Aspects of Understanding: On Multiple Perspectives and Representations of Linear Relations and Connections Among Them*

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INTRODUCTION

The algebra standard for middle school mathematics in the NCTM Curriculum and Evaluation Standards (1989, p. 102) includes the following statement:

In Grades 5–8, the mathematics curriculum should include explorations of algebraic concepts and processes so that students can

- understand the concepts of variable, expression, and equation;
- represent situations and number patterns with tables, graphs, . . .
- develop confidence in solving linear equations using concrete, informal, and formal methods.

The Standards for Grades 9–12 extend the desired competencies in significant ways. All students are expected to:

- use tables and graphs as tools to interpret expressions . . . (Standard 5, Algebra, p. 150);
- represent and analyze relationships using tables, verbal rules, equations, and graphs (Standard 6, Functions, p. 154);
- translate among tabular, symbolic, and graphical representations of functions (Standard 6, Functions, p. 154);
- analyze the effects of parameter changes on the graphs of functions (Standard 6, Functions, p. 154).

*This chapter was a collaborative effort. The order in which the authors are listed was determined by a random choice procedure.
These are statements of expected performance. In contrast, the reality of current student achievement is indicated by the following quotation from *The State of Mathematics Achievement* (Executive Summary), a distillation of results from the 1990 National Assessment of Educational Progress:

When the mathematics became at all complicated, performance fell off dramatically, even for twelfth graders. For example, high school seniors had considerable difficulty with the following set of questions.

![Graph with equation y = 2x - 5](image)

a. On the axes above, draw a line parallel to \( y = 2x - 5 \) that goes through the origin \( O \).

b. On the line below, write an equation of the new line.

Equation: ____________________________

Only 32% of the high school seniors drew the new parallel line on the graph, when a correct response essentially required the ability to find the origin \( O \) on the graph, the ability to find the existing line on the graph, and an understanding of the term "parallel." Sixteen percent of the twelfth graders answered both parts of this question correctly. (Mullis et al., 1991, p. 11)

Because there is significant variation in state mathematics requirements across the country, some of the students who took the test may not have had formal instruction in the mathematics of the Cartesian plane. However, given the large number of students who have studied the relevant material and the relative simplicity of the question, the fact that only 16% of the high-school seniors who took the test were able to answer both parts (a) and (b) correctly is distressing. From the perspective of people who are comfortable with the properties of algebraic functions and their graphs, the question is straightforward. The skills required to answer part (a) correctly have already been noted. A correct though telegraphic explanation of (b) is as follows: “The desired line has the same slope as the line \( y = 2x - 5 \), so its slope is 2; it passes through the origin, so its \( y \) intercept is 0; hence its equation is \( y = 2x \).”

The level of complexity of the NAEP task is increased substantially (and the task becomes more interesting and more mathematically important) if in its statement “the origin” is replaced by any other point in the plane. One would expect that the percentage of graduating seniors who could solve Problem 1, which follows, would be quite small.

Problem 1. Determine an equation of the line that is parallel to \( y = 2x - 5 \) and that goes through the point \((1, 4)\).

This chapter explores the complexity of Problem 1 and of a family of related problems concerned with different symbolic representations (algebraic expressions, tabular representations, and graphs) of linear relations. Our analysis indicates that there is more complexity to the domain than would appear at the surface level: Consistent with other research in the domain (see later discussion), we indicate that students must come to grips with connections across representations (e.g., the meanings of algebraic parameters in a geometric context) and, depending on context or interpretation, with different perspectives regarding the functions themselves. Those perspectives are as follows.

From the process perspective, a function is perceived of as linking \( x \) and \( y \) values: For each value of \( x \), the function has a corresponding \( y \) value.\(^1\) From the object perspective, a function or relation and any of its representations are thought of as entities—for example, algebraically as members of parametrized classes, or in the plane as graphs that, in colloquial language, are thought of as being “picked up whole” and rotated or translated.

The recent literature (see, e.g., Even, 1990; Schwarz & Yerushalmi, 1992; Sfard, 1992) makes it clear that coming to grips with both the object and the process perspectives is an essential part of learning about functions and graphs. The following quote indicates aspects of the process-object distinction.

Consider now the two functions

\[
x + 3 \quad \text{and} \quad 4 + x - 1.
\]

\(^1\)We thank Ed Dubinsky for pointing out that much of our discussion applies to relations (in which an \( x \) value is not necessarily linked with a unique \( y \) value) as well as to functions.
From the point of view of the process that is carried out with the recipe, these are two different recipes. If, however, one were to plot the output of each of these recipes against its input on a Cartesian plane then the two recipes would be indistinguishable. We see that the symbolic representation of function makes its process nature salient, while the graphical representation suppresses the process nature of the function and thus helps to make the function more entity-like. A proper understanding of algebra requires that students be comfortable with both of these aspects of function. (Schwartz & Yerushalmy, 1992, p. 265)

Breidenbach, Dubinsky, Nichols, and Hawks (1992) described the two perspectives as follows:

A process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. . . A function is conceived of as an object if it is possible to perform actions on it, in general actions that transform it. (Breidenbach et al., p. 263)

Both the process and object perspectives shed light on the behavior of functions, in every representation, but the perspectives are differentially useful, in that one perspective may be usefully invoked in some problem contexts and not in others. In part, we argue that developing competency with linear relations means learning which perspectives and representations can be profitably employed in which contexts, and being able to select and move fluently among them to achieve one's desired ends. We illustrate these perspectives with the analysis of one approach to solving Problem 1. Note that many of the processes that experts use to solve such problems are automatic. We do not claim that people who solve Problem 1 consciously invoke all of the information described, but that such knowledge and perspectives do underlie a competent solution.

To begin, one knows that Problem 1 can be solved because any two independent pieces of information (e.g., two points on the line, the slope and the value of the y intercept, and so on) are enough to determine a line, in any of its representations. This general knowledge cuts across representations and perspectives. However, since the problem asks for an equation of a line parallel to one expressed in the form \( y = mx + b \), it seems reasonable to use that form. One expects to write the equation of the line (call it \( L \)) in the form \( y = mx + b \), where \( m \) and \( b \) are parameters whose values must be determined.

The object perspective is natural for determining \( m \). One attribute of a line as a whole is its slope. Parallel lines have the same slope, so \( L \) has the same slope as the line whose equation is \( y = 2x - 5 \). One can read the slope of that line directly off its equation, as the coefficient of \( x \). Hence \( m = 2 \), and the equation of \( L \) is given by \( y = 2x + b \). Now the value of \( b \) needs to be determined.

Here a change of perspective is in order. The second piece of information in the problem statement is that the graph of \( L \) passes through the point (1,4). Exploiting this information depends on using the crucial information contained in the (deceptively simple) statement of the Cartesian Connection “A point is on the graph of the line \( L \) if and only if its coordinates satisfy the equation of \( L \).” From the process perspective, (1,4) lies on the graph of \( L \), so the equation for \( L \) must produce the \( y \) value of 4 when the corresponding \( x \) value is 1. Hence \( 4 = 2(1) + b \), and \( b = 2 \). Thus the equation of \( L \) is \( y = 2x + 2 \).

We make some preliminary comments. First, the preceding analysis may seem like an exercise in overkill. One might have described the solution to Problem 1 in just a few lines. Is all of the complexity described in the previous two paragraphs really necessary? We argue that it is, especially to capture (and facilitate) the learning process. By way of crude analogy, consider all the things one must learn when first learning to ride a bicycle. From the perspective of someone who, having had much practice, simply hops on a bike and rides off, bike riding could hardly seem simpler. But watch a child first struggling to master a two-wheeler, training wheels and all, and the full complexity of the domain is revealed. Skills, connections, and coordinations that are quite difficult to develop may seem trivial once they have been mastered. Similarly, unraveling the complexity of the domain serves a useful pedagogical function. On the basis of prior research (e.g., Schoenfeld, Smith, & Arcavi, in press), we can assert that some aspects of the domain that we take to be trivial are major stumbling blocks for students. And, knowing what the underlying skills and perspectives actually are can serve as a guide to developing curricula.

In a narrow sense, then, this chapter seeks to elaborate the theme announced in its title—to elaborate aspects of an understanding of linear relations that correspond to the ability to move flexibly between the process and object perspectives in a variety of representations (our focus here being on algebraic, tabular, and graphical representations). In a broader sense, we view this effort as part of a research and development program whose

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2We note that in invoking these three representations we are not necessarily invoking objects that are well defined or well understood. Dan Chazan has remarked to us, for example, that equations have different entailments depending on the ways we think of them (e.g., as "number recipes" or transformations); our descriptions of the objects within the representational categories must become more nuanced to be fully useful.
intention is to map out understandings of complex domains and to construct curricula that help students come to grips with that complexity. We place our main efforts much more along the lines of “seeing and exploiting connections” than on procedural knowledge. The next section of this chapter briefly provides background and context for the body of research that led to this particular study and for the research perspective that was sketched in this section. We focus on a 2 by 3 matrix or framework (the two perspectives and the three representations just discussed) and illustrate how the framework can be used to help construct or assess curricula. The remainder of the chapter is devoted to two “data stories” taken from a series of ongoing tutoring studies, where the curriculum for the studies was informed by our preliminary ideas about the Cartesian Connection.

We hope that the tight focus of this chapter helps to cast some issues in high relief. The elaboration of the process and object perspectives in the case of linear relations points to some loci of conceptual difficulty for students, as seen in our data stories. Generally speaking, our notion of domain competence includes having the ability to use mathematical ideas to deal with somewhat novel and complex problems—problems much more complex and mathematically interesting than the NAEP example that started this chapter. Our analysis of this domain indicates that such competence rests in part on the ability to know which representations and perspectives are likely to be useful in particular problem contexts, and to switch flexibly among representations and perspectives as seems appropriate. The framework delineated in this chapter, suitably expanded, can highlight the kinds of connections one needs to make. Such a detailed delineation can serve both as a means of assessing curricula and as a heuristic frame for curriculum construction. The framework can also serve as a guide for interpreting and understanding students’ solutions to problems in this domain.

Two caveats are appropriate here. The first is that we make no claims for the completeness of the framework. This chapter does not discuss verbal representations, for example, or the construction of mathematical models (in any representation) that capture the great variety of real-world situations that embody linear relations. Nor does it discuss other central mathematical ideas (e.g., proportionality) on which an understanding of linearity depends. Here we elaborate on what it means to understand part of a domain—not all of it. However, one can easily envision extensions of the framework that deal with such issues. Second, our data stories are intended to be illustrative and suggestive, and are rather sketchy in consequence. The reader will not find here the welter of detail that typifies our cognitive analyses (see Schoenfeld et al., 1993; Schoenfeld et al., in press).

4. ASPECTS OF UNDERSTANDING

BACKGROUND

Our intention in this section is to provide the reader with some perspective on the task in which we are engaged and on the issues we consider important. The Functions Group at Berkeley has been engaged since 1985 in a series of studies related to students’ understanding of functions and graphs. Early on, we constructed a computer-based microworld called GRAPHER (Schoenfeld, 1990), designed to help students come to grips with aspects of the domain. Ultimately the research group spent a year and a half engaged in the very fine-grained analysis of 7 hours of videotape of one student working with GRAPHER—the goal of that analysis being to understand precisely how her understanding of functions and graphs changed over the period she worked in our lab (Schoenfeld et al., in press). The analysis resulted in our description of the Cartesian Connection, a characterization of the understandings possessed by people who are knowledgeable in the domain. The analysis also indicated that students who appear to be competent in the domain can, indeed, miss fundamental connections. For example, students can treat the algebraic and graphical representational domains as though they are essentially independent. Although the $m$ in the equation form $y = mx + b$ is typically referred to as the slope and a student may refer to it as such, the student may not attribute any slope-related graphical properties to equations that have differing values of $m$. Similarly, the student may refer to the $b$ value of the equation as the $y$ intercept but may not know that the point $(0, b)$ lies on the graph of the equation—even though the student uses the term $y$ intercept when referring to properties of the graphs. Or the student may not realize that the parameters $m$ and $b$ in the form $y = mx + b$ are independent—that is, that one can change one of the parameters while leaving the other constant, and generate a family of lines with specific properties (see also Moschkovich, 1989, 1990).

Subsequently, the research group organized a collection of problems designed to focus on aspects of the Cartesian Connection and to serve as the basis for a curriculum introducing students to linear functions and graphs. We are now engaged in the extended analyses of videotapes of students and tutors working through that tutoring curriculum, with the goals of (a) extending the cognitive analyses in Schoenfeld, Smith, and Arcavi (in press), (b) delineating the complexities of the tutoring process and constructing a tutor model (Arcavi & Schoenfeld, 1993; Schoenfeld et al., 1993), and (c) constructing a trial curriculum for technology-based classroom instruction on linear functions.

One result of the research has been to delineate the complexity of what, unexamined, might appear to be absolutely straightforward. For example, many educational researchers and developers seem to have the belief that
once things are shown clearly on the computer screen (as opposed to the rude drawings we produce by hand) then students will understand." The work of Goldenberg (1988), discussed elsewhere in this volume, showed all too clearly that such assumptions are unwarranted. Here is another example that points to the dangers of naive curricular assumptions.

As part of a curriculum development project, Magidson (1989) developed guided discovery unit in which student volunteers were introduced to near equations and their graphs. Early in the unit, pairs of students who worked on the curriculum were instructed to use the available graphing software (Green Globs on an Apple II) to work the following problem.

Clear the screen and type in these equations, one at a time:

\[
y = 2x + 1
\]

\[
y = 3x + 1
\]

\[
y = 4x + 1.
\]

What do you notice?

How are these lines similar?

How are they different?

What do you think will happen if you type in \( y = 5x +1 \)? Sketch your prediction on this empty graph [which was provided on a work sheet] and then try it on the computer.

What happened?

The intent of the problem should be obvious, as were Magidson's expectations of what the students should see: All of the lines pass through the point \((0,1)\), and the larger the coefficient of \( x \), the steeper the line (see Fig. 4.1) But then again, we know what to look for. Here are the responses from one pair of students, reproduced verbatim:

What do you notice? The higher the number you are multiplying by \( x \) the more upright the line.

How are these lines similar? All practically the same angle.

How are they different? They're not the same angle.

What do you think will happen if you type in \( y = 5x +1 \)? Sketch your prediction on this empty graph and then try it on the computer. (The students' sketch is given in Fig. 4.2.)

FIG. 4.1. The graphs of \( y = 2x + 1, y = 3x + 1, y = 4x + 1 \).

What happened? It went more upright than the other lines but less upright than we thought it would be.

Note that the students' sketch passes through the point \( (0,5) \). The students had failed to notice that all the lines graphed by the computer had passed through \( (0,1) \)! At least, these students did notice the relationship between the coefficient of \( x \) and the steepness of the line. In general the students who worked the problem made two kinds of observations, neither of which had anything to do with the common point of intersection. The first kind was related to the position of the line. The observations sometimes dealt with steepness, as above. But they frequently dealt with the manner in which the lines appeared on the screen. The software produces any graph in order of increasing \( x \) values. Thus when they appeared, all of the lines "started" at the bottom of the screen, moving upwards and to the right. Many of the students made only the following type of observation: "As the numbers get bigger, the lines start further to the right."
domain as students do, and to help them develop the understandings that allow them to perceive and understand the objects in it in a manner consistent with our perceptions and understandings.

THE FRAMEWORK

As indicated in the introduction, working competently in this domain involves thinking along at least two dimensions. One dimension refers to available means of representing linear functions (our focus here being on the three most common symbolic representations, algebraic, tabular, and graphical); the second refers to the perspective from which a linear function is seen or operated on. These two aspects are represented in Table 4.1. The solutions to tasks or problems may reside solely within one cell of the table; they may move across representations within one perspective or across perspectives within one representation; or they may move across both dimensions. As will be seen, the main curricular and mathematical interest is in tasks whose solutions call for moving both horizontally and vertically across the cells of Table 4.1. We begin with simple illustrations, building up in complexity.


<table>
<thead>
<tr>
<th>Perspective</th>
<th>Tabular</th>
<th>Algebraic</th>
<th>Graphical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Process</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Object</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Process Perspective

In the process perspective one's attention is directed to the relationship between the $x$ and $y$ values of a linear relation. The relation itself may be represented in tabular form [a list of $(x,y)$ pairs], as an algebraic equation, or as a graph (where, thanks to the Cartesian Connection, the $(x, y)$ coordinates of points on the graph are seen as "satisfying" the equation and corresponding to possible entries in the tabular form). The focus is on the $x$ and $y$ values and the relationship between them, on the variables in an equation that stand for those numbers, or on the sets of individual points in the Cartesian plane that, collectively, constitute lines. To begin with a trivial example, the solution to

Problem 2. Given the equation $y = 3x + 2$, find $y$ when $x = 5$. 

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3We say "in essence" because we have paraphrased the students' responses. One of the clearer student statements, verbatim, was: "When you try an equation with smaller numbers the line gets straighter; when you type higher numbers the line gets thicker."
Involves only the process perspective within the algebraic representation. It is easy, though boring, to find tasks whose solutions reside within just one cell of Table 4.1; we will give no further examples. Here is a slightly more interesting problem.

Problem 3. Given the following table of values, find the corresponding linear equation.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
</tbody>
</table>

Solving this problem or its dynamic version, Guess My Rule, is a bit more complicated. Obtaining a solution involves making a connection between two representations, the tabular and the algebraic. Typically, students do so entirely within the process perspective: The goal is seen as finding, by guesswork or some previously developed algorithm, the algebraic expression with the property that when the $x$ values in the table are input into the function, the corresponding $y$ values are produced. Although it certainly is possible to think about this problem from the object perspective, we suggest that students do not.

Similarly, the following two problems call for aspects of the Cartesian Connection linking the algebraic and graphical representations at the process level.

There is a subtle distinction to be made here. One might think that in saying “the desired function has the form $mx + b$” and guessing at the values of $m$ and $b$, the student is necessarily invoking the class of linear functions with $m$ and $b$ as parameters—hence invoking the object perspective. In fact, in many situations where students appear to be dealing with variables or parameters, they are not. There is an extensive literature (see, e.g., Wagner & Kieran, 1989) indicating that in introductory algebra classes, students perceive the task in the problem “solve the equation $x + 3 = 5$” to be that of “determining an unknown” (i.e., a single, predetermined value) and do not conceive of the $x$ in the equation as a variable. Likewise, determining which rule of the form $y = mx + b$ will generate the given table can be seen as a task of determining particular, predetermined values of $m$ and $b$: $m$ and $b$ themselves may not be perceived as parameters, and the function as an as-yet-not-specified process.

Problem 4. Why is the number 4 in the equation $y = 3x + 4$ the $y$ intercept of its graph?

Problem 5. Find the $x$ intercept of the graph of the equation $y = 2x - 2$ and explain how you did so.

The Object Perspective

The situation becomes more complex when one considers the following:

Problem 6. The table on the left represents specific values of the function $f(x) = x^3 - 3x^2 + 2x$. Fill in the table on the right, which represents the function $y = x^3 - 3x^2 + 2x + 1$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
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<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
<td>1</td>
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<td>2</td>
<td>2</td>
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<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

$y = x^3 - 3x^2 + 2x + 1$

There are at least two ways to approach this problem. One could complete it without referring at all to the table on the left. The function under consideration is $y = x^3 - 3x^2 + 2x + 1$, and one can simply calculate the $y$ values that correspond to $x = 0, 1, 2, 3,$ and $4$. That, of course, is treating the function as a process. But it's also a lot of work. People with more mathematical sophistication are likely to approach the problem by saying (at least implicitly): “If I call the two functions $f(x)$ and $g(x)$ respectively, then $g(x) = f(x) + 1$. I can complete the table for $g(x)$ by simply adding 1 to each of the entries for $f(x)$.” Here the two functions are treated as objects, and the transformation “adding 1” as an operator that converts one object to the other. The values of $g$ are determined without direct computation.

Two more obvious examples at the object level are the following:
Problem 7. What are possible values for the slope of a line that lies entirely in the shaded region of the figure below?

\[ y = mx + b \]

Problem 8. Write an equation that characterizes an arbitrarily chosen line from those illustrated in the figure below.

\[ y - y_1 = m(x - x_1) \]

In both of these problems, candidate lines are thought of as being indexed by their slope. In Problem 5, the lines are of the form \( y = mx \); the region is bounded by the \( x \)-axis and the line \( y = x \), which have slope 0 and 1 respectively; and, because lines of increasing slope increase in steepness, the slopes must lie between 0 and 1. For Problem 6, one notes that the family suggested by the figure is “all of the lines passing through \((0,2)\)” or perhaps “the set of lines you get rotating a line through \((0,2)\)” and that the corresponding algebraic form is \( y = mx + 2: m \in R \).

Connections

Connections between perspectives allow for the possibility of flexibly switching from viewing a line (or an equation) as an object that can be manipulated as a whole, to viewing a line (or an equation) as made up of individual points (ordered pairs). This is an especially crucial flexibility when explaining why objects in the domain behave in the manner that they do. For example, consider the following question:

Problem 9. Why is the graph of \( y = 3x \) steeper than the graph of \( y = 2x \)?

What about \( y = 4x \), \( y = 5x \), \( y = 10x \)?

One first answer could be “because it has a larger slope.” However, this still requires an explanation of why a larger slope number corresponds to a steeper line. There are various ways in which one can answer this question, but no matter how one goes about it, the answer involves making connections across representations and perspectives. As in the previous two examples, each of the lines is indexed by its slope; the value of the slope parameter determines which lines are being considered. Here the lines are considered as entities, and the object perspective is employed in the algebraic and graphical representations. But to explain how the values of \( m \) correspond to the steepness of the individual lines requires the process perspective. Here is the standard way:

Consider any line \( L: y = mx + b \) in the plane. Let \((x_1,y_1)\) and \((x_2,y_2)\) be any two distinct points on \( L \). Some algebraic hocus-pocus results in the following:

\[ y_2 - y_1 = (mx_2 + b) - (mx_1 + b) = m(x_2 - x_1) \text{ so that } m = \frac{y_2 - y_1}{x_2 - x_1}. \]

The key to interpreting this ratio for \( m \) graphically lies in the Cartesian Connection. By convention the expressions \((y_2 - y_1)\) and \((x_2 - x_1)\) represent directed line segments in the plane. Those line segments have magnitude [the lengths \(|y_2 - y_1|\) and \(|x_2 - x_1|\), respectively] and direction [up or down, respectively, if \((y_2 - y_1)\) is positive or negative, and to the right or left, respectively, if \((x_2 - x_1)\) is positive or negative]. It follows from these conventions that \( m \) is positive if and only if \( x_2 > x_1 \) implies \( y_2 > y_1 \).

One way to determine the steepness of the line \( L \) is as follows. Take any two points on \( L \) whose \( x \) coordinates differ by 1. Then \((y_2 - y_1) = m(x_2 - x_1) = m(1), \) so \( y_2 = y_1 + m \). If \( m \) is positive, the graph of \( L \) rises \( m \) units for each unit change in \( x \). Thus larger (positive) \( m \) corresponds to steeper slope.
As noted above, there are a number of ways to solve Problem 9. Here is another, briefer solution. All lines of the form \( y = mx \) pass through the origin. The lines \( L_1: y = m_1x \) and \( L_2: y = m_2x \) pass respectively through the points \((1, m_1)\) and \((1, m_2)\); hence if \( m_2 > m_1 \), \( L_2 \) rises more steeply (see Fig. 4.3). Even in this condensed solution, however, one can see aspects of both the process and object perspectives, in the algebraic and graphical representations. From the object perspective, the individual equations and lines are considered as members of the parametric family \( \{ y = mx: m \in \mathbb{R} \} \). But using points on the graphs, and determining their coordinates using the equations of the lines, employs the process perspective.

A third solution, suggested by Ed Dubinsky, is as follows. In \( y = mx \), a change of value in \( x \) (run) causes a change in \( y \) (rise). The larger the value of \( m \), the larger will be the rise for the given run. This solution might be considered the compiled, process version of the first “standard” solution.

![Graphical Illustration](image)

**Problem 10.**

(a) What can you say about the slopes of these two lines?

(b) What can you say about the \( y \) intercepts of these two lines?

(c) The following list includes the equations of the two lines. Match each line with its equation.

\[
\begin{align*}
y &= 2x + 6 \\
y &= 2x - 2 \\
y &= -2x - 2 \\
y &= -2x + 6
\end{align*}
\]

(d) Find the coordinates of points A, B, C, and D, knowing that the line segments CD and EF are parallel to the \( y \) axis.

(e) If the \( x \) coordinate of point E is 5, find its \( y \) coordinate and the coordinates of point F. (Does it look right on the graph?)

(f) Find the lengths of segments EF, CD, and AB. Does your result make sense? Why?

(g) Draw another segment connecting the two lines which is parallel to the \( y \) axis. Can you predict its length without knowing the coordinates of the endpoints? Explain. Would the equation help in this task? Why?

A Particularly Rich Curricular Example

The following problem is adapted from Resnick (1987, p. 158). To solve it calls for moving back and forth between graphs and equations using the Cartesian Connection, and both the process and object perspectives. We find the problem particularly rich in connections across cells in the framework, and think it provides a rich context for assessing the degree to which students have made such connections (as well as for inducing them).
representations come into play. It is also worth noting how many different ways there are to solve the problem. Examples of student work on it are discussed in the next section.

**NAVIGATING THROUGH THE CURRICULUM: TWO DATA STORIES**

As explained above, part of our research and development efforts involved the creation of a tutoring curriculum designed to introduce students to the ideas represented in the framework. Here we describe some of our rationale for curricular construction, and the work of some students who worked through the tutoring curriculum. Our intention is to show how this framework (and more generally, how a detailed analysis of what it means to understand a domain) can be used as a heuristic guide to curriculum development, and also to illustrate, with samples of student work, the complexity of knowledge development within the domain. We have two main story lines. The first illustrates a sequence of activities designed to have students become familiar with the object perspective. The second illustrates our emphasis on “flexible competence” rather than procedural mastery as an indicator of what it means to understand a domain. Here we paint with a broad brush. Detailed analyses of the student work (replications and extensions of Schoenfeld et al., in press) and of the character of the tutoring interactions (Arcavi & Schoenfeld, 1993; Schoenfeld et al., 1992) are in progress.

**Data Story 1. Coming to Grips With the Object Perspective**

It may appear that there is a natural progression from the process perspective to the object perspective, that one must first learn to “put a function together” and see how it works at the process level before one is capable of thinking of the function as an object. That is, how could one think about the graph of a function as an object until one could produce it—and how could one produce it except as a rule (in algebraic representation) or by plotting points (in graphical representation), using the process perspective? Indeed, absent the current technology, that may have been the only learning path available to students. Schoenfeld (1990, pp. 285-286) wrote autobiographically as follows:

In school, I learned to draw graphs (for concreteness, say the graph of \( y = x^2 + x - 3 \)) by calculating the \( y \)-values for different \( x \)-values (usually \( x = 0, \pm 1, \pm 2, \pm 3 \), etc.; more points if more detail was necessary), making a table of values, plotting the points from the table, and joining the plotted points with somewhat curvilinear segments. Early on the “overhead” for all the subsidiary operations was tremendous; after the trouble of calculating, plotting, and drawing, there was hardly the focus to reflect on the curves or their properties. . . . Over time and with extensive experience, I came to abstract the idealized mathematical curve from the empirical procedure. . . . Even so, this was only the first step in a long progression.

Consider what I saw when I compared the graphs of \( y = x^2 + x - 3 \) and \( y = x^2 + x - 7 \). The two looked similar. Moreover, having composed the tables for both, I know that the latter had the property that each of its \( y \)-values was precisely four units below the corresponding \( y \) value for the former. Nonetheless, this was a “point by point” comparison . . . . The mathematician also thinks of the two curves as representing the graphs of entities, where \( y = x^2 + x - 7 \) is the function (note the singular; it is one object!) obtained by “subtracting four” from the function \( y = x^2 + x - 3 \), and the graph of \( y = x^2 + x - 7 \) (again, singular; it is perceived of as a whole, single entity) is obtained by shifting the graph of \( y = x^2 + x - 3 \) four units downward in what is technically called vertical translation. At some point, I developed this understanding based on my ability to think of the functions and graphs themselves as conceptual entities—objects that, despite their complexity, I could think of as single, concrete manipulable objects. Later, I could begin to imagine the transformations as dynamic manipulations. Having graphed \( y = ax^2 \) for many values of \( a \), often on the same sheet, I could eventually “see” the idealized family of parabolas \( y = ax^2 \) vary continuously as \( a \) varied.

This description of conceptual development may make it seem as though there is a natural progression from process to object perspective, in which (a) one must master the former before grappling with the latter, and (b) the object perspective ultimately supersedes the process perspective, and mathematical “adepts” will work solely at the object level. We believe that neither assertion is correct. Regarding (b), we point to the solution of Problem 1 discussed in the introduction to this paper, noting that both the process and object perspectives were components of its solution. On that point, Kieran (1991, p. 252) wrote as follows:

The acquisition of structural conceptions by which expressions, equations, and functions are conceived as objects and are operated on as objects does not eliminate the continued need for the procedural conception . . . . Both play important roles in mathematical activity. However, very few studies have addressed the issue of the role and interaction of both conceptions in doing algebra. . . . The challenge to classroom instruction is to . . . . develop the abilities to move back and forth between the procedural and structural conceptions and to see the advantages of being able to choose one perspective or the other—depending on the task at hand.
Regarding (a), a main point in the design of GRAPHER was that with the help of the technology, students no longer had to follow the path just described.

The path we wish to make smoother with the help of the technology is a "learning trajectory" that results in the kinds of competence held by people who are fluent in the domain. This learning trajectory via the technology need not recapitulate the learning trajectory taken by those without it. With the help of the technology, students might (a) have different experiences with the mathematics, and (b) be able to deal with it in a different order or in different ways. For example, one need not start by having students graph very simple functions, if the computer will display complex functions that the students can analyze. Different sequencing made possible by the technology may allow for different kinds, and orders, of "scaffolding." (Schoenfeld, 1990, p. 285)

Our tutoring curriculum was designed to provide students with a set of experiences, early on, in which they manipulate graphs as objects. As such, it was intended to allow students to develop an intuitive feel for such manipulations—to give them an intuitive base for understanding slope as a parameter, and the correlation between values of $m$ and the orientation of lines of slope $m$. Later, when students encountered the formal definition of slope, that definition might help explain phenomena about which the students had an intuitive grasp.

As we see in the discussion of student work, coming to grips with the object perspective takes time. Being able to think of functions of the form $y = mx + b$ as members of a two-parameter family in which the parameters $m$ and $b$ (a) are independent, (b) determine the position of a line graphically, and (c) each "move" the graph of a line in particular ways as they are varied, requires making a lot of connections. There are two "morals" in the data story that follows. First, giving students access to the object perspective in the ways suggested in the previous paragraph seems to be of some use. Second, it is easy to read either too much or too little into student actions as one observes them working in the domain. Saying when a student actually "has" the object perspective is not a simple matter. It is not a yes/no kind of knowledge, but one of degrees, and the process of learning is not one of simple monotonic growth, but one that includes a fair amount of oscillation.

In the first unit of our tutoring curriculum, students were introduced to standard conventions of the Cartesian plane. They were then given a number of point-plotting exercises that resulted in the graphs of straight lines, such as "plot five points for which the $y$ coordinate is twice the $x$ coordinate plus 1. What do you notice?" These were followed by exercises that made similar links between tables and graphs (given a table, abstract the rule; write the equation, graph the function), tables to algebraic expressions (Guess My Rule), graphs to tables and equations (given a graph, make a table and find the equation), etc. The overall lesson of that first unit was this: "Certain tables, verbal statements of relations between $x$ and $y$ values, verbal and algebraic expressions of the form $y = (something)x + (something)$, and linear graphs in the plane, are different ways of representing the same thing."

The second unit of the curriculum begins with the Starburst problem (Magidson, 1993). Students are presented with Fig. 4.4. Their task is to reproduce the Starburst. They can type in equations of the form $y = mx$, and GRAPHER will produce the graphs of those equations. Note that this activity is used before students have seen the formal definition of slope. It is motivational and engages students for long periods of time; although it provides little formal structure, it allows students to make numerous observations about the relationship between the value of $m$ and the properties of the line.

![Fig. 4.4. A Starburst.](image)

Almost all students observe that larger positive values of $m$ produce steeper lines, but that the resulting spacing for $m = 1, 2, 3, 4, 5, \ldots$ is not uniform. Many observe that $m$ must be between 0 and 1 to produce a line between $0^\circ$ and $45^\circ$; many observe that the line $y = -mx$ produces the vertical reflection of $y = mx$, so that solving "half the problem" (getting even spacing for lines that have positive slope) will solve the whole problem. And they can do all this without knowing what slope is—thus laying an informal, empirical foundation for the formal definition.
As an example, AK, an eighth-grade student for whom our curriculum was the first introduction to the Cartesian plane, spent almost an hour working on Starburst. With the guidance of his tutor CK, he became aware of all of the issues just mentioned: that integer values of \( m \) produce nonuniform spacing, that values of \( m \) between 0 and 1 are required to produce lines that make angles between 0° and 45° with the positive x axis, and that lines of slope \( -m \) are the vertical “mirror images” of lines with slope \( m \). Asked “What does the number before the \( x \) do?” he wrote the following:

The number determines where the line goes. The smaller the number gets the closer the line is to the \( x \)-axis (when positive). And when it’s negative the bigger the number is the closer the line gets to the \( x \)-axis. When big and positive [it] gets closer to \( y \)-axis. When smaller and negative gets closer to \( y \)-axis.

Once one understands AK’s language use (note that his use of “larger” and “smaller” for negative numbers does not refer to absolute value but to position on the number line; he correctly refers to \(-0.1\) as a “large” negative number and \(-10\) as a “small” one), it is clear that AK’s statement is correct. And, AK has not only located the position of various lines as determined by their slopes, but he has used the language of change; it certainly appears that he has a sense of parametric variation. Hence, there is a strong temptation to say that AK has—at a correlational level rather than one of mechanism, because his conclusion is based on pattern recognition and he does not have any idea why particular values of \( m \) correspond to particular orientations—a good sense of the role of \( m \) as a parameter.

That was the goal of the Starburst task, of course. However, one must be careful in what knowledge one attributes to a student at any particular time. For example, the best case attribution for AK is that he understands one-parameter variation. As it happens, that attribution is too generous in some ways. A task given shortly after Starburst asked AK to “predict what the equations \( y = 10x \) and \( y = -10x \) would look like.” His sketch appears in Fig. 4.5.

Note that in labeling the point \((1, 10)\), AK used his knowledge of the process perspective: the equation \( y = 10x \) produces a \( y \) value of 10 for \( x = 1 \). The same holds for his graph of \( y = -10x \), which unambiguously passes through \((1, -10)\).

The next part of the problem asked: “Now graph the lines \( y = 10x \) and \( y = -10x \) using the computer. Was your prediction correct? Do you have any ideas that explain what happened?” AK’s response, “10 is the opposite of \(-10\) so they slant down in opposite directions,” indicates clearly that he had observed that the two lines, of opposite slope, are symmetric with regard to the \( y \) axis. Yet his graph, and his response, miss what to adepts in the domain may be the most salient feature of Starburst—the fact that lines of the form \( y = mx \) must pass through the origin! We stress that Starburst did a good deal of what it was intended to do in helping AK develop an intuitive sense of \( m \) as a parameter, but also that the observer must be cautious, for it is easy to read more than is warranted into student performance. It is most accurate to say, here, that AK had taken some important steps toward the development of an understanding of one-parameter variation.

We now fast-forward through two more units of the curriculum. In those two units, AK explored the properties of slope and \( y \) intercept. He encountered the informal definition of slope as “rise over run” and learned to use the slope formula \( m = \frac{y_2 - y_1}{x_2 - x_1} \); he learned to “read” the \( y \) intercept of a graph from the equation, and also from a table of values (that is, he understood that the \( y \) intercept occurred at the point on the graph when
x = 0, which corresponds to the tabular entry when x = 0); and he learned that parallel lines have the same slope. Toward the end of his sixth tutoring session, AK encountered Problem 11:

Problem 11. Draw different lines which cross exactly 2 quadrants.
- What can you say about the slope of all these lines?
- What can you say about the y intercept of all these lines?

The intent of the curriculum designers was to have the student conceive of the answer in terms of parametric families. Three families of lines satisfy the given condition: all vertical lines except for the y axis, all horizontal lines except for the x axis, and all lines except the two axes that pass through the origin. Algebraically, those families are:

\[ x = a \quad (a \neq 0) \]
\[ y = b \quad (b \neq 0) \]
\[ y = mx \quad (m \neq 0) \]

There is a subtle distinction to be made in understanding the solution to this problem, one that parallels the distinction between "variable as specific but unknown quantity" and "variable as quantity that takes on a range of values" discussed in footnote 4. It is one thing to characterize a given set of lines by referring to its parameter values; it is yet another to invoke a class of lines by referring to a range of parameter values. That is, in working Problem 7, students may observe that every specific line they generate that lies within the region has a slope value between 0 and 1, and say "all the lines have slope between 0 and 1." Although this statement is true, it does not necessarily invoke the family of lines of the form \( y = mx \) (where \( 0 < m < 1 \)). What we see in AK's work on Problem 11, discussed briefly below, is the way in which he comes to realize that a set of parameter values can invoke a family of lines.

AK began Problem 11 by interpreting it in a way that is typical of student approaches, illustrating the problem's tacit use of the conventional mathematics register (Pimm, 1987). Although the lines the student is told to draw are intended to be general (in that the student is intended to draw general conclusions from them), the problem statement permits a literal interpretation that is not general. Hence AK answered the first part of the problem by drawing \( y = x \) and \( y = -x \), and said their slopes were plus and minus 1 respectively. He noted that the y intercept of both lines is 0. AK then reread the last part of the problem, focusing on the world "all." After saying

"What can you say about the y intercept of all these lines?" he realized that vertical lines (except the y axis) also pass through precisely two quadrants. He realized that he was dealing with more than one family of graphs, and that caused some difficulty, as indicated by the following segment of dialogue with his tutor, CK:

CK: If the line is straight up and down what can you say about the y intercept?
AK: It doesn't have one.
CK: What if the line slants?
AK: It has one.
CK: Aha, so what kind of y intercept is it?
AK: It's a sometimes y intercept . . . it intercepts sometimes.

One can only speculate about mechanism, for there is scant evidence on the videotape, but it appears that the observation about vertical lines (all vertical lines save for the y axis meet the condition of the problem) led AK to realize that the two lines \( y = \pm x \) were only two examples of a class of lines that meet the two-quadrant criterion. From this point on he appears to be referring to the families of lines as partial solutions of the problem. And, he seeks a way to characterize those families. He and CK focus on nonhorizontal lines. When reminded of the problem condition, that the lines must pass through precisely two quadrants, he says:

So it's always going to be—two of them [that is, two of the lines that meet the desired criterion] are only gonna be \( y = -x \) and \( y = x \) . . . and you can't get it any other way . . . \( y = x \) equals something without a plus or minus, you can't have a plus or minus. So \( y = \pm x \) will go through the origin and work.

Here AK has come up with a verbal formulation of the relevant algebraic class: Implicitly he refers to equations without a constant term ("you can't have a plus or minus"), and in which the coefficient of \( x \) can vary (\( y = x \) and \( y = -x \) are prototypes, but \( y = \pm x \) will go through the origin and work"). Having classified the set of lines that pass through the origin, AK turns his attention to a classification of vertical lines. He has some difficulty, and then says:

When it's going vertical . . . any number going vertical will not touch the y axis except when it's \( x = 0 \), so . . . \( x \) can be anything, it's \( x \) equals anything [writes "\( x = \) anything" on the work sheet], any equation like that. Let's say
$x$ equals $n$ [writes "$x = n" on the work sheet]. So $x = n$, it will go through two quadrants.

In this segment of dialogue we see AK developing the understanding that the family of lines can be invoked by its parametric description. He is still a long way from understanding the family $y = mx + b; m, b \in R$ as a two-parameter family, where $m$ and $b$ are independent, but he has made tremendous strides. We pursue the issue of AK's learning trajectory—more generally, the issue of what it means to come to grips with subtle notions such as parametric representations—in the concluding discussion.

Data Story 2. On Flexible Understanding: The Importance of "Making Connections" Above and Beyond "Learning Procedures"

Here we discuss two students' work on Problem 10, which was introduced in the section entitled The Framework. We wish to highlight the facts that (a) there are multiple ways to solve the problem, and (b) solving the problem calls for making connections across representations and for employing both the process and object perspectives. These are two of the main reasons that the task is mathematically rich and interesting. Problem 10 begins the fifth unit of our tutoring curriculum. AK, with whom the reader is familiar from the discussion in Data Story 1, worked the problem at the beginning of his sixth tutoring session. CK was his tutor.

Absent the equations for the two slanted lines in Problem 10, parts (a) and (b) can only be answered in qualitative terms: The two lines have (the same) negative slope; one has a negative $y$ intercept, the other a positive $y$ intercept. On the basis of that information AK was able to make the proper selection from the candidates given in part (c) of the problem. He identified the lines correctly as $L_1: y = -2x - 2$ and $L_2: y = -2x + 6$.

It is interesting to note that AK then had some trouble with part (d). Having dealt with the lines holistically as objects, he found himself at a loss when asked to determine the coordinates of point A—even though he had just determined the equation of $L_2$. Looking at the picture, he said:

AK: It's zero something, it's $(0, x)$, but there's no way you can tell what $A$ is.

AK's problem was not caused by a lack of knowledge, but rather by a difficulty in seeing and pulling together the relevant information. When CK reminded him that he had determined the equations of the lines:

CK: Well . . . you know the equation for these lines, right? . . . 'cause you just did that.

AK's response was almost immediate:

AK: Oh, so it's six . . . so this is zero, six [writes down coordinates for A].

AK then estimated the $x$ coordinate of point D as $-1$ ("because it looks that way") and computed its $y$ coordinate as 8. He then turned to compute the coordinates of point B. Despite having just gone through the process, AK found himself at a loss once again: He needed to be reminded that he had determined the equation of $L_1$, and that he knew one of the coordinates of point B. Things progressed more smoothly for points C and D. AK computed the $x$ coordinate of C by solving the equation $-2x - 2 = 0$, and substituted that value into $L_2$ to obtain the $y$ coordinate of D. Moreover, he commented that it was sure to be right, as opposed to his earlier guess.

What we find notable here is that AK had all of the relevant information at his disposal, but that he found this part of the problem difficult. We believe that it is because the diagram leads one to focus on features of the lines as objects, while determining the coordinates of points A, B, C, and D requires a switch to the algebraic representation of the function, and to the process as well as object perspective.

AK began part (e) of Problem 10 by substituting $x = 5$ into the equation of $L_2$. Then, after having some difficulties with the coordinates of the various points he had labeled in the figure, he reverted to geometric reasoning. Observing that the vertical distance between the points C (-1,0) and D (-1,8) is 8, he drew several segments parallel to EF, saying:

AK: Eight . . . eight . . . eight . . . once I knew that this one [CD] was gonna be eight, I knew the other one [EF] was gonna be eight.

His tutor, pushing for him to make the connection to the equations, asked:

CK: Could you see it [the distance of 8] from the equations?

and AK responded as follows.

AK: Look, they are 8 apart [points to the two equations] they are 8 apart because 2 and $-6$ is 8 apart.

In brief: Although AK had much of the "within representation" and "within perspective" knowledge required for this problem, solving it calls for moving flexibly across representations and perspectives where necessary or appropriate. Such flexibility does not come easily. Much of AK's difficulty came when he had to shift gears—or more precisely, when he needed to shift from one perspective or representation to another.
We turn now to a second student-tutor pair. AU was an eighth grader, the one student who entered the curriculum with some prior knowledge of graphs of linear functions (she often worked mathematics problems with her father in the evenings). Her tutor was AS, one of the authors of this paper.

AU moved smoothly through parts (a) through (c) of Problem 10. She, like AK, was at first stymied by part (d). An interjection by the tutor pointed her to the equation:

AS: Let's take one point at a time. Given what you know, which is the equation of this line [points to L1] and that line [points to L2], can you tell me what the value of A is?

AU: It's 6. I get it now. This [the y coordinate of A] is 6, so this [points at D] would be... well...

AS: Let's take them one at a time. A was 6. The next one is B...

AU: Negative 2.

AS: How about C?

AU: Negative 1 I think.

AS: How did you get that?

AU: Negative 2x negative 1 is 3 minus 2 equals... Wait, no. I mean if you substitute x for negative 1 [in y = -2x - 2] that equals 3 - Wait... equals 2 minus 2, equals zero.

AS: So that's (-1,0) [pointing to C on the work sheet].

AU took a different approach when identifying the coordinates of point J. She observed that the vertical distance from C to D is the same as the vertical distance from A to B (which is 8); hence, the y coordinate of D is 0 + 8 = 8, and the coordinates of D are (-1, 8). And she employed yet another method for part (e) of the problem, using the slope formula (!) to obtain the y coordinate of the point (5,y) on L2. [She knew the slope of L2 o be -2; since the points (0,6) and (5,y) lie on L2, it follows that -2 = (y - 6)/5, and that y = -4.] Parts (f) and (g) of Problem 10 were solved using geometric reasoning.

We make two observations. First, AU's difficulty came at a point in the problem that called for changing perspectives: She had dealt with parts (a) through (c) of the problem using the object perspective, but she needed to move to the process perspective (substituting into the equations) to solve part (d). AU, like AK, was stymied at a point that called for a shift in perspective. Second, a major cause of AU's overall success was that, in general, she managed to move flexibly between the process and object perspectives. She solved or substituted into equations (process perspective) where necessary to determine the coordinates of particular points, but exploited global properties of the lines (object perspective) when considering their slope, or the distance between them. As discussed next, such flexibility is a hallmark of competence.

CONCLUDING DISCUSSION

Our first major goal in writing this chapter was to introduce and elaborate the framework for understanding functions that was outlined in schematic form in Table 4.1. There we pointed to two ways of viewing functions (the process and object perspectives) and the three most prominent representations of functions (in tabular, graphical, and algebraic form). We hope to have indicated that competence in the domain consists of being able to move flexibly across representations and perspectives, where warranted: to be able to "see" lines in the plane, in their algebraic form, or in tabular form, as objects when any of those perspectives is useful, but also to switch to the process perspective (in which an x value of the function "produces" a y value), where that perspective is appropriate. As the data indicate, developing such flexibility is difficult: AK and AU, good students both, faltered at points in Problem 10 where a change in perspective was called for. Again, we stress that Table 4.1 is incomplete; there is much that it leaves out. Despite this limitation, however, the table has much to offer.

For us, now augmented by columns representing "real world contexts" and "verbal representations," it serves both as a heuristic guide to curriculum development (Does any curriculum we propose make adequate connections across representations and perspectives? If not, it had better be revised) and for understanding and assessing student learning (Can the student move flexibly across representations and perspectives when the task warrants it?).

We suggest that the approach illustrated here—seeing understanding as making connections, and analyzing content domains to see what kinds of connections competent practitioners make—will be a profitable approach for both curriculum development and (student and curriculum) assessment. Indeed, our second major goal has been to focus the reader's view on this version of understanding and away from more procedurally oriented

5As far as we can tell, AU guessed/intuited the solution after she examined the equation -2x - 2 = 0, and she then checked her conjectured solution by seeing if x = -1 satisfied the equation.

6Note that although our discussion has focused primarily on linear functions, everything suggested in Table 4.1 applies to functions that can be expressed in closed form.
definitions such as "The student understands linear functions when she/he has mastered and can use the point-slope formula, the two-point formula (etc.) when appropriate." One of the points we hope emerged from the discussion of AK's and AU's work is that they produced delightfully different solutions to parts of Problem 10, and that such variation should be treasured and encouraged. Procedural competence is a component of understanding, but it should not be mistaken for the real thing. As a case in point, we highlight recent work by Dowker (1991, 1992). Dowker asked 44 mathematicians to perform a range of numerical estimation tasks. Not only did she discover a huge amount of variation in the procedures employed by the mathematicians—as many as 23 different strategies for one problem!—but she found that the experts were not consistent in the ways they approached the problems. Eighteen of the mathematicians were tested on the same problems a few months later. The mathematicians frequently used different strategies to solve the same problems: a minimum of 9 alternate strategies on the 20 problems, a maximum of 17 out of 20. In short, real competence consists of being able to get the job done easily, not in doing it the same way over and over again. We should teach for connections and understanding, not merely for procedural skills.

A third goal was to remind the reader of the subtlety and complexity of the learning process. The work here, both in research methodology and in its view of the domain, is grounded in the data provided in Schoenfeld et al. (in press). In particular, we believe that the growth and change of knowledge (i.e., learning) is a slow and complex affair. Coming to grips with parametrization, for example—recall Data Story 1—is not easy regardless of the visual assistance that computer-based technology can offer. Further exegeses of the learning process in this domain, which were only hinted at in Data Story 1, are in progress.

A fourth and final top-level goal was to try to place the use of instructional graphing software in a reasonable perspective. It should be clear that we have an investment in such media, and believe they can help change things for the better: GRAPHER as a tool, and tasks such as Starburst, offer students the opportunity to deal with some aspects of functions, and develop some intuitions, in ways (and sequences) that were simply inaccessible prior to the availability of computer-based tools. Specifically, the software allows students to operate on equations and graphs as objects, and that may facilitate the development of the object perspective in ways not possible before the existence of such technologies. Having said this, however, we must repeat the caution discussed in the background section: The action is not what takes place on the screen, but is what the individual student puts together for herself or himself. And coming to grips with complex mathematical notions will take time and experience. In short, we are somewhat technophilic but hope not to be naive technophiles. We hope to develop, in a principled way, a deeper understanding of what it means to understand complex mathematical domains, and to exploit available resources to help students develop such understandings. Perhaps the work described here will be a step in that direction.

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