’ travel? Be sure to consider the following questions:
   A. How far did the group travel in the day? How much time did it take them?
   B. During which interval(s) did the riders make the most progress? The least progress?
   C. Did the riders go further in the first half or the second half of the days’ ride?
3. By analyzing the table, how can you find the time intervals when the riders made the most progress? The least progress? How can you find these intervals by analyzing the graph?

FIGURE 3  Problem: From Atlantic City to Lewes.
FIGURE 4  Carlos’s graph: whole view (top) and in close up (bottom).
FIGURE 5  David’s graph: whole view (top) and in close up (bottom).
The students displayed an orientation to a common understanding of how classrooms work: Problems are expected to have solutions, the solutions are expected to be solvable using local resources, and there is only one correct solution. If two students arrive at differing solutions, at least one of them is wrong. We can hear this orientation in David’s statement, “Here’s my graph. Did it come out like yours?” (line 22); in Carlos’s response, “I don’t know” (line 24); and in Carlos’s remarks about the differences between the two graphs as he explained, “It’s because you did it upwards” (line 27). By prefacing his utterance in line 29 with “oh,” saying “Oh, you did” (line 29), David marked it as claim of new information (Heritage, 2005) about Carlos’s graph. Carlos’s and David’s questions (lines 30–33) regarding how they were “supposed” to do the graph are further evidence of the students’ expectation that the graphs would look the same.

The students started out with the goal of evaluating the shape of the graphs, probably to decide whether they had the right answer. The teacher joined the discussion at the students’ request (lines 30–33). She initially set the goal of finding similarities and differences between the two graphs and later set the goal of considering the impact of the scale on the shape of the graph. She next asked the students to compare how they had labeled their axes and consider what was different about the two graphs. She suggested that David and Carlos look at their numbers and the way they had placed their numbers and concluded that they both had put time on the x axis. It is at this point in the discussion that the students started using phrases of the form “I went by” (line 40).

Multiple Meanings for “I Went By”

Carlos first introduced the phrase “I went by twos” (line 42) to describe his own y-axis scale and the phrase “he went by one” (line 42) to describe David’s y-axis scale.

Excerpt 1

40. Teacher: You both put time here ((referring to the x axes)). OK, so what’s different about these? I don’t think it’s, I don’t think it’s the positioning of them. Look at your numbers, the way you placed your numbers.
41. Teacher’s gesture: ((Points at the x axis on both papers, back and forth. Then turns David’s paper in the same direction as Carlos’s paper.))
42. Carlos: Oh, that’s true, ‘cause I went by twos, I went 1, 2 ((0.2 s)) and then I put that one (—-) he went by one.
43. Carlos’s gesture: ((Begins counting with his right finger following the numbers on his paper.))

Carlos had labeled his axes by making a tick mark at every two-grid segment (see Figure 4). In contrast, David had labeled the axes of his graph by making tick marks on every grid segment (see Figure 5). Although David and Carlos had la-
beled their axes differently, David claimed that he also “went by twos,” first in line 56 (not shown in the excerpts) and then again in line 61.

Excerpt 2

61. David: I went by twos.
62. Teacher: You went by twos and you went b:y- ((0.2 s))
63. Carlos: I went by twos. You (didn’t) you went by ones!
  °What are you talking about. °

64. Teacher: No, here on the y axis.
65. Teacher’s gesture: ((Points to the axis.))
66. Carlos: Oh, I went by fives.

The teacher first accepted David’s claim that he “went by twos” and proceeded to describe Carlos’s scale. Carlos disagreed with David’s claim, insisting that whereas he “went by twos,” David “went by ones” (line 63). Carlos explicitly disagreed with David’s claim that David had gone by twos. He also noticed that they might not all be talking about the same thing and said, “What are you talking about?” (line 63). Carlos then changed the description of his own scale, marking new information (Heritage, 2005) and saying “Oh, I went by fives” (line 66). These conflicting descriptions and this disagreement served as resources by providing a reason for the students to examine the scales more closely and engage in the subsequent discussion of the scales.

The teacher first agreed with Carlos’s claim but then, after a short pause, she disagreed and proposed that Carlos had not gone by fives but by “two and a halves.” In response, Carlos proposed that if he “went by two and a halves,” then David “went by one” (line 69).

Excerpt 3

67. Teacher: You went by fives. ((0.2 s)) No, actually you didn’t go by fives. You actually went by two and a halves because you’d, you did every two spaces was five.
68. Teacher’s gesture: ((She points to Carlos’s paper while she explains.)) ((0.3 s))
69. Carlos: Then he only went by one.
70. Carlos’s gesture: ((Carlos points to David’s paper.)) ((0.2 s))
71. Teacher: Every one space was two of his. You see, they’re almost the same. If you look at the next two- ((Puts down her notebook and points to the graphs.))

72. Carlos: Wait! But I don’t get what you’re saying.
73. Teacher: OK.
74. Carlos: ‘Cause I went by fives. ((David stands up.))
75. Teacher: OK, your numbers, right, the numbers you have are by five ((0.2 s)) OK ((0.1 s)). If you look at one line here, what number is he at?
76. Teacher’s gesture: ((Takes David’s paper and places it next to Carlos’s paper then points to David’s graph.))
As the teacher compared the spaces on the two scales, she explained that on David’s scale, each space had a value of two units: “Every one space was two of his” (line 71). At this point, Carlos said that he did not understand the teacher’s explanation (line 72) and returned to claiming that he “went by fives” (line 74). With the series of actions and utterances at the end of Excerpt 3, the teacher set the goal of comparing the two scales in more detail. She put her notebook down and pointed to the two graphs. In response to these actions, David reoriented himself, standing up (line 74) probably so that he could see both graphs. The teacher then took the students’ papers in her hands, turned the two graphs so that they were facing her, and pointed to David’s graph.

The multiple meanings for the phrase “I went by,” the competing claims, and the disagreements evident in these three excerpts provided an opportunity for a mathematical discussion around a question set by the students themselves. Carlos was most actively involved in responding verbally to the teacher. At one point he even interrupted the teacher, saying “Wait! But I don’t get what you’re saying” (line 72). Although David talked less than Carlos did, he participated in the discussion in several ways: He looked intently at the graphs, he seemed to follow the teacher’s pointing gestures toward the graphs, and he stood up to look at the two graphs. David continued to look intently at the two graphs during the teacher’s explanation, shown later in Excerpt 4.

How Students Coordinated Meanings and Views

We see how each student coordinated meanings for “I went by” and views of the two scales by looking more closely at selected lines (lines 56, 63, 66, and 67 listed in Table 2).

Table 3 lists how three meanings for “I [or you] went by” can be coordinated with different ways to focus attention on the inscriptions. One meaning refers to the value of the interval between tick marks, the second to the number of segments between tick marks, and the third to the value of each segment between tick marks. In the excerpts above, it seems clear that Carlos was using the first and second meanings, whereas David might have been using the first or third meanings.

It is difficult to unequivocally interpret David’s utterances to describe his scale (lines 56 and 61), because grid segments coincide with tick marks on his scale (see Figures 5 and 6). When David said “I went by twos” (line 56) to describe his own graph, he could have been referring to the number labels he had placed on the tick marks on his scale, to the value of each grid segment on his scale, or to both (see Figure 6). Because on David’s scale tick marks corresponded to grid segments, the number label on the tick mark was also the value of each grid segment. Therefore, it is not possible to tell whether he was referring to the number labels on the tick marks or to the value of each grid segment. David might have been using the phrase
“I went by twos” (line 56) to mean that the value of each tick mark on the y axis increased by 2, that each grid segment had a value of two units, or both.

Because Carlos labeled his scale differently than David and because he described both his own and David’s graphs, it is easier to interpret Carlos’s descriptions. On Carlos’s scale tick marks did not correspond to grid segments, because Carlos had labeled only every other grid segment with a tick mark and a number (see Figure 4). In one instance, he used the phrase to refer to how many grid segments there were between tick marks on his graph. In two other instances, he used the phrase to refer to the value between the tick marks on the y axis of his graph.

Initially, Carlos seemed to be focusing on the placement of the tick marks on his scale that appeared every two segments (lines 42 and 63, “I went by twos”) and on David’s scale that appeared every single segment (line 42, “he went by one”). Carlos alternated between describing his own scale by saying he “went by twos” (lines 42 and 63; Figure 7) and “went by fives” (lines 66 and 74; Figure 8). In the first case (lines 42 and 63; Figure 7), Carlos seemed to be referring to how

<table>
<thead>
<tr>
<th>Lines from Transcript 1</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>David</td>
<td></td>
</tr>
<tr>
<td>56. David: I went by twos.</td>
<td>David is referring to the scale on his graph.</td>
</tr>
<tr>
<td>Carlos</td>
<td></td>
</tr>
<tr>
<td>42. Carlos: Oh, that’s true, ‘cause I went by twos, I went 1, 2 ((0.2 s)) and then I put that one (—-) he went by one.</td>
<td>Carlos is referring to the placement of the tick marks along either the x or the y axis. Carlos had placed a tick mark every two grid segments, whereas David had placed a tick mark on every grid segment.</td>
</tr>
<tr>
<td>63. Carlos: I went by twos. You (didn’t) you went by ones!</td>
<td>Carlos is now using “went by” to refer to the value between the tick marks on the y axis of his graph.</td>
</tr>
<tr>
<td>66. Carlos: Oh, I went by fives.</td>
<td>Carlos is referring to the placement of tick marks along the y axis of David’s graph. There was a tick mark placed every 1 segment.</td>
</tr>
<tr>
<td>69. Carlos: Then he only went by one.</td>
<td>Carlos returns to referring to the value between the tick marks on the y axis of his graph.</td>
</tr>
<tr>
<td>74. Carlos: ‘Cause I went by fives.</td>
<td>Carlos returns to referring to the value between the tick marks on the y axis of his graph.</td>
</tr>
<tr>
<td>Teacher</td>
<td></td>
</tr>
<tr>
<td>62. Teacher: You went by twos and you went by ((0.2s))</td>
<td>The teacher is referring to the value of the segments in David’s graph.</td>
</tr>
<tr>
<td>67. Teacher: No, actually you didn’t go by fives. You actually went by two and a halves because you’d, you did every two spaces was five.</td>
<td>The teacher is referring to the value of one segment in Carlos’s graph.</td>
</tr>
<tr>
<td>71. Teacher: Every one space was two of his. You see, they’re almost the same.</td>
<td>The teacher once again refers to the value of each segment (2 units) in David’s graph.</td>
</tr>
</tbody>
</table>
he had labeled the axes of his scale so that tick marks appeared every two grid segments. This use was consistent with how Carlos described David’s scale in line 42 as “he went by one” (see Figure 9 for David’s scale) so that tick marks appeared every one grid segment. In the second case (lines 66 and 74), Carlos was using “I went by fives” to refer to the value between the tick marks on the scale.

There are several ways to interpret Carlos’s utterance, “then he only went by one” (line 69). One is that Carlos was referring to how many grid segments correspond to a tick mark on David’s scale. The other is that Carlos was taking the value between tick marks (2) and dividing by 2 because that was what the teacher had done for Carlos’s graph (dividing 5 by 2 to obtain 2.5).

This analysis of student descriptions of the scales reveals not only different meanings for the phrases of the form “I went by …” but also different ways to view the grid segments, tick marks, and number labels on the scale. Student meanings for these phrases were closely coordinated with different ways of viewing and fo-
cusing joint attention on the scales. The same utterance can have different meanings depending on the view of the inscription. The phrase “I went by twos” can refer to the action used to construct the scale by making tick marks every two segments and mean “I made tick marks every two segments” (see Figure 1). This description is coordinated with a focus of attention on how many segments are marked by the tick mark that has a number label but need not refer to the quantity represented by each segment marked or to any units.

The same utterance, “I went by twos,” could also be used to describe the number of units represented by the chunk created between two tick marks, as in “I made tick marks at every segment and each segment represents two units” (see Figure 2). This meaning is coordinated with a focus of attention not only on the marks and the number of segments marked but also on the number of units represented in any marked segment. Using this focus of attention for the phrase “I went by,” one would describe the scale in Figure 2 by saying “I went by twos” but describe the scale in Figure 1 by saying “I went by half (or point five).” Although in both of
these cases the phrase “I went by” refers to an actor who constructed the scale, in the second case the phrase connects an account of how the graph was constructed to the units represented by the marks.

The Teacher’s Role in the Discussion

The analysis of student interpretations presented above shows that students used phrases of the form “I went by” that were ambiguous, had multiple and shifting meanings, and were coordinated with different views of the scales. The teacher did not respond to these multiple interpretations and ambiguous meanings as obstacles. Instead, she built on student contributions in several ways. First, she used students’ work as the basis for the discussion rather than producing another graph. Because both graphs were correct, the teacher accepted the two graphs as solutions to the problem. Second, she used phrases of the same form the students had used, say-
ing “you went by” herself to refer to each scale. She did not provide a formal definition of the phrase “I went by,” ask students to define this phrase, or use a more formal phrase. Third, the teacher accepted some aspects of Carlos’s descriptions, saying, for example, “the numbers you have are by five” (line 75), and contested others, saying “No, actually you didn’t go by fives” (line 67). Lastly, she connected the multiple meanings, using all three meanings (see Table 3) for the phrase “I went by.”

Although the teacher built on student contributions, she also provided new conceptual content and connected student interpretations to a canonical practice. She did this by (a) using canonical mathematical practices related to graphs; (b) distinguishing among labels, quantities, and measures; (c) moving from actions to nominalization; and (d) connecting the meaning of the phrase “I went by” to a unitized view of the scales.

The teacher’s contributions to this discussion involved several canonical mathematical practices described in the literature on understanding graphs (Kaput, 1998; Leinhardt et al., 1990; Moschkovich, 2004; Moschkovich et al., 1993; Nemirovsky & Monk, 2000). She set the goal of looking at the scales rather than looking at the shape of the curves (lines 37–40). We see that the students started out treating the two curves as objects when they compared the shapes of their graphs (lines 27–36). Although this is an important canonical mathematical practice and a central aspect of competence in this domain, to really understand why two graphs look different it is necessary to examine the scales. A curve will be stretched or shrunk depending on the scale on each axis. The slope of a curve can appear different depending on the scale on each axis of a graph. Because slope is an intensive quantity that compares the ratio of vertical change to horizontal change, a difference in either the horizontal or the vertical scale can change how the slope looks. To compare the slopes of two curves it is necessary to consider both the horizontal and the vertical scales.

Initially, the students focused on the curves as objects, comparing the curves in terms of their overall shape, saying that a curve went “upwards” (lines 27 and 30) or “up” (line 35) in the case of David’s graph, or that a curve was “crooked” (line 32) or “towards the side” (line 35) in the case of Carlos’s graph. The teacher proposed and set a new goal, describing the axes rather than describing each curve. This new goal involved implicit conceptual knowledge that the two scales would have an effect on how the slope of a curve appeared on the graph. The teacher thus helped the students shift from a view of the scales as objects to a focus on the scales.

As the teacher compared the values of the grid segments on the two scales, she shifted from talking about one quantity to comparing two quantities and made distinctions among labels, quantities, and measures of units. In Excerpts 2 and 3, the teacher shifted from referring to one quantity, to relating two quantities on one scale, and then to comparing quantities on the two scales. She first used the phrase “you went by” (lines 62 and 67) to refer to one quantity on one scale; then shifted
to relating the number of segments to the value of each segment on one scale, saying “you did every two spaces was five” (line 67); and then shifted to comparing the value of the spaces on the two scales, saying “every one space was two of his” (line 71). The teacher also explicitly distinguished between the labels that go “by five(s)” (increase by five units) and the unit value of the grid segments, saying “you actually went by two and a halves” (line 67). In the second case, the phrase “you went by” refers to the unit value of one grid segment and is an instance of unitizing. She also marked these differences in the concluding explanation (see line 75 in Excerpt 4).

The teacher’s descriptions in Excerpts 2 and 3 built on the students’ descriptions while also moving toward more canonical forms. The teacher initially (lines 62 and 67) started out describing the scales using the same phrase “you went by” that the students had introduced and were using. She then shifted to referring to the action of marking the scales as “you did every two spaces was five” (end of line 67). She then shifted to comparing how many spaces were between number labels on each scale, saying “every one space [on Carlos’s scale] was two of his [on David’s scale]” (line 71). The teacher thus started with a description parallel in form to those of the students, describing an action taken to construct the scales. By looking at her concluding explanation (Excerpt 4), we see that she then moved away from describing a personal action taken to construct the scales toward nominalized forms describing quantitative relationships and comparing two quantities. However, even while moving to the more canonical nominalizations, she did not completely abandon the students’ perspectives. The analysis below shows how in Excerpt 4 she wove together descriptions of the scales using nominalizations, quantities, and units with descriptions from an actor’s point of view, as if one were on the graph, saying “If you look at one line here, what number is he at?” (line 75) and “What number would you be at if you had a number here?” (line 78).

In Excerpt 4 the teacher highlighted a view of the scale anchored in unitizing and provided the students with an opportunity to use this view. During her explanation in Excerpt 4, the teacher introduced and maintained a meaning for the phrase “I went by” that was coordinated with a view of the marks on the scales as representing units. This meaning and view of the scales anchored the teacher’s descriptions in the canonical practice of unitizing. In response to Carlos’s claim that he “went by fives” (Excerpt 3, line 74), the teacher one again distinguished between the number labels on the scale and the unit value of each segment, saying “the numbers you have are by five” (line 75).

Excerpt 4

75. Teacher: OK, your numbers, right, the numbers you have are by five ((0.2 s)) OK. ((0.1 s)) If you look at one line here, what number is he at?
76. Teacher’s gesture: ((Takes David’s paper and places it next to Carlos’s paper then points to David’s graph.))
77. Carlos: Two.
78. Teacher: What number would you be at if you had a number here?
79. Teacher’s gesture: ((Points to Carlos’s graph.))
80. Carlos: Three.
81. Teacher: Almost, two and a half.
82. Carlos: Yeah.
83. Teacher: Because that’d be half way to five. OK. ((0.1 s)) At this point, after 1, 2, 3, he’s got 6. For you after three, 1, 2, 3, you’d be at 7 and a half.
84. Teacher’s gesture: ((She counts the squares with her pencil.))
85. Carlos: OK.
86. Teacher: See what I mean? So it’s actually two and a half. The numbers you wrote are by fives but since you skipped a line in between, each one is two and a half.
87. Teacher’s gesture: ((Raises her hand and in the air uses thumb and index finger to show interval.))

Through her questions (lines 75 and 78), the teacher asked Carlos to shift from focusing on how the number labels increase to focusing on the value for the $y$ coordinate on the curve that is produced by the same $x$ coordinate. When $x$ is 3, David’s curve is at $y$ coordinate 6 and Carlos’s curve would be at $y$ coordinate 7.5. Carlos’s utterance “three” (line 80) in response to the teacher’s questions is difficult to interpret. If we assume that Carlos knew how to divide 5 by 2 to obtain the correct answer, then his answer that the point on the graph halfway between 0 and 5 is 3 may be evidence that Carlos did not at this point interpret each grid segment as units corresponding to lengths of equal value. Although it is not possible to tell how convinced Carlos really was at this point, he did seem to begin to agree with the teacher, saying “OK” very slowly (line 85). The teacher concluded by claiming

![Carlos's graph](image)

**FIGURE 10** Teacher describes Carlos’s scale as “You went by two and a halves.”
that although the number labels that Carlos wrote were “by fives,” the value of each grid segment was two and a half (line 86).

Through these questions the teacher provided the students with an opportunity to use a *unitized* view of the marks on the scales. She set a new problem, determining the $y$ coordinate of a point on the two graphs given the same $x$ coordinate (3). As she and Carlos jointly estimated the $y$ coordinates for the same $x$ coordinate on the two graphs, she engaged him in talking about and viewing the scales from a canonical point of view.

During this explanation, the teacher’s descriptions alternated between referring to actions and to nominalized quantities. On the one hand, she used verbs to describe actions a student took or would take, saying “if you look” (line 75), “would you be at?” (line 78), “he’s got” (line 83), “you’d be at” (line 83), “you wrote” (line 86), and “you skipped” (line 86). On the other hand, she also described the marks as nominalized quantities, saying “that’d be half way to five” (line 83), “it’s actually two and a half” (line 86), and “each one is two and a half” (line 86). The teacher’s explanation thus connected students’ own descriptions of an action (“went by”) to a nominalization (“every one space was two of his”) comparing quantitative relationships and using units. This weaving of multiple views is one of the central characteristics of instructional conversations (Goldenberg, 1991; Tharp & Gallimore, 1988, 1991).

By the end of the explanation in Excerpt 4, the teacher had invited the students to jointly focus their attention on and talk about the *units* represented by the marks on the two scales. The question remains whether the participants moved closer to a shared meaning for “I went by.” Even if Carlos and the teacher did not reach total agreement on this day, this first discussion may have provided a foundation for Carlos to begin to view and describe graphs in new ways.

**Epilogue: Carlos Explains a Graph**

Three months later, this time in a whole-class setting, Carlos led an explanation of the scale on a graph and the impact of the scale on the shape of the curve. Students had recorded data on a graph as they dropped a ball from different heights, and small groups presented their graphs in front of the class. The teacher and the students were looking at several graphs that students had constructed showing the relationship between drop height and bounce height. Carlos provided an explanation of a graph posted on the blackboard. This coconstructed explanation parallels the earlier small-group discussion and showed Carlos using a unitized view of the labels on the scale.

Carlos began by once again using “went by” to refer to the labels on the scale. He described two different ways to label the scale on the $y$ axis. One way to label the $y$ axis would be to use the actual drop heights (see Figure 11). These were the numbers the students had used to label the $y$ axis of their graph (1, 3, 5, 8, and 9).
Carlos described how using that set of numbers, “they just put the numbers” (line 19) and “only the answers that they got” (line 29), would (always) produce a straight line or, in his words, “would only show that it went up in a straight line” (line 21; see Figure 11). The teacher asked students to focus on the y axis (line 22), used a marker to make that scale darker (line 23), and made X marks where data points would fall if one used the revised scale (see Figure 12).

Much in the same way that the teacher had asked Carlos to look at the numbers on the scales in the small-group discussion in Excerpts 1 through 4, Carlos asked the other students to look at the numbers on the scales. He asked students to focus on the numbers on the y axis (1, 3, 5, 8, 9, and 12), which did not mark equal intervals. He compared the labels on the y axis to a set of unitized labels, saying “in-
stead of putting like 2, 4, 6, 8 to 20, they put only the answers that they got” (line 29). Carlos then described how using a set of units to mark equal intervals (2, 4, 6, 8, and so on up to 20) to label the y axis would produce a different curve, one that would not be a straight line but “would have been like up and down” (line 34; see Figure 12). Carlos repeated his explanation, stating once again that using a set of numbers that are not units produces a straight line and that using labels as units produces a curve that would not be a straight line (line 37).

In Transcript 2 (see Appendix B), Carlos proposed an explanation and communicated this explanation with the teacher’s support. Carlos’s explanation parallels the teacher’s contributions during the small-group discussion in Excerpts 1 through 4. In this explanation Carlos focused on the units on a scale, explicitly
compared two views of the graph, and used a unitized view of the scales. The claim is not that there was a cause-and-effect relationship between the teacher’s and Carlos’s explanations, but rather that this second transcript provides some evidence that Carlos was talking about and viewing the scale on the graph in ways that parallels the teacher’s earlier explanations.

CONCLUSIONS

The analysis provided here describes how two students and a teacher used multiple meanings and views to describe the scales on two graphs. The small-group discussion involved ambiguous meanings for utterances and changing views of the inscriptions, as the utterances referred to imagined segments and units. This mathematical discussion exemplifies how such discussions involve situated meanings for utterances, rather than static meanings for words, and how mathematical Discourse practices involve not only utterances but also views of inscriptions.

We can expect learners to generate multiple interpretations of inscriptions, use multiple meanings for words, and focus on different aspects of inscriptions. This classroom discussion provides an example of how teachers, instead of seeing students’ interpretations as obstacles, can build on student interpretations and connect them to canonical mathematical practices and concepts. Multiple, ambiguous, and shifting meanings did not prove detrimental to this discussion. Instead, the teacher used students’ multiple interpretations of the marks on the scales as resources. This teacher wove these meanings and views together and connected them to a canonical mathematical practice. Students will use their own meanings and views when talking about mathematical situations. This study shows how a teacher, rather than seeing student interpretations as instances of nonmathematical ways of talking and viewing, can build on student interpretations and connect them to canonical mathematical Discourse practices and important conceptual content.

The teacher introduced new meanings for utterances and views of the scales that were anchored in unitizing, a canonical mathematical practice for using scales. She focused attention on the units on a scale, and her descriptions highlighted a unitized view of the scales. She thus introduced the practice of unitizing: viewing, talking about, and acting as if the marks on the scale represent quantities that can be unitized in multiple ways. Providing a unitized view of the scales can seem deceptively procedural and simple, when in fact it is a deeply conceptual and complex contribution to a discussion of slope.

Although the analysis shows how the teacher provided a unitized view of the scales, her contribution to the discussion of the graphs went far beyond unitizing. The teacher’s unitized view of the scales was a deeply conceptual mathematical move connecting the roots of the rate concept to the roots of the slope concept. This
view moved the discussion away from the idea that “steeper is faster” to a discussion of comparing slopes only if graphs have comparable axes, because the slope is just a compact way to coordinate the changes occurring on each axis. The teacher engaged the students in moving away from a way of viewing the graph that has been described as a major misconception ("if it looks steeper, it must be faster") to a more sophisticated understanding connecting the slope of a line segment with each of the axes. Steepness (or speed) is not simply what appears on the graph; it is a coordination of changes on the $x$ axis with changes on the $y$ axis. In order to interpret and compare slopes, this teacher led the students to pay attention to how the graph represented those changes in each dimension.

Teachers need to be prepared to respond to students’ multiple ways of interpreting, talking about, and viewing inscriptions. Although pedagogical principles can guide teachers in implementing this type of instruction, it is also important to provide detailed examples of such discussions. Examining such examples can be useful for developing teaching practices and illustrating pedagogical principles. We need many more examples of the two features of teaching that supports both procedural and conceptual discussions. One feature is that teachers and students pay explicit attention to concepts; the other is that students themselves wrestle with important mathematics (Hiebert & Grouws, 2007). Teachers face a considerable challenge balancing these features in their teaching, as it is easy to emphasize one feature over the other or attend to only one. The first feature, that teacher and students pay explicit attention to concepts (Hiebert & Grouws, 2007), can be understood and enacted in multiple ways. It can be interpreted to mean that teachers should explicitly define slope as “the ratio of $y$ over $x$” and explicitly tell students “you cannot compare the slopes of two lines unless you have looked at the scales on the two axes.” Although these may seem like important ways of paying explicit attention to concepts, these strategies do not require that students themselves wrestle with important mathematics and thus do not address the second feature. This classroom discussion shows that it is possible for teachers and students to pay explicit attention to concepts in ways other than providing definitions or stating general principles. The analysis presented here provides one example of a way to attend to concepts through a discussion grounded in inscriptions and questions generated by students as they themselves wrestle with mathematical concepts.

In order to develop communication skills, students need to participate in negotiating meaning (Savignon, 1991) and in tasks that require output (Swain, 2001). Based on these findings, mathematics instruction should provide opportunities for students to actively use mathematical language to communicate about and negotiate meaning for mathematical situations. Instructional conversations have been proposed as especially useful for supporting the development of students’ oral communication: “The critical form of assistance is dialogue, the questioning and sharing of ideas that happen in conversation” (Tharp & Gallimore, 1991, p. 3). This classroom mathematical discussion is an example of an instructional conver-
sation in which a teacher built on student contributions and wove multiple meanings and views into an explanation.

Mathematical topics, such as graphing, that involve students generating, understanding, and discussing inscriptions they have generated themselves, rather than inscriptions presented in texts or generated by the teacher, may be particularly fruitful situations for instructional conversations because they involve multiple perspectives, yet claims can be grounded on common artifacts. Graphing involves developing particular ways of viewing inscriptions that are connected to conceptual and canonical practices. Instructional conversations can be especially useful in introducing such canonical and conceptual practices. Further analysis of instructional conversations about mathematical topics will need to address issues such as what the differences are between instructional conversations occurring in whole-class and small-group settings, what knowledge of mathematical concepts teachers need to lead instructional conversations, and what knowledge of student thinking in particular topics teachers need to prepare for productive instructional conversations.

ACKNOWLEDGMENTS

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REFERENCES


APPENDIX A

TRANSCRIPT CONVENTIONS

Single parentheses ( ) Talk for which transcriber doubt exists.

Double parentheses (( )) Transcript annotations

Brackets [ ] Marks the beginning or end of temporal overlap among utterances produced by two or more speakers.

Equal sign = Indicates the end or beginning of two ‘latched’ utterances that continue without an intervening pause. Where necessary, can be combined with brackets.

Timed pause (1.8) Measured in seconds, this symbol represents intervals of silence occurring within and between speakers’ utterances.

Period . Indicates a falling pitch or intonation at the conclusion of an utterance that may or may not mark the completion of a grammatically constructed unit.

Question mark ? Rising vocal pitch or intonation at the conclusion of an utterance that may or may not have the grammatical structure of a question.

Exclamation point ! Marks the conclusion of an utterance delivered with emphatic and animated tone. The utterance itself may or may not be an exclamation.

Comma , Indicates a continuing intonation with slight upward or downward contour that may or may not occur at the end of a grammatical phrase.

Hyphen - An abrupt halt between syllables or words.

Colon(s) : One or more colons indicate sustained enunciation of a vowel, consonant, or syllable.

Degree signs ° ° Marks speech produced more softly or at a lower volume than surrounding talk.
Transcript 1

22. David: Here’s my graph. Did it come out like yours?
23. David’s gesture: ((Turning his paper toward Carlos.))
24. Carlos: I don’t know.
25. Carlos’s gesture: ((He looks at both papers and compares graphs.)) ((0.2 s))
27. Carlos: = It’s because you did it upwards.
28. Carlos’s gesture: ((Sweeping his right hand, with his finger pointing up.))
29. David: Oh, you did. ((0.2 s)) ((Teacher approaches the group.))
30. Carlos: Were we supposed to do the graph upwards? Or to-
31. Carlos’s gesture: ((Sweeping his hand upward again and then moving his hand hor-izontally in the air.))
32. David: Or, or or crooked like this? Whatever.
33. Carlos: Or horizontally? ((0.2 s))
34. Teacher: Doesn’t matter. ((shaking her head))
35. Carlos: ‘Cause like, like when we look at our graphs his is going up and mine is going towards the side.
36. Carlos’s gesture: ((Moving his right hand upward and then moving his right hand horizontally in the air.)) ((0.2 s))
37. Teacher: Did you have the same, did you put both of the same things on the x axis and on the y axis?
38. Teacher’s gesture: ((Looking through students’ graphs.))
39. Carlos: No, ((0.2 s)) yes, actually.
40. Teacher: You both put time here (referring to the x axes <AQ33></AQ>). OK, so what’s different about these? I don’t think it’s, I don’t think it’s the positioning of them. Look at your numbers, the way you placed your numbers.
41. Teacher’s gesture: ((Points at the x axis on both papers, back and forth. Then turns David’s paper in the same direction as Carlos’s paper.))
42. Carlos: Oh, that’s true. ‘cause I went by twos, I went 1, 2 ((0.2 s)) and then I put that one (—-) he went by one.
43. Carlos’s gesture: ((Begins counting with his right finger following the numbers on his paper.))
44. Teacher: Aha, You skipped one. So how does that change how it looks?
45. Carlos: ‘Cause it doesn’t go up as far, it only goes, it’s more steeper. It looks more steeper.
46. Carlos’s gesture: ((Moving his right hand outward. Then moving his right hand straight up.))
47. Teacher: Remem-. Similar to the difference between this one and ((0.1 s)) and this one here. Right?
48. Teacher’s gesture: ((Makes a sign with right thumb and index finger of her hand to show interval differences on their papers. Then she points to a graph on the black-
board. Next she points to a second graph on the blackboard. The first and second graphs have different scales on the $x$ axis so that the second graph is compressed along the $x$ direction.)

49. David: That one.
50. Carlos: Yeah.
51. Teacher: Here the numbers are closer together so it looks steeper. Other than that, are they the same graph?
52. Teacher’s gesture: ((Makes a sign with thumb and index finger. Then she gestures upward with her right hand.)) ((0.1 s))
53. Carlos: No, also here in the $x$ axis.
54. Carlos’s gesture: ((Carlos point to the $x$ axis on his paper.))
55. David: (the distance) ((Points to the axis on his paper.))
56. David: I went by twos. =
57. Carlos: = This is the $x$ axis. Right?
58. ((Carlos points to the axis.))
59. Teacher: This is the $y$ axis, ((0.1 s)) this is the $x$ axis.
60. Teacher’s gesture: ((Sweeps her pencil vertically to represent the $y$ axis and then horizontally to represent the $x$ axis.))
61. David: I went by twos.
62. Teacher: You went by twos and you went b:y- ((0.2 s))
63. Carlos: I went by twos. You (didn’t) you went by ones! °What are you talking about.°
64. Teacher: No, here on the $y$ axis.
65. Teacher’s gesture: ((Points to the axis.))
66. Carlos: Oh, I went by fives.
67. Teacher: You went by fives. ((0.2 s)) No, actually you didn’t go by fives. You actually went by two and a halves because you’d, you did every two spaces was five.
68. Teacher’s gesture: ((She points to Carlos’s paper while she explains.)) ((0.3 s))
69. Carlos: Then he only went by one.
70. Carlos’s gesture: ((Carlos points to David’s paper.)) ((0.2 s))
71. Teacher: Every one space was two of his. You see, they’re almost the same. If you look at the next two- ((Puts down her notebook and points to the graphs.))
72. Carlos: Wait! But I don’t get what you’re saying.
73. Teacher: OK.
74. Carlos: ‘Cause I went by fives. ((David stands up.))
75. Teacher: OK, your numbers, right, the numbers you have are by five ((0.2 s)). OK. ((0.1 s)) If you look at one line here, what number is he at?
76. Teacher’s gesture: ((Takes David’s paper and places it next to Carlos’s paper then points to David’s graph.))
77. Carlos: Two.
78. Teacher: What number would you be at if you had a number here?
79. Teacher’s gesture: ((Points to Carlos’s graph.))
80. Carlos: Three.
81. Teacher: Almost, two and a half.
82. Carlos: Yeah.
83. Teacher: Because that’d be half way to five. OK. ((0.1 s)) At this point, after 1, 2, 3, he’s got 6. For you after three, 1, 2, 3, you’d be at 7 and a half.
84. Teacher’s gesture: ((She counts the squares with her pencil.))
85. Carlos: O.K.
86. Teacher: See what I mean? So it’s actually two and a half. The numbers you wrote are by fives but since you skipped a line in between, each one is two and a half.
87. Teacher’s gesture: ((Raises her hand and in the air uses thumb and index finger to show interval.))

APPENDIX B

Transcript 2 (3 months later)

1. Maria: Okay. We dropped it ((referring to the ball)) every four inches.
2. Francine: And most of the – well, first, all of the bounce backs were uneven numbers. I was just going to say that.
3. Maria: I’m at the end right here.
4. Francine: Yeah, except for the –
5. Maria: Except for the 8, and then 12, 14, 16, 18.
6. Francine: At the end, it began to go even. It went 12, 14, 16, 18, 20.
7. Teacher: Okay. Did you see any patterns to the increase?
8. Francine: That’s the pattern.
9. Teacher: What was the pattern? Can you describe the pattern?
10. Francine: Well, 12, 14, 16, 18, 20, it went by twos.
11. Teacher: It increased by twos?
13. Teacher: And the drop height increased by fours?
15. Maria: Yeah.
16. Teacher: Okay. How is it represented –
17. Carlos: They didn’t – they didn’t go like two, four, six. They went only the numbers.
18. Teacher: Say that again?
19. Carlos: On the y axis, they just put the numbers – they just put the numbers that are up there.
20. Stephanie: That are up here.
21. Carlos: Weren’t they supposed to like, kind of like – put like two, four, six, eight? Because then that would only show that it went up in a straight line if they did that.
22. Teacher: Uh huh. Okay. Do you hear what he’s talking about? He’s talking about the y axis.
23. Teacher: Okay. You know what I’m going to do? I’m going to make this, excuse me, just a little darker so we can see this. Can you just stand like to the side there?
24. Steve: It didn’t go by twos.
25. Maria: I’m going by fours.
26. Teacher: Carlos, would you come up here and point out what you are describing?
27. Stephanie: You probably need those numbers.
28. Teacher: Okay.
29. Carlos: What they did is they, instead of putting like 2, 4, 6, 8, to 20, they put only the answers that they got. So that just showed that it went up in an even straight line.
30. Stephanie: Oh, I get it. They didn’t even it out.
31. Student: Yeah.
32. Student: Yeah, but it’s going in a pattern. So what does it matter what numbers you use?
33. Stephanie: Yeah, but you even it out though, don’t you?
34. Carlos: Because, because if she evened it out, then it would have been like up and down like that. But she —
35. Student: It would have still been a straight line.
36. Student: No, it would have showed the way it really was.
37. Carlos: Because look at the numbers. It’s three, five, and then that would — this only shows that it went up in a steady, straight line. But if they put it like two, four, six, it wouldn’t have gone up in a straight line like that.