Moving Up and Getting Steeper: Negotiating Shared Descriptions of Linear Graphs

Judit N. Moschkovich
Institute for Research on Learning
Menlo Park, California

This study examines mathematics learning in the context of peer discussions by focusing on students' descriptions of lines graphed on a computer screen. The article describes how these discussions provided a rich context for negotiating shared descriptions, supported conceptual change, and resulted in convergent meanings through reciprocal elaborations and clarifications. The participants in the study used graphing software to explore the connections between linear equations and their graphs with a peer. The article presents the analysis of three case studies examining how students negotiated shared descriptions of lines. These conversations supported students' construction of shared descriptions, but not necessarily by presenting conflicting ideas or through one student guiding another. Rather, negotiation functioned through local conversational resources such as the use of reference objects, spatial metaphors, and coordinated gestures and talk. These case studies also point to an important role for instruction in orchestrating and supporting peer discussions by modeling how to resolve negotiations and maintaining students' focus on mathematically productive learning trajectories.

Curriculum guidelines and researchers in mathematics education have endorsed peer discussions as a context for improving conceptual learning in mathematics (Brown & Pallincsar, 1989; Mathematics framework for California, 1992; National Council of Teachers of Mathematics [NCTM], 1989; Resnick, 1989). Working with peers is supposed to provide an environment in which students can "explore, formulate and test conjectures, prove generalizations, and discuss and apply the results of their investigations" (NCTM, 1989, p. 128). Although there are many

Requests for reprints should be sent to Judit N. Moschkovich, Institute for Research on Learning, 66 Willow Place, Menlo Park, CA 94025.
possible ways in which a conversation with a peer might support learning, there are few detailed descriptions of such conversations focusing on conceptual learning. This article presents evidence that conversations between peers can support the construction of shared descriptions of mathematical objects, describes how students refined their descriptions of linear graphs in such conversations, and examines the resources students used to construct shared meanings for their descriptions.

The article presents the analysis of three case studies showing how students negotiated the meaning of descriptions of lines graphed on a computer screen and explores the following questions: How did students describe and compare lines? How did students negotiate and construct shared descriptions of lines? How did students refine their descriptions? What resources did students use to elaborate and clarify their descriptions?

During the discussions, students grappled with problematic or ambiguous descriptions, contesting each others’ understandings and engaging in repeated dialogues about these descriptions. As the discussions progressed, the students elaborated their descriptions, clarified meanings, and constructed shared descriptions. In clarifying their descriptions, they used conversational resources such as everyday spatial metaphors, coordinated gestures and talk, and reference objects. In the process of contesting, elaborating, and clarifying their descriptions, students refined the meaning for many terms and developed more precise descriptions of lines.

These three case studies show that peer discussions can create the need for clarification and provide a rich context for negotiating shared meanings. The negotiation and construction of shared descriptions were important aspects of how these students made sense of lines and their equations. The negotiation of shared meanings that students engaged in, the fact that most of the students arrived at shared descriptions, the ways that students refined their descriptions, and the fact that many students’ descriptions came to reflect more conceptual knowledge, show that the negotiation of descriptions is an important aspect of learning through peer discussions.

These three cases also show that conversations between students can support conceptual change and that this progress does not necessarily happen through conflict with a peer’s perspective or guidance by a more advanced other. Instead, students constructed shared meanings using constraints and resources local to the conversation such as spatial metaphors, reference objects, and coordinated gestures and talk. Although students can and do reach agreement and make conceptual progress during a conversation with a peer, neither resolution nor conceptual convergence are guaranteed. Although some conversations with a peer are shown to support progress, the last case study shows that there is also an important role for instruction or guidance. Teachers can support peer discussions by modeling ways to resolve negotiations, coaching students on how to reach agreement, and helping students to focus on mathematically productive paths.
THEORETICAL FRAMEWORK

The study begins with the assumption that knowledge is socially constructed and that this construction is mediated by language (Vygotsky, 1978, 1987). I also assume that competence in a complex domain, such as linear functions, involves more than textbook formulas, procedures, or the use of technical terms. Following Solomon (1989), I view competence as knowing how to act in specific situations involving lines and their equations, including knowing how to use language. The framework for the study draws on three areas in current theory and research: learning through collaboration, conceptual understanding of linear functions, and the relation between language and learning mathematics.

Working collaboratively with peers is one possible context for supporting learning; at the very least it does not hinder learning, and it improves attitudes about subject matter and peers (Brown & Pallincsar, 1989; Davidson, 1985; Doise, 1985; Sharan, 1980; Webb, 1985). Researchers have begun to consider specifically how conversations between peers might support conceptual learning in mathematics (Forman, 1992; Forman & McPhail, 1993). There are two main paradigms for peer collaboration. A neo-Piagetian perspective emphasizing sociocognitive conflict and a neo-Vygotskian perspective emphasizing guidance by a more advanced peer. This study explores a third alternative that, rather than emphasizing conflict or difference, focuses on negotiation and shared construction through conversational, and thus inherently social, resources (Roschelle, 1992).

Although it is beyond the scope of this article to discuss different perspectives on the relation between language and concepts (Lucy & Wertsch, 1987; Vygotsky, 1987), clearly the two are related in intricate and complex ways. I assume that the relation between language use and conceptions is a complex and dialectical one, rather than unidirectional or deterministic, without addressing the details of this relation. I also assume that learning to participate in mathematical discourse is part of learning mathematics (Durkin & Shire, 1991; Pimm, 1987). Mathematics discourse includes the mathematics register, argumentation rules and styles, values, and beliefs (Richards, 1991). Learning to participate in mathematical discourse is not merely or primarily a matter of learning vocabulary definitions. Instead, it involves learning how to use language while solving and discussing problems in different contexts.

Several studies exploring the relation between language and learning mathematics (Cocking & Mestre, 1988; Durkin & Shire, 1991; O'Connor, in press; Pimm, 1987; Richards, 1991) have focused on one aspect of mathematical discourse, the mathematics register (Halliday, 1978). Halliday defined register in the following way:

A register is a set of meanings that is appropriate to a particular function of language, together with the words and structures that express these meanings. We can refer to
the "mathematics register," in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p. 195)

In light of this work, the analysis also considers possible differences between the everyday and mathematical registers for this domain and shows how students' language use moved closer to the mathematics register by becoming more precise and reflecting more conceptual knowledge.

The Domain of Linear Equations and Graphs

Linear functions is a complex domain where the development of connected pieces of conceptual knowledge is essential for competence. In such a complex domain, social interaction and language can play a crucial role in the development of conceptual understanding. Conceptual understanding in this domain involves more than using procedures to manipulate equations or graph lines; it involves understanding the connections between the two representations (algebraic and graphical), knowing which objects are relevant in each representation, and knowing which objects are dependent and independent.

The complexity of descriptions of linear equations and graphs is a reflection of the conceptual complexity of this domain. Competence in the domain of linear functions involves not only using precise descriptions but also understanding the conceptual entailments associated with these descriptions. There are two initial student conceptions documented in this domain (Moschkovich, 1992) that are relevant to student descriptions of lines: (a) The x-intercept is relevant for equations of the form $y = mx + b$ (i.e., it should appear in the equation, either in the place of $m$ or in the place of $b$), and (b) $m$ and $b$, or rotation and translation, are not independent (i.e., if you change $m$ in the equation, the $y$-intercept might change in the graph; if you change $b$ in the equation, the slope might change in the graph; if

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1 Students have also been reported as seeing the effect of changing the $b$ in an equation as making lines move from left to right (or right to left), rather than up and down (Goldenberg, 1988; Moschkovich, 1992). Changing the $b$ in an equation does, in effect, move lines along the $x$-axis (as well as along the $y$-axis), so that these descriptions are not necessarily wrong. What is important about these descriptions is that experts usually choose to focus on movement along the $y$-axis as a result of changing $b$. This choice reflects the fact that lines move exactly $b$ units up or down the $y$-axis when, for example, an equation is changed from $y = mx$ to $y = mx + b$. Although it is possible to relate the parameter $b$ in an equation to the movement of a line along the $x$-axis, this is a more complicated correlation. Lines move either $-b$ units along the $x$-axis, in the case of $m = 1$, or $(-b)/m$ units, in the case of $m \neq 1$. Focusing on movement along the $y$-axis is the simplest possible correlation between the two representations for equations of this form and is thus not an arbitrary choice.
a line is translated up or down on the y-axis, this is a result of changing \( m \); if a line is rotated about the origin or the y-intercept, this is a result of changing \( b \). As students explore this domain, they build on and refine these initial conceptions.\(^2\)

Students’ initial descriptions reflected these two conceptions. The use of the x-intercept was reflected in initial descriptions of lines as moving “right” and “left,” either instead of or in addition to “up” and “down,” as a result of changing \( b \) in the equation \( y = mx + b \). The conflating of the role of the parameters \( m \) and \( b \) was reflected in initial descriptions in several ways. Initially, students did not associate a change in \( m \) with a change in the steepness of a line (or a change in \( b \) with a translation along the y-axis). Also, they did not separate rotation and translation as independent properties, describing the effect of a change in one parameter, say \( m \), as possibly generating both a rotation and a translation. There were many dialogues involving initial misunderstandings and negotiation of the meaning of the phrases “the line is steeper/less steep,” “the line moved up/down the y-axis,” and “the line moved left/right.” Students refined the meaning of relational terms such as steeper and negotiated these two conceptual aspects of their descriptions, the separation of rotation and translation as independent movements and the focus on horizontal or vertical translation.

These students’ descriptions did not involve technical terms such as slope and intercept. Despite the absence of technical terms, the students still discussed and negotiated the meaning of the less technical descriptions used in the discussion problems. Thus, the students did not simply learn to use the technical terms slope and y-intercept. Instead, they constructed shared descriptions and refined the everyday meanings of terms using conversational resources such as gestures, reference objects, and spatial metaphors for clarification.

**RESEARCH DESIGN**

The students who participated in the discussions were from an exemplary pilot 1st-year algebra course (see Table 1). The students worked with a peer of their choice, using graphing software to explore linear equations and their graphs while being videotaped. This article reports on the analysis for three of these pairs. Protocol analysis of the videotaped discussion sessions was used to explore how students negotiated the meaning of their descriptions and how these descriptions changed.

These students attended an urban school that has about a 90% minority population of working class and lower middle class families. The students in this course

\(^2\)Elsewhere (Moschkovich, 1992), I argue that these are not misconceptions but should be seen as “transitional” conceptions that are reasonable, useful, and part of learning trajectories.
TABLE 1  
Data Sources

<table>
<thead>
<tr>
<th>Data Sources</th>
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<tbody>
<tr>
<td>Classroom observations</td>
</tr>
<tr>
<td>Observed two 6-week chapters, the first covering linear functions (fall) and</td>
</tr>
<tr>
<td>the second covering quadratic functions (spring).</td>
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<tr>
<td>Peer discussion sessions</td>
</tr>
<tr>
<td>Duration: There were no time constraints. Sessions lasted from 2 to 4 hr over</td>
</tr>
<tr>
<td>a period of at least 2 days and at most 4 days. Sessions were conducted in a</td>
</tr>
<tr>
<td>classroom after school.</td>
</tr>
<tr>
<td>Data: Videotapes of all sessions.</td>
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<tr>
<td>Written assessments.</td>
</tr>
</tbody>
</table>

were mostly ninth graders, although a few were tenth graders. They were neither honors nor remedial students, and the classes were heterogeneous in terms of previous math achievement scores. The six students discussed here speak English and one other language in their home (Fred and Harold speak Chinese, Marcela and Giselda speak Spanish, and Monica and Denise speak Tagalog). These students were “mainstreamed” because they were officially considered proficient and fluent in English, and they have experienced either all or most of their mathematics education in English. Thus most, if not all, of their mathematical conversations have been in English.3 These three pairs from three different language backgrounds, different classroom achievement levels,4 and different scores on written assessments for this domain encountered similar difficulties with descriptions of lines and had similar discussions. Therefore, these particular discussions of descriptions do not seem to be linked specifically to speaking any one other language, to achievement in mathematics, or to the students’ scores on the written assessments.

The students were from two classrooms observed earlier in the school year (Moschkovich, 1990) during two chapters on functions, the first on equations and graphs of linear functions and the second covering quadratic functions. The two chapters included modeling of real world situations, use of graphing calculators and computer software, and student group work with some whole-class discussions. The curriculum was designed to include exploration and discovery, focus on mathematics as a process rather than on results or answers, support work in groups, and encourage students to discuss their ideas.

Their classroom work had focused on applications of linear and quadratic functions to a problem from science and developing a qualitative understanding of the connections between equations and graphs. Their classroom experiences did not focus on the use of technical terms such as slope and intercept, or on the

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3 Marcela and Giselda were the only students who used a language other than English during their discussion session. In the two cases when they spoke Spanish to discuss their answers for the problems, the Spanish and an English translation are provided in the transcript excerpts.

4 Determined by their course grades and a qualitative evaluation by their teacher.
definition of slope as the ratio of rise over run. In designing the problems for the peer discussions, I purposefully used the terms *steeper* and *less steep*, rather than slope, to describe the difference between two lines of different slopes, and the phrases *move up on the y-axis* and *move down on the y-axis*, rather than *y-intercept*, to describe the difference between two lines with different *y*-intercepts.

Discussion Problems

In the discussion sessions, the students explored slope and intercept using SuperPlot (Steketee, 1985), a graphing utility that allows students to graph equations, and problems designed by the researcher. The problems addressed the specific conceptions noted in the classroom observations: using the *x*-intercept and conflating the effect of changing *m* and *b*. Because the discussions were meant to resemble classroom discussions between peers as closely as possible, intervention by the researcher was kept to a minimum.

The following example illustrates the basic format used for all the problems. In the first part of each problem (such as Problem 3a shown in Figure 1) students were given the equation *y = x* and its graph and were asked to predict what changing the equation from *y = x* to a target equation (in this case *y = x + 5*) would do to the line. In the second part of each problem they were given the graph of *y = x* and a second line and were asked to predict what change in the equation *y = x* would generate the target graphed line. In some problems, Choice C read: “The line would flip to the other side of the y-axis” to address the effect of a negative slope. All the problems discussed in the transcripts have the same format as the problem in Figure 1.5

The introduction to the discussion sessions included a review of basic coordinate graphing skills, an explanation of how the computer graphs equations (using a table of values generated by the students), a description of how they were being asked to discuss the problems, and an explanation of key words and phrases used in the problems (*steep, steeper, less steep, origin, move up or down on the y-axis*, etc.) using examples.

To structure dialogue and discussion of different conjectures and predictions, the students followed an instructional sequence similar to the Itakura method for classroom discussions in science (Hatano, 1988; Inagaki, 1981; Inagaki & Hatano, 1977):

1. Students were presented with a question and several alternative predictions (Questions A, B, and C).
2. Each student was directed to choose and record a prediction on the paper.

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5 Figure 1 is included with the first transcript. For subsequent transcripts, I only include the target equation.
3a. If you start with the equation $y = x$ then change it to the equation $y = x + 5$, what would that do to the graph?

A. Make the line steeper

Why or why not? YES NO

Why or why not? YES NO

B. Move the line up on the y axis

Why or why not? YES NO

Why or why not? YES NO

C. Make the line both steeper and move up on the y axis.

Why or why not? YES NO

Why or why not? YES NO

FIGURE 1 Problem 3a.
3. The pair was asked to explain and discuss their choices before graphing.
4. The pair was directed to test their predictions using the computer to graph an equation.
5. The students were asked to then choose an agreed on answer and explanation once again after graphing.

Students were told that they did not have to agree on their choices before graphing and that their individual choices would be recorded on the videotape, but that they had to agree on their choices after graphing. Students did follow these instructions and thus the conversations that ensued are labeled *discussions*, in keeping with Pirie’s (1991) definition of mathematical discussion as “purposeful talk on a mathematical subject in which there are genuine pupil contributions and interactions” (p. 143).

**Analysis**

The protocol analysis of the videotape data was conducted by case studies for pairs of students. During the classroom observations, I had noted that students had difficulties describing what they saw on the computer screen (Moschkovich, 1990). One analysis of the videotapes addressed this issue by coding students’ descriptions of lines. For the three case studies I noted each instance where a student described a line, its movement, or compared two lines. This analysis led to a second coding where I analyzed the instances where students negotiated their descriptions. These instances involved several relational and directional terms—either the ones used in the problems or others generated by students. The last coding of the data traced the changes in students’ descriptions by comparing these descriptions during two problems at the beginning of the sessions, during several problems in the middle, and during the last two problems in the discussion sessions.

The transcript materials are divided into excerpts for each problem. Each student turn is numbered consecutively within a problem excerpt. Students’ gestures, the referent of a pronoun, or clarifications are included within brackets. For the pair of students who at times spoke Spanish (Marcela and Giselda), the English translation is provided in brackets following the Spanish. Any participation by the researcher is labeled *Int.* for *interviewer.*

There are several factors affecting the refinement of students’ descriptions, not all of which can be attributed solely to the peer discussion sessions. The first is that for two semesters these students had participated in classroom work where verbal and written communication about mathematics problems was encouraged and supported. Thus, they had learned to participate in discussions where they described their solutions in detail, attempted to understand other students’ explanations, and tried to reach agreement. Also, during the discussions in the classroom, the teachers
focused on coordinating the algebraic and graphical representations in their descriptions, referred only to vertical translation, and treated rotation and translation as independent transformations on a line.

Nevertheless, because these students’ descriptions did change during the discussion sessions, there are also aspects of these discussions that impacted their language use. The introduction by the researcher, the presence of lines graphed on the screen, the descriptions used in the problems, and the fact that several problems focused on the two conceptions described earlier all provided some constraints on student descriptions. However, as is seen in the case studies, the constraints seem to lie more in the nature of the conversations than in the presence of an authoritative text.

CASE STUDY 1: USING COORDINATED GESTURES AND TALK TO NEGOTIATE A MEANING FOR STEEPER

The term steeper was the focus of many of the discussions and an important aspect of the construction of shared descriptions. Students in all three case studies discussed the term steeper and negotiated the meaning of this term. Some students used the term to refer to the translation as well as the rotation of a line. In this case study, I show how one pair of students struggled with the meaning of steeper. These two students initially showed some confusion about the meaning of steeper and eventually focused on the task of explicitly clarifying these meanings to each other. By the end of their discussion session they used a shared meaning for steeper as well as referred to rotation and translation as independent properties of a line.

Fred and Harold were working on the problem shown in Figure 1 provided for them on paper. They had been instructed to first predict whether changing the equation $y = x$ to the equation $y = x + 5$ would make the new line steeper or not, to graph the equations on the computer, and to decide whether their prediction was right after graphing. While making their predictions, they had left the answer to Question A blank and answered Question B as yes. After they discussed their interpretations of Question C, Fred chose no for the answer to C, whereas Harold thought the right choice was yes (lines 14–25, transcript not included here). Next they graphed the equation $y = x + 5$ and returned to answer Questions A through C after graphing. At this point, they were looking at the line for the equation $y = x + 5$ on the screen (see Figure 2). Excerpt 1 shows how the term steeper was a problematic aspect of their descriptions and highlights how Harold and Fred used coordinated gestures and talk as resources for constructing and negotiating their descriptions.
FIGURE 2 Fred—line 3: “I think it’s steeper right here.”
Excerpt 1: Fred and Harold (Problem 3a)

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>26 Harold: [Reading] After graphing,</td>
<td>Harold asks whether the line is steeper on the screen than the line y = x.</td>
</tr>
<tr>
<td>is it steeper?</td>
<td>Fred questions whether the line is steeper and proposes that it is not.</td>
</tr>
<tr>
<td>27 Fred: Isn’t it steeper? No.</td>
<td></td>
</tr>
<tr>
<td>28 Harold: It’s not steeper, is it?</td>
<td>Harold questions whether the line is or is not steeper.</td>
</tr>
<tr>
<td>[moves hand to the screen]</td>
<td></td>
</tr>
<tr>
<td>29 Fred: Are we talking about the</td>
<td>Fred asks for a clarification.</td>
</tr>
<tr>
<td>same thing?</td>
<td></td>
</tr>
<tr>
<td>30 Harold: Yeah ...</td>
<td></td>
</tr>
<tr>
<td>31 Fred: I think it’s steeper right</td>
<td>Fred elaborates the meaning of steeper, proposing that the line is steeper</td>
</tr>
<tr>
<td>here [points to the y-intercept of</td>
<td>and justifying this claim by pointing to two distances along the axes.</td>
</tr>
<tr>
<td>the line y = x + 5].</td>
<td></td>
</tr>
<tr>
<td>Cause look at it ... 1, 2, and 1, 2</td>
<td></td>
</tr>
<tr>
<td>and 1, 2 [counting up to 5 on the</td>
<td></td>
</tr>
<tr>
<td>y-axis and then to 5 on the x-axis,</td>
<td></td>
</tr>
<tr>
<td>the axes are labeled with a slash</td>
<td></td>
</tr>
<tr>
<td>every two units]. This is the</td>
<td></td>
</tr>
<tr>
<td>same.</td>
<td></td>
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</tbody>
</table>

As Fred elaborated on his description, he proposed that the line $y = x + 5$ is steeper at the $y$-intercept. His justification was based on the fact that the distance from the origin to the line along the $y$-axis is the same as the distance from the origin to the line along the $x$-axis (see Figure 2).

During this elaboration, Fred was not only proposing that the line $y = x + 5$ is steeper than the line $y = x$, he was also using an object usually associated with translation, the $y$-intercept, in his description of a steeper line. He referred to steepness as a property associated with a point, saying “it’s steeper right here” as he pointed to the $y$-intercept. In this elaboration, Fred’s use of coordinated gesture and talk helped unpack what he meant by steeper: A line is steeper than another at a specific location, the point where the line crosses the $y$-axis.

As they continued discussing their descriptions of the line $y = x + 5$, Harold proposed that the line $y = x + 5$ was going up, whereas Fred continued to wonder whether the line was steeper or not. Up to this point it is not always clear whether Fred and Harold were referring to Question A (“make the line steeper”) or Question
Negotiating Shared Descriptions of Linear Graphs

C (“make the line both steeper and move up on the y-axis”). Next, Fred proposed they explore how to make a line steeper on the graph. As they took on this new problem, they began using a pen to represent steeper lines in front of the screen.

Dialogue

35 Fred: Can you make it deeper ... steeper?

36 Harold: [Demonstrating with the pen.] Steeper it might go like that [rotating the pen counterclockwise from the line y = x] or like that [moving the pen up to (0,5) and then rotating the pen counterclockwise].

Commentary

Harold shows two ways to make a line steeper, using a pen placed on the computer screen.

In line 36, Harold proposed two ways to make the line steeper: rotating the pen counterclockwise from the line y = x or translating the pen up the y-axis first and then rotating it about the y-intercept. He was referring to lines that are steeper than the line y = x, which he used as a reference object, even though it was not graphed on the screen. There are several ways to interpret this proposal. He may have been saying that rotation is what makes a line steeper, regardless of what the y-intercept is. On the other hand, he may have been saying that “making a line steeper” also refers to translating it up on the y-axis. In this demonstration, Harold used coordinated gestures and talk concurrently to clarify the meaning of steeper.

Dialogue

37 Fred: Steeper like that [grabs the pen in Harold’s hand and moves it below the x-axis so that it is in the position of a line that is steeper than y = x and also has a negative y-intercept]. This way right?

Commentary

Fred proposes that a line that is below the origin and steeper than the line y = x would be steeper.

In Fred’s elaboration in line 37 it is difficult to tell whether he was including translation down the y-axis as part of the description “steeper than the line y = x”
or showing that a line that has a negative $y$-intercept can also be steeper, regardless of what the $y$-intercept might be. However, it is striking that Fred was moving the pen in Harold’s hand down the $y$-axis as he was saying “steeper like that.” This coordinated gesture and utterance suggests that he was associating steeper with the translation down the $y$-axis.

**Dialogue**

38 Harold: Steeper is like this [puts the pen in the position of a line steeper than the line $y = x$] but more like this [keeps the pen at an inclination steeper than $y = x$ as he moves the pen up and down the $y$-axis] ...

**Commentary**

Harold shows a steeper line first as a line steeper than $y = x$ and then also as a line that moves up and down the $y$-axis, presumably meaning a line that is translated up or down as well as rotated (see Figure 3).

In this third elaboration, Harold seemed to be proposing that a steeper line is one that is rotated counterclockwise about the origin, saying “like this” as he rotated the pen, as well as a line that is translated up and down the $y$-axis, saying “more like this” as he moved the pen up and down the $y$-axis. In these three elaborations, Fred and Harold used coordinated gestures and talk to expose the details of their understanding of the meaning of steeper. However, they did not yet seem to have reached a clear agreement on the meaning of this term. They returned to answering the questions:

**Dialogue**

40 Harold: So we’re going up on the $y$-axis … and “make the line both steeper and move on the $y$ axis” [referring to Question C] … You don’t want it steeper you just want it to move up on the $y$-axis … so … yes or no? [Looks at Fred.]

41 Fred: Mm … no.

**Commentary**

They agree that the line for $y = x + 5$ would not be steeper, that it would move up on the $y$-axis, and that the answer to Question C was no because they didn’t “want the line to be steeper.”

In sum, during Excerpt 1, Fred and Harold were working on several concurrent problems: answering the questions, deciding whether a line can become steeper as well as move up (or down) on the $y$-axis, and exemplifying what situations can be described by the term steeper. In the conversation just given, they began to negotiate how to use the term steeper through elaboration and clarification. They elaborated
FIGURE 3  Harold—line 38—"... but more like this."
on examples of what situations the description steeper refers to and clarified when they each thought the description would apply. Even while looking at the line \( y = x + 5 \) on the screen (and apparently knowing where the line \( y = x \) would have been located) these two students spent a considerable amount of time discussing and representing the term steeper. One way to interpret their dialogue and gestures is that at different points in the conversation each of them used steeper to sometimes refer to translation, sometimes to refer to rotation, and other times to refer to both movements.

The dialogue just given exemplifies the use of gestures and talk interactively to disambiguate the meaning of a description. Through repeated gestures representing a line (or lines) on the screen and coordinated talk describing these lines, each student elaborated their understanding of the situations in which the description steeper would apply. This excerpt highlights the importance of gestures in general (McDermott, Gospodinoff, & Aron, 1978) and specifically when describing graphical objects. Gestures were an integral part of these students’ descriptions of graphs, and their language use might have been interpreted differently without the videotape as a source of data. Another resource for elaborating descriptions is the line \( y = x \), which even though it is not graphed on the screen is implicitly present in each example of a steeper line and serves as a reference object in several descriptions. Both students implicitly referred to this line as a reference object, which, because it is shared, supports the construction of a shared description.

By the end of Excerpt 1, Fred and Harold did not seem to have produced an example that unequivocally communicated what they each meant by steeper to their partner or settled on a shared understanding of what constitutes steeper lines. The next excerpt shows how they continued their negotiation of the meaning of the term steeper.

Excerpt 2: Fred and Harold (Problem 9a)

**Target Equation** \( y = 10x \)

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 Fred:</td>
<td>Before graphing the equation both Fred and Harold predict that the line will be steeper.</td>
</tr>
<tr>
<td>Ten ( x ) [mumbles] ( y = 10x ) will be up here . . . so it will be steeper, right?</td>
<td></td>
</tr>
<tr>
<td>3 Harold:</td>
<td>It will be steeper, right . . .</td>
</tr>
<tr>
<td>4 Fred:</td>
<td>Yeah . . . yes [writes the answer for Question A].</td>
</tr>
<tr>
<td>5 Together:</td>
<td>Steeper.</td>
</tr>
<tr>
<td></td>
<td>They agree on their prediction.</td>
</tr>
</tbody>
</table>
During these first exchanges Fred and Harold check their predictions with each other and agree that the line $y = 10x$ will be steeper than the line $y = x$. Next they discuss whether the line $y = 10x$ would move on the $y$-axis:

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>6</strong> Harold:</td>
<td>The line will move on the $y$-axis.</td>
</tr>
<tr>
<td><strong>7</strong> Fred:</td>
<td>No ...</td>
</tr>
<tr>
<td><strong>8</strong> Harold:</td>
<td>No?</td>
</tr>
<tr>
<td><strong>9</strong> Fred:</td>
<td>What do they mean on the $y$-axis?</td>
</tr>
<tr>
<td><strong>10</strong> Harold:</td>
<td>$y$-axis.</td>
</tr>
<tr>
<td><strong>11</strong> Fred:</td>
<td>No, it would be still on that ... like that always cross the $y$ ... I mean the $x$-axis.</td>
</tr>
</tbody>
</table>

In this dialogue, although Fred asked for a clarification of the meaning of the phrase “on the $y$-axis,” Harold did not respond. They moved on to graphing the equation.

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>12</strong> Fred:</td>
<td>$Y$ equals ten $x$ [he graphs the equation $y = 10x$] ... This is ten $x$ [points to the line $y = 10x$ on the screen].</td>
</tr>
<tr>
<td><strong>13</strong> Harold:</td>
<td>Yeah, so it’s steeper.</td>
</tr>
<tr>
<td><strong>14</strong> Fred:</td>
<td>Yeah.</td>
</tr>
</tbody>
</table>

After graphing the equation, Fred and Harold easily agreed that the line $y = 10x$ was steeper. This ease stands in contrast to the extended negotiation in Excerpt 1. They moved on to deciding whether the line moved on the $y$-axis or not.

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>19</strong> Harold:</td>
<td>Did it go on the $y$-axis?</td>
</tr>
</tbody>
</table>
Dialogue

20 Fred: No ...

21 Harold: Did it or did it not [looks at Fred]? ... It didn’t right? So it’s on it ... passing, right ...

22 Fred: The point’s here [pointing to the origin] ... See the point ... [pointing to another point on the same line but below the origin] ... There’s a point here.

Commentary

Fred proposes that it did not. Harold looks to Fred for an answer and then proposes that it did not. Fred offers an elaboration involving the origin and another point on the line.

Even after graphing the equation, Harold remained unsure of whether the description “moved up on the y-axis” was correct or not, referring to movement “on the y-axis” and dropping the word up. Fred presented a clarification using the points on the line. It is not clear exactly what Fred’s elaboration involved, whether Harold understood this explanation, or whether Harold was convinced that the line had not moved up. This lack of agreement stands in contrast to their recent ease in agreeing that the line y = 10x was steeper in lines 12 to 14, showing that although they had moved closer to a shared meaning for the term steeper, they had not yet accomplished this for the phrase “move up on the y-axis.”

During Excerpt 2, Fred and Harold used several resources to negotiate the meaning of their descriptions. Once again, Fred used gestures coordinated with talk to clarify the meaning of the phrase “move on the y-axis.” He also used reference objects, the origin and points on a line, as resources for his explanation. However, in contrast to Excerpt 1, where Harold and Fred used only one term for steepness, in Excerpt 2 they used several different phrases to describe the line: “it will always cross,” “go on the y-axis,” “on it”, and “passing.”

Fifteen minutes later, they worked on the last problem of their discussion session, where a line is translated down the y-axis. This last excerpt shows how Fred and Harold now easily reached agreement on a description without resorting to challenges, elaborations, or clarifications.

Excerpt 3: Fred and Harold (Problem 14a)

Target Equation y = x - 100

Dialogue

1 Harold: [Reading Question A] “Steepness would change.”
The dialogue in this last excerpt was very straightforward, especially in comparison to the initial negotiation of the meaning of *steeper* in Excerpt 1, and to the later negotiation of the phrase “move on the y-axis” in Excerpt 2. During this last problem each of these students seemed confident of their descriptions and seemed to assume that their partner readily understood these descriptions. Their conversation proceeded without elaboration or clarification, both when they were making a prediction and when they were describing a line on the screen. The students thus seemed to have constructed shared understandings of the meaning of the term *steeper* and the phrase *move on the y-axis*. These shared meanings allowed them to move with ease through the cycle of predicting, graphing, and checking their descriptions with each other. Fred and Harold arrived at these shared meanings by explicitly taking on the task of clarifying their descriptions to each other. During this clarification, they used gestures, talk, and reference objects as coordinated resources for unpacking the meaning of a description.

Toward the end of their discussion, not only did Harold and Fred’s descriptions flow more easily, their descriptions also reflected some conceptual refinement. In the beginning of their conversation it was not always clear whether or when Harold and Fred were considering the effects of changing $m$ and changing $b$ as generating independent movements on a line. For example, Fred used the y-intercept to explain why a line was steeper. Moreover, both Fred and Harold initially thought the line $y = x + 5$ might be steeper. They also had to clarify whether this change in the equation would make a line both become steeper as well as move up on the y-axis. By the end of their discussion session, they easily and confidently referred to the

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 Fred: It …</td>
<td>They agree in their prediction that the steepness will not change.</td>
</tr>
<tr>
<td>3 Together: It won’t change.</td>
<td></td>
</tr>
<tr>
<td>4 Harold: “Move on the y-axis” [reading Question B].</td>
<td>Fred proposes that the line will move down on the y-axis.</td>
</tr>
<tr>
<td>5 Fred: Yeah, cause it’s coming down.</td>
<td>They graph the equation $y = x - 100$.</td>
</tr>
<tr>
<td>6 Fred: X minus 100 …</td>
<td>Harold describes the steepness as not changing and the line as having moved on the y-axis.</td>
</tr>
<tr>
<td>7 Harold: Steepness will not change … it will move on the y-axis.</td>
<td>Fred specifies that the line moved down.</td>
</tr>
<tr>
<td>8 Fred: It will come down.</td>
<td></td>
</tr>
</tbody>
</table>
effects of changing $m$ and changing $b$, rotation and translation, as separate properties of lines.

CASE STUDY 2: USING REFERENCE OBJECTS TO CLARIFY AND JUSTIFY DESCRIPTIONS

Students' initial descriptions reflected the use of everyday meanings—meanings that sometimes proved problematic for the conversations. Although these everyday meanings for descriptions of lines can be ambiguous, they can also serve as resources for constructing shared descriptions. Case Study 2 shows how two students used a shared metaphor from everyday experience and reference objects to elaborate the meaning of their descriptions. The students in this case study developed shared descriptions of lines by relying on a metaphor comparing lines to hills and by repeatedly using reference objects to justify their descriptions.

Marcela and Giselda began their discussion with repeated disagreements about the meaning of the term *steeper*. I provided them with an explanation of steepness comparing two lines on the board to two hills, explaining that steeper lines are harder to climb. This metaphor proved to be useful for the students as they constructed a shared understanding of the meaning of *steeper*. Marcela persisted in using this metaphor to explain to Giselda why a line was steeper or less steep than another. She first referred to the $x$-axis as "the ground" and later used both axes as reference objects for justifying her descriptions. Reference objects were also an important resource for this conversation. Marcela introduced the use of several reference objects, such as the $x$-axis, the $y$-axis, and the origin to clarify the meaning of her descriptions to Giselda. Giselda followed Marcela's lead in using these resources and later independently used these reference objects in her own descriptions.

When working on the problem where the target equation was $y = x + 5$ (Problem 3a, See Figure 1), Marcela and Giselda started out disagreeing on whether the line $y = x + 5$ would be steeper than the line $y = x$ or not, and whether it would move on the $y$-axis. Marcela first proposed that the line would "go up five more." Giselda proposed an alternative description, saying that the line would "touch the middle." They then drew the line $y = x + 5$ on their paper. Marcela wrote down "That will make the line go up five more and be steeper" on the paper and checked yes for the answers to Questions A, B, and C. This written answer was not the same as Marcela's first proposal that the line would "go up five more," because it included a statement that the line will be steeper. At this point, Marcela explicitly disagreed with the written answer:
Excerpt 4: Marcela and Giselda (Problem 3a)

Target Equation $y = x + 5$

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marcela: No, it's not steeper!</td>
<td>Marcela challenges the written answer.</td>
</tr>
<tr>
<td>Giselda: Look ... how come you put yes [referring to the answers to Question A and C]?</td>
<td>Giselda questions why Marcela had initially answered the questions yes,</td>
</tr>
<tr>
<td>Marcela: Because it's the same line.</td>
<td>Marcela justifies her new answer, describing the line $y = x + 5$ as &quot;the same line.&quot;</td>
</tr>
<tr>
<td>Giselda: What do you mean the same line?</td>
<td>Giselda asks for a clarification of the phrase &quot;the same line.&quot;</td>
</tr>
</tbody>
</table>

Presumably, Marcela meant that the line $y = x + 5$ has the same slope as $y = x$. They moved on to the problem of clarifying what the phrase \"the same line\" meant by pointing to the examples on the board. There were two examples drawn, $y = x$ and $y = 8x$ as an example of steeper, and $y = x$ and $y = x + 6$, as examples of move up on the y-axis. Although they tried to address the meaning of \"same line,\" they did not seem to reach any overt agreement on this aspect of Marcela's description. They then moved on to considering the relation between a steeper line and the origin. Giselda asked whether a line that is steeper than another line has to cross the origin and Marcela insisted that it does not.

At this point, I intervened, asking them, \"If two lines are parallel, do you think one is steeper than the other one?\" Marcela answered no and Giselda said she did not know. They both agreed that the two lines on the board, $y = x$ and $y = x + 6$, were parallel. I attempted to clarify the meaning of the term steeper making a comparison between lines and hills and saying that steeper lines or hills are harder to climb:

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Int: Do you think this one [pointing to the line $y = x + 6$ on the board] is steeper than this one [pointing to the line $y = x$]? If you had</td>
<td></td>
</tr>
</tbody>
</table>
Dialogue

to climb up this hill [pointing to the line \( y = x + 6 \)], would it be harder?

26 Giselda: Yeah, the top one would be harder.

27 Marcela: Why?

28 Giselda: Because it’s sleeper, I mean steeper [laughs].

29 Marcela: Why is it steeper?

30 Giselda: Because it is, look!

Commentary

Giselda identifies the line \( y = x + 6 \) as harder to climb. Marcela asks for a justification. Giselda identifies the line \( y = x + 6 \) as steeper. Marcela again asks for a justification. Giselda again proposes that the line \( y = x + 6 \) is steeper.

In this conversation, Giselda explicitly identified the line \( y = x + 6 \) as steeper or harder to climb several times. Although Marcela asked Giselda to justify her description several times, Giselda provided only the line itself as evidence. After I asked Giselda if she thought that steeper meant higher and she agreed, I went on to clarify the difference between these two descriptions:

Dialogue

So if you had to climb this hill you think that would be harder? [pointing to \( y = x + 6 \)]

33 Int: [Nods her head in agreement.]

34 Giselda: [Nods her head in agreement.]

35 Int: Because …

36 Giselda: Well, I thought that steeper means to like high … higher.

37 Int: It doesn’t. This is the same steepness as this one [pointing to \( y = x \) and \( y = x + 6 \)].

Commentary

Giselda again identifies the line \( y = x + 6 \) as harder to climb than the line \( y = x \). Giselda again identifies the line \( y = x + 6 \) as steeper. Giselda clarifies her understanding of the term steeper.

Giselda seemed to accept the proposal that there is a difference between the meaning of the term steeper and the meaning of the term higher. She also changed her description of the line \( y = x + 5 \):
Dialogue
38 Giselda: So it's the same, it's going in the same position, so it's not steeper ... OK.

Commentary
Giselda compares the lines $y = x$ and $y = x + 5$.

Giselda described the line $y = x + 5$ as "the same," clarifying that this means "in the same position" and "not steeper," presumably meaning that the line $y = x + 5$ has the same slope as $y = x$. They went on to graph the equation $y = x + 5$ on the computer and answered all the questions correctly.

In the next excerpt, Marcela continued using the metaphor that lines are like hills, comparing the $x$-axis to the ground. She also began using reference objects to clarify and explain her descriptions, a move that Giselda followed in her later descriptions. During this problem, Marcela and Giselda initially disagreed as to whether the line for $y = -0.6x$ was less steep than the line $y = x$. Giselda initially thought that the steepness would not change, and Marcela twice asked her to make sure that was her answer. They then graphed the equation on their paper, answered the questions, and Marcela proceeded to check Giselda's answers. Giselda had answered that the line would be steeper.

Excerpt 5: Marcela and Giselda (Problem 8a)

Target Equation $y = -0.6x$

Dialogue
22 Marcela: No, it's less steeper ...
23 Giselda: Why?
24 Marcela: See, it's closer to the $x$-axis ... [looks at Giselda] ... Isn't it?
25 Giselda: Oh, so if it's right here ... it's steeper right?
26 Marcela: Porque fijate, digamos que este es el suelo. Entonces, si se acerca más, pues es

Commentary
Marcela corrects Giselda's answer.

Marcela clarifies the meaning of "less steeper," using the distance from the line to the $x$-axis.

Marcela introduces the metaphor that the $x$-axis is like the ground.

---

6Choice C in this problem reads: "The line would flip to the other side of the $y$-axis" to address the effect of a negative slope on the line.
After Marcela proposed that the second line was “less steeper,” she clarified the meaning of this phrase first by using the x-axis as a reference object (line 24) and then by using the metaphor that lines are like hills (line 26), referring to the x-axis as the ground as well as using the distance to the x-axis. Marcela continued to describe lines using the axes (line 30). She also explained the meaning of “less steep” using a comparison of the distances from the line to the x- and y-axes (line 32):

**Dialogue**

30 Marcela: ... 'cause see this one [referring to the line $y = x$] ... is ... está entre el medio de la $x$ y de la $y$ [is between the $x$ and the $y$]. Right?

31 Giselda: [Nods in agreement.]

32 Marcela: This one is closer to the $x$ than to the $y$, so this one is less steep.

33 Giselda: All right.

**Commentary**

Marcela describes the steepness of the line $y = x$ using the axes as reference objects. Giselda agrees. Marcela repeats her clarification that the other line is less steep because it is closer to the $x$ than to the $y$-axis. Giselda agrees.

As the discussion progressed, Marcela repeatedly clarified her understanding of the terms and phrases steep, steeper, less steep, move on the y-axis, move up, and move down to Giselda. Marcela continued to use the $x$- and $y$-axes as reference objects for describing the steepness of lines, as illustrated in Excerpt 5. In subsequent discussions about steepness, Marcela continued to use the metaphor that lines are like hills to clarify the meaning of her descriptions to Giselda. Marcela also alternated between using the “ground” and the “x-axis” or “the x” as reference objects to clarify or justify her claims about the steepness of a line. During four other problems, Marcela explained to Giselda why a line had not “moved on the y-axis” and why a line had not changed steepness. Each time Marcela used the axes as reference objects, describing a line that was less steep as “closer to the x-axis” and a steeper line as having moved “closer to the y-axis.”

By Problem 9a, although Marcela predicted that the line for the equation $y = 10x$ would be “almost straight up,” Giselda was initially not sure whether the line would be steeper or not and then proposed that the line would be “less steeper.” Next they
worked on the question of whether the line would move on the y-axis. Giselda first proposed that this line would move on the y-axis, and Marcela disagreed with her. Marcela referred to the examples on the blackboard to clarify what this question meant. After discussing how a negative coefficient for x affects the line, they returned to considering whether the line would move on the y-axis:

Excerpt 6: Marcela and Giselda (Problem 9a)

<table>
<thead>
<tr>
<th>Target Equation $y = 10x$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dialogue</strong></td>
</tr>
<tr>
<td>27 Marcela: The line will not move on the y-axis. Why?</td>
</tr>
<tr>
<td>28 Giselda: Because it's not a negative number!</td>
</tr>
<tr>
<td>29 Marcela: No! It's because it's still crossing the x point, the origin point.</td>
</tr>
<tr>
<td>30 Giselda: Aquí es [here it is] because what?</td>
</tr>
<tr>
<td>31 Marcela: Here it's because it got closer to the x-axis ... I mean to the ...</td>
</tr>
<tr>
<td>32 Together: y-axis!</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Commentary</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Marcela uses the origin as reference object to justify her claim that the line will not move on the y-axis.</td>
</tr>
<tr>
<td>Giselda asks for an explanation for Question A: &quot;Will the line be steeper or less steep?</td>
</tr>
<tr>
<td>Marcela explains why the line is steeper using the x-axis and corrects herself to say y-axis at the same time that Giselda corrects her. They graph the equation $y = 10x$.</td>
</tr>
</tbody>
</table>

During this conversation, Marcela used two reference objects in her descriptions. First she used the "x point" or "the origin point" to explain why the line had not moved on the y-axis. Second, she used the y-axis to describe a steeper line. This last description was co-constructed by Marcela and Giselda together. As Marcela incorrectly described the line as closer to "the x-axis," they changed this to "y-axis" in unison.

Giselda later independently generated descriptions using the same reference objects originally introduced by Marcela into the conversation. For example, in the next excerpt Giselda used the origin as a reference object for describing steepness as well as vertical translation and described steepness in terms of the distance from
the two axes. These descriptions were built on the use of the reference objects that Marcela had earlier introduced.

**Excerpt 7: Marcela and Giselda (Problem 11a)**

Target Equation $y = 0.1x$

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Giselda: If you change the equation to $y = 0.1x$ …</td>
<td>Marcela describes the line using the x-axis as a reference object.</td>
</tr>
<tr>
<td>2 Marcela: Tiene que ser bien pegadita. [It has to be very close.]</td>
<td></td>
</tr>
<tr>
<td>3 Giselda: Right here? No, it doesn’t have to cross …</td>
<td>Giselda describes the line using the origin as a reference object.</td>
</tr>
<tr>
<td>4 Marcela: It is! It will cross.</td>
<td></td>
</tr>
</tbody>
</table>

In this dialogue, Marcela implicitly used the x-axis to describe the steepness of the line, saying it had to be “very close” (to the x-axis). Giselda used the origin in her description of the line, saying that it “doesn’t have to cross” (the origin).

As Marcela drew the line on the paper, they continued describing the line and answering Question A: “The steepness would change” (the choices are steeper or less steep). Giselda predicted that the line would be less steep and Marcela agreed. Marcela then provided an explanation:

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 Giselda: It’s less because …</td>
<td>Marcela describes the line’s steepness using the x-axis as a reference object.</td>
</tr>
<tr>
<td>17 Marcela: The line is closer to the x-axis …</td>
<td></td>
</tr>
<tr>
<td>18 Giselda: Let me do it … is closer to the x-axis and the … and is further from the y-axis.</td>
<td>Giselda completes the description using the y-axis as a reference object.</td>
</tr>
</tbody>
</table>

They moved on to deciding whether the line would move on the y-axis:

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>19 Marcela: The line will move on the y-axis?</td>
<td>Marcela reads Question B.</td>
</tr>
<tr>
<td>20 Giselda: No.</td>
<td></td>
</tr>
</tbody>
</table>
NEGOTIATING SHARED DESCRIPTIONS OF LINEAR GRAPHS 265

Dialogue
21 Marcela: No, no, because it will still ...

22 Giselda: Because it would still cross the origin.

23 Marcela: Yeah.

Commentary
Marcela begins a justification.
Giselda finishes Marcela’s justification using the origin as a reference object.

Giselda was, at that point, also using the axes and the origin as reference objects in her descriptions (lines 18 and 22). These two descriptions were again co-constructed. In lines 14 and 16, Giselda provided the description “less steep,” Marcela began the justification in line 17, and Giselda completed the justification in line 18. For the answer to Question B, Marcela began a justification (line 21), and Giselda completed it (line 22). Since the dialogue in Excerpt 5, Marcela and Giselda had moved toward co-constructing shared descriptions. Initially, Marcela provided both the descriptions and the justifications, as seen in Excerpt 5. During Excerpt 6, Giselda started to generate a description and corrected Marcela’s justification. In Excerpt 7, both Giselda and Marcela generated descriptions and justifications; Giselda also completed justifications started by Marcela. The repeated use of the same reference objects in their descriptions and justifications was an important resource for moving toward these shared descriptions.

By problem 13a, although neither student could predict what the line $y = x + 100$ would look like before graphing (they wrote “we have no idea” for all answers), once they had graphed the equation, they easily described the line and agreed on their descriptions. After they graphed the equation $y = x + 100$, Giselda correctly described the line as having the same steepness and continued to describe translation using the origin as a reference object:

Excerpt 8: Marcela and Giselda (Problem 13a)

Target Equation $y = x + 100$

Dialogue
1 Marcela: “The steepness would change.”

2 Giselda: No … it’s still the same … it didn’t pass through the cross, cross through the origin point … it went up because …

3 Marcela: [Writes the explanation for Question B: Because it’s

Commentary
Marcela reads Question A.
Giselda uses the origin as a reference object to propose that the steepness is the same. She also proposes that the line went up.
Case Study 2 shows that metaphors from everyday experience can be useful resources for constructing shared descriptions of mathematical objects. The comparison of lines and hills provided a way for Marcela to justify her understanding of the meaning of steepness. This case study also shows how students used reference objects as resources for clarifying the meaning of their descriptions and for justifying their descriptions. Marcela introduced a metaphor comparing the $x$-axis to the land and repeatedly used reference objects to explain her descriptions to Giselda. Marcela used the $x$-axis as a reference object to argue why a line was less steep, described the line $y = x$ as being “between the $x$ and the $y$,” and defined the term less steep as meaning “closer to the $x$ than to the $y$.” Marcela also introduced the use of the origin as a reference for vertical translation. As the discussion progressed, Giselda also came to use these reference objects in her descriptions, initially completing, correcting, or justifying Marcela’s descriptions, and later generating similar descriptions on her own.

Case Studies 1 and 2 show that peer discussions can successfully support the construction of shared descriptions of mathematical objects. In each of the case studies previously presented, students’ conversations at the end of the discussions reflected the construction of shared descriptions of lines. That is, an individual’s description was not contested by the partner, and the pair seemed to agree on one description. Conversations were no longer interrupted to negotiate, elaborate, or clarify meanings. Although the style of the conversations in Case Study 1 and 2 are different, both discussions resulted in the construction of shared descriptions. Marcela seemed to take on the role of explainer, whereas Giselda rarely explained a description to Marcela. In contrast, Harold and Fred seemed to participate more equally in providing both explanations and elaborations. Although the discussion styles were different, both pairs moved toward less elaborations, less contested descriptions, and more shared meanings.

Not only did these students construct shared descriptions, their descriptions also became more precise, and thus more mathematical, and came to reflect important conceptual pieces. Three of the four students in Case Studies 1 and 2 refined their descriptions so that they reflected important conceptual knowledge about linear functions. By the end of the discussions, Harold, Fred, and Marcela explicitly referred to translation and rotation as independent properties of a line. For example, in the last problem of the discussion, Marcela explicitly described rotation and translation as independent properties in her written answer “it only moved down.” Although Giselda’s descriptions were more tentative, she had also started to
separate these two properties. Thus, these conversations with a peer supported students’ conceptual change as well as the construction of shared descriptions.

CASE STUDY 3: UNRESOLVED ALTERNATIVE DESCRIPTIONS

This last case study is presented as a contrast to Case Studies 1 and 2. Although the students in Case Study 3, Monica and Denise, attempted to negotiate the meaning of their descriptions, they did not resolve initial ambiguities or move toward shared meanings for their descriptions. One characteristic of the conversations in Case Study 3 is that Monica and Denise generated many alternative descriptions, rather than focusing on clarifying one or two descriptions. Moreover, they did not resolve conflicts in the meaning of these alternative descriptions. For example, Monica and Denise alternatively described lines as having moved “to the left,” “to the right,” and “to the side(s).” The descriptions using left and “side(s)” were especially ambiguous and problematic for the conversations. Monica and Denise alternated between using these two terms to sometimes refer to rotation about the origin and other times to refer to horizontal translation.

Although choosing the axes as reference objects to describe and justify the steepness of a line, as Marcela did in Case Study 2, might seem natural for describing the steepness of lines that cross the origin, Monica and Denise did not settle on any one choice of reference objects to describe steepness. Monica initially introduced the terms left and right to refer to rotation. This choice of terms made the discussion problematic, because these two students later also used these terms to refer to horizontal translation.

The first use of the terms left and right occurred during Problem 4a. In the dialogue immediately preceding this excerpt, Monica and Denise had predicted that the answer for Question A (“Would that make the line steeper?”) before graphing the equation $y = 3x$, was yes, and Denise had explained that this was “Because when you times it (the line) goes steeper.” They then proceeded to graph the equation $y = 3x$ and attempted to describe what happens to the line when $b$ changes (“when you add”) and when $m$ changes (“when you times”):

Excerpt 9: Monica and Denise (Problem 4a)

Target Equation $y = 3x$

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 Denise: 'Cause when you add it goes on ... on ... some-</td>
<td></td>
</tr>
</tbody>
</table>
Dialogue
where on the line [points to the negative side of the x-axis].

17 Monica: But we weren't adding! We were timesing.

18 Denise: When you're timesing it stays right in the middle [points to the origin].

19 Monica: It moves this way [rotates hand clockwise and counterclockwise] to the left or to the right.

Commentary
Denise describes addition as affecting a line by moving it along the x-axis.

Denise proposes that multiplication does not change where a line crosses the origin.

Monica uses a gesture and the phrase "to the left or to the right" to describe the effect of changing $m$ on the line.

When Denise described the effect of addition, she first used the description "it goes somewhere on the line" as she gestured toward the x-axis. Monica accurately gestured with her hand to represent the effect of multiplication as rotating a line about the origin and described this movement as "to the left" or "to the right." Next, they tried to clarify what they each meant by the phrase "to the left":

Dialogue

20 Denise: It moved that way [moves hand counterclockwise between the two lines on screen].

21 Monica: It moved to the left, right? [they both point to the lines on the screen].

22 Denise: Yeah, it moved to the left ... It moved clockwise [the line actually moved counterclockwise].

23 Monica: Put yes ...

24 Denise: 'Cause when ...

Commentary
Denise uses a counterclockwise gesture to clarify her description, labeling this movement that way.

Monica proposes that the line moved left.

Denise agrees and clarifies the meaning of left as clockwise.

Monica proposes that the answer to Question A is yes.
Dialogue

25 Monica: 'Cause when ... you ... multiply the line gets steeper ... it moves more to the left and makes it steeper ... moves to the left.

Commentary

Monica proposes the explanation that multiplication moves the line to the left and makes it steeper.

This dialogue revolved around the negotiation of the meaning of the phrase “to the left.” In line 19, Monica had moved her hand clockwise and counterclockwise as she described rotation as “to the left or to the right.” Denise initially accepted Monica’s definition of rotation as “to the left” (line 22) and then added that the line moved “clockwise” when the line had in effect moved counterclockwise. Monica’s concluding description described a change in $m$ as having the effect that “it moves more to the left and gets steeper.” Although in line 19 Monica used “to the left or to the right” to refer to rotation, as evidenced by her gesture, it is not clear how Monica was using either left or steeper in line 25. She may have been using left to describe horizontal translation. If she was using left to describe rotation, then her last description referred to rotation twice.\(^7\)

The use of “to the right or to the left” became problematic when Denise subsequently used right or left to refer to translation as they answered Question B: “Does it move the line up on the y-axis?” They initially went back and forth, disagreeing as to whether the line had or had not moved up on the y-axis. Monica insisted that the line had moved up on the y-axis. Denise insisted that the line had not moved up on the y-axis because it still crossed the origin. Monica proposed that the line had “moved on x,” and Denise answered that the line “didn’t move on nothing.” Monica seemed to reluctantly accept Denise’s description, and they argued back and forth about who would write down the answer. As they returned to deciding whether the line had moved up on the y-axis, they again disagreed on the meaning of the phrase “move left”:

\(^7\)There are two aspects of this dialogue that are related to students’ conceptions. One is that Denise was beginning to use the connection between a change in the equation and a change in the line in her explanations: “When you add it goes on ... somewhere on the line [axis] (line 16), and “when you’re timesing it stays right in the middle [origin]” (line 18). Monica, however, neither initiated this sort of explanation nor was she convinced by Denise’s use of this connection between the two representations (lines 11–15). The second aspect is that Denise focused on horizontal, rather than vertical, translation (line 16).
### Excerpt 10: Monica and Denise (Problem 4a, Continued)

#### Dialogue

<table>
<thead>
<tr>
<th>Line</th>
<th>Monica</th>
<th>Denise</th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>Denise:</td>
<td>It didn’t move up or down or right or left, it just got steeper.</td>
</tr>
<tr>
<td>57</td>
<td>Monica:</td>
<td>Yes, it did move left [pointing to the line]! It didn’t ... Oh, gosh! [writing on the paper]. It didn’t move up or down ... because it didn’t move up or down, it just got steeper.</td>
</tr>
</tbody>
</table>

#### Commentary

<table>
<thead>
<tr>
<th>Line</th>
<th>Monica</th>
<th>Denise</th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>Monica</td>
<td>Denise uses right or left to describe horizontal translation.</td>
</tr>
<tr>
<td>57</td>
<td>Denise</td>
<td>Monica first proposes that the line did in effect move left, then changes her description.</td>
</tr>
</tbody>
</table>

In the preceding dialogue, although Denise used left to refer to horizontal translation along the $x$-axis (line 56), Monica insisted that the line had moved left (line 57), presumably meaning the line had gotten steeper and using the meaning for left established in their dialogue in Excerpt 9. Although Monica’s last answer (line 57) might seem to indicate that they had reached agreement on the meaning of their description, later conversations show that this was not the case. Monica and Denise continued to use several alternative descriptions where the same term referred to two movements.

During the next problem, Denise introduced another phrase to describe both rotation and translation, “to the side” (or sides), which also proved problematic for the construction of a shared description. Denise initially used the term *side* to refer to horizontal translation, describing the effect of addition as moving a line “to the side” as she pointed to the segment of the $x$-axis left of the origin. During this problem, Monica used *left* and *right* in association with addition, even though she had recently used the phrase “to the left” to refer to rotation in Excerpts 9 and 10. They attempted to clarify their descriptions of the effect of addition and agreed that “when you add it moves to the sides and parallel,” thus using “to the sides” to refer to translation. However, this agreement on the meaning of the phrase “moves to the sides” was transitory, because Denise later also used the phrase “to the side” to describe rotation.

From this point on, Monica repeated and wrote what Denise said, without either contributing any descriptions of her own, asking for clarification, or contesting a description. After working on 10 problems together, they had not yet arrived at shared descriptions of lines. This lack of resolution was due in part to the ambiguous use of several alternative phrases to refer to both translation and rotation. In sum, these two students alternated between using “to the left” to refer to rotation (Monica and Denise, Excerpt 9) and “move right or left” to refer to horizontal translation (Denise, Excerpt 10; Monica, transcript not shown). They also used alternative
descriptions using the term *side(s)* to refer to rotation, such as "tilts to the side" (Denise) as well as horizontal translation, "to the side" (Denise), "to the sides" (Monica and Denise). The discussion between Monica and Denise continued to be characterized by this repeated use of alternative descriptions without agreement on shared meanings.

During subsequent problems, Monica and Denise generated several more alternative ways to describe a line, but failed to agree on the meaning of these descriptions or settle on shared descriptions. For example, while describing the line \( y = x + 6 \), Denise used the phrase "it will just go down," focusing on vertical translation, whereas Monica said the line would "just move over," focusing on horizontal translation. Denise described this line as having "the same angle," whereas Monica described the line as staying "on the axes." Monica introduced yet another description using the term side. She described the effect of changing the coefficient of \( x \) as making the line "go to the other side," which could refer to translation, rotation, or reflection about the y-axis. Monica also introduced the term *lower*, which can be interpreted as referring to either translation or rotation, to describe the effect of multiplication by a negative number. By the end of their discussion sessions, Monica and Denise had not agreed on shared descriptions for either rotation or translation and had not resolved the conflicting uses of their many alternative descriptions involving the terms *left* or *side(s)*.

Although Monica and Denise did not arrive at shared meanings, their descriptions did come to reflect some conceptual changes. Their later descriptions reflected an increased coordination between the algebraic and graphical representations. They described a change in the equation as generating a change in the line, saying "add and it will go down," or "multiply it will go down and to the other side." In her later descriptions, Denise focused on vertical translation as the result of a change in \( b \). She also began to separate rotation and translation as independent properties, for example, stating that "when you add it went up, maybe when you subtract it will go down, it [the steepness] will be the same" and saying that a line with a different y-intercept would "just go down."

Monica's descriptions also reflected a greater coordination of the two representations. For example, she described the effect of a coefficient that is less than 1 as "it [the coefficient] will make it steeper but not that much because multiplying changes the steepness, right?" On the other hand, Monica continued to focus on horizontal translation as the result of a change in \( b \), to refer to horizontal and vertical translation concurrently, and at times combined references to rotation and translation in her descriptions.

The conversations in this last case study illustrate the importance of not only negotiating meanings but also resolving these negotiations. Although these two students engaged in repeated negotiations, they did not resolve the conflicts among their alternative descriptions. Unlike the students in Case Studies 1 and 2, who addressed conflicts directly and resolved their negotiations by the end of the
discussion session, Monica and Denise persisted in using either different descriptions or the same description with different or ambiguous meanings.

When compared with the students in Case Studies 1 and 2, Monica and Denise generated many more alternative descriptions for the same situation, rather than focusing on elaborating and clarifying one or two descriptions. Although the students in Case Studies 1 and 2 generated some alternative descriptions, they usually returned to the descriptions provided in the problems (steeper, less steep, and "moves up/down on the y-axis") to describe the lines on the screen. Moreover, Monica insisted on focusing on horizontal translation even though the problems only referred to vertical translation. The students in the first two case studies thus stayed on mathematically productive paths by focusing on only a few descriptions and using the descriptions provided in the problems. These comparisons point to three important characteristics of a conversation with a peer, whether and how students (a) address conflicts, (b) resolve negotiations, and (c) maintain a focus on mathematically productive paths.

CONCLUSIONS

The analysis presented in this article shows that peer discussions can successfully support the construction of shared descriptions of mathematical objects. These conversations can create the need for clarification and provide a rich context for negotiating shared meanings. Students use many resources to elaborate and clarify their descriptions: everyday meanings and metaphors, reference objects, and coordinated gestures and talk. Although students can and do reach agreement during a conversation with a peer, neither resolution nor conceptual convergence is guaranteed. The role of instruction in orchestrating and supporting peer discussion lies in modeling how to resolve negotiations and in maintaining students’ focus on mathematically productive paths.

Reaching conversational clarity and moving toward agreement were important goals during these discussions. Students initially used descriptions that were sometimes ambiguous and other times problematic for their conversations. They often did not use terms referring to the translation and rotation of lines with the same meaning as that intended by the researcher or, perhaps more importantly, by their partners. Students’ descriptions were sometimes problematic, as in the case of the term steeper to mean higher (Case Study 2) or to refer to translation concurrently with rotation (Case Study 1). Other instances were the uses of the terms left or side(s) to refer to both translation and rotation (Case Study 3). In the first two case studies, as the discussion proceeded, the students increasingly settled on shared descriptions of the lines on the screen.

The negotiation of meaning that students engaged in, the fact that two of the pairs arrived at shared descriptions, and the ways that some of the students refined their descriptions all show that peer discussions can be a productive context for
transforming students’ language use. Peer discussions may motivate the refinement of students’ descriptions by creating situations in which their descriptions are not clear or precise enough to communicate successfully with another student. In much the same way that there must be some motivation for changing or giving up one’s initial conceptions about a domain (or the conceptions that work in everyday situations), so also, there must be some motivation for changing the everyday language one uses to describe objects in that domain. If the initial language used is not precise enough or is too ambiguous to communicate successfully with another student, reaching conversational clarity can be a motivation for the negotiation of meaning, the elaboration of descriptions, and the refinement of language use.

Although the constructs of sociocognitive conflict or guidance by a more advanced other have contributed to the understanding of peer discussion, the analysis presented in this article focused on conversations and conversational resources as a way to understand the process of learning through peer discussions. One important reason for focusing on conversational processes is that conversations are inherently social phenomena. This move shifts learning from an individual location, as in the sociocognitive conflict model, to a social site. Students used several local conversational resources to elaborate and disambiguate descriptions such as the coordination of talk and gestures, the use of reference objects, and the use of spatial metaphors from everyday experience.

The analysis of peer conversations presented here draws on neo-Vygotskian theories in some important ways. The assumptions that learning is mediated by language, that social interaction is integral to the learning process, that learning involves the construction of socially shared meaning, and that learning in school involves a shift from everyday to “scientific” concepts are all central to neo-Vygotskian perspectives. However, this analysis also diverges from these frameworks in taking a perspective that makes the co-construction of shared meanings as central as guidance. Thus, rather than identifying who is the more advanced participant or privileging the contributions of one participant as the source of expertise, the contributions of each participant are considered equally in the negotiation and construction of meanings.

This account of learning mathematics shows that understanding the connection between the algebraic and graphical representations of linear functions includes refining descriptions. The negotiation and refinement of students’ descriptions were an important aspect of making sense of lines and their equations. This learning process involved, in part, a shift from everyday to more mathematical and precise descriptions. One important difference between the everyday and the school mathematics registers may be the meaning of relational terms such as steeper and less steep, and phrases such as moves up the y-axis and moves down the y-axis. Meanings for these terms and phrases that may be sufficiently precise for everyday purposes proved to be ambiguous for describing lines in the context of a mathematical discussion.
Although the difference between the everyday and mathematical registers may sometimes be an obstacle for describing lines in mathematically precise ways, everyday meanings and metaphors can also be resources for understanding mathematical concepts. Rather than emphasizing the limitations of the everyday register in comparison to the mathematics register, it is more important to understand how the two registers serve different purposes and how everyday meanings can provide resources for conceptual change.

Each of the students discussed here refined their descriptions of lines in at least some conceptual ways. This refinement in students’ descriptions can be understood as a movement toward the mathematics register, where descriptions of lines are precise and reflect conceptual knowledge central to this domain. However, the mathematics register transcends the use of technical terms and does not consist only of technical terms such as slope and intercept. These students did not simply learn to use the technical terms slope and y-intercept. Instead, they refined the meaning of their descriptions by connecting even nontechnical phrases such as “the line will be steeper” or “the line will move up on the y-axis” to conceptual knowledge about lines and equations.

Mathematical descriptions of lines involve conceptual knowledge such as the interdependency of the two representations, what is necessary and sufficient for describing lines and their movement, and which properties of lines are dependent or independent. Some of the core assumptions one makes when using the terms and phrases steeper, less steep, moves up, or moves down to describe the movement of lines are:

1. Rotation and translation are necessary and sufficient to describe the movement of all lines.
2. Rotation and translation are independent of each other.
3. The movement of lines is described in terms of a preferred reference object. In the case of \( y = mx + b \), the preferred reference objects for describing the effect of changing \( b \) are points on the y-axis. The preferred reference objects for describing the effect of changing \( m \) are one or both of the axes.

These assumptions are embedded in mathematical descriptions of lines and their movement in a plane. Students in this study refined their descriptions so that they reflected some aspects of these conceptual pieces. The refined descriptions of five of the six students reflect the following conceptual knowledge: an increasing coordination between the algebraic and graphical representations, a separation of the parameters \( m \) and \( b \) (and the corresponding movement of lines), omitting horizontal translation, and focusing on vertical translation as a result of changing \( b \).
One model for supporting the refinement of mathematical descriptions in the classroom might be to present vocabulary items and explain these explicitly to students. However, this study suggests that there may be important differences between a discussion with a peer and a presentation by an adult. One of the differences between the peer discussions and the presentation by an adult was that, when working with a peer, students had many opportunities to generate their own descriptions, elaborate the meanings of these descriptions, and negotiate shared descriptions. Thus, one of the beneficial processes in peer discussions may be the occurrence of such conversational cycles of elaboration and clarification.

However, some discussions were more successful than others. Although negotiation was an important process, the resources students bring to bear on these negotiations and the character of their discussions seems to be related to the construction of shared descriptions. The two students in Case Study 3 did not reach agreement, resolve discrepancies in their descriptions, or move toward conceptual convergence. These two students also seemed to generate many alternative ways to describe situations rather than persevere at understanding a few descriptions, like the students in Case Studies 1 and 2. This difference in the nature of these peer conversations points to the important role of instruction in orchestrating and supporting peer discussions. This role lies in modeling how to resolve negotiation and in maintaining students' focus on productive questions.

The analysis presented here raises questions regarding how students learn with peers, specifically in terms of the role of authority and language use. If one of the peers is identified as an authority, then peer discussions are much like adult guidance and can be described in terms of peer tutoring, scaffolding, and so on. But these discussions were also different from adult guidance. Students used their own terms and meanings, and they engaged in extensive discussions of the meaning of terms. These two activities may be in contrast to the way that students engage in discussions with adults. Exploring the differences between adult guidance and peer discussions, especially in terms of how language is used, elaborated, and clarified, is an important focus for further research.

There are several other issues raised by this study that merit further investigation. There are important questions in terms of language use and the mathematics register. The differences between the vernacular and mathematical uses of terms need to be explored in more detail. What the advantages and disadvantages of the use of everyday spatial metaphors in mathematical contexts might be remains an open question. In particular, further research should address how students who speak a language other than English develop competence in the mathematics register in English. Although the metaphor that “learning mathematics is like learning a second language” may be useful, it is not clear what the similarities and differences might be between learning a second language, learning mathematics, and learning mathematics in a second language.
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REFERENCES


