Supporting Mathematical Reasoning and Sense Making for English Learners

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English learners are a large and growing population in U.S. schools. In 2001, 4.5 million K–12 students in public schools (9.3 percent) were labeled as English learners (Tafoya 2002). Between 1979 and 2006, the number of school-aged children (aged 5–17 years) in the population who spoke a language other than English at home more than doubled, increasing from 3.8 million (9 percent) to 10.8 million (20 percent) (Planty et al. 2008). Most English learners in the United States are Latinos/Latinas. In 2006, about 72 percent of school-aged children who spoke a language other than English at home spoke Spanish (Planty et al. 2008). In some states the numbers are even greater. For example, in California, 25 percent (1.5 million) of the children in public school in 2001 were labeled English learners, and 83 percent of those children spoke Spanish as their primary language (Tafoya 2002).

As the population of English learners increases in U.S. public schools, so do concerns with the needs of these students in mathematics classrooms. Although language issues are important to consider in all mathematics classrooms, we seem most concerned with issues of language as they arise in classrooms with students who are learning English as a second language. Perhaps language issues seem more salient when the teacher and students obviously do not share a common language for instruction. However, as many mathematics teachers know, English learners and native speakers share some of the same challenges. For example, many students who are native English speakers have trouble understanding and solving word problems. We also know that students who are native English speakers may use words with meanings that differ from the mathematical meanings of words in a textbook. A native English speaker might respond to the question “Is a square a rectangle?” by saying “No, it’s not,” because the student is not using the mathematical meaning of the word square.

We could imagine that the solution to the problem of mathematics instruction for English learners involves a quick fix: new manuals for teachers, a new piece of software, a new teaching method, and so on. Unfortunately, such solutions risk reinforcing myths about language, how we learn a second language, and how we learn mathematics. This chapter’s goal is to suggest recommendations for teaching practices that are based on research rather than on myths. A commitment to improving mathematics learning for all students—and especially for students who are learning English—prompts these recommendations. I begin with the assumption that language is not the problem. Although the chapter does not offer recipes for teaching or point to a quick fix, I hope that
these recommendations will help to guide teachers in developing their own approaches to supporting mathematical reasoning and sense making for students who are learning English.

Two issues specific to English learners are crucial to consider for discussing instruction and assessment policies for this student population. First, the label “English learner” as currently used in the United States is vague, has different meanings, is not based on objective criteria, does not reflect sound classifications, and is neither comparable across states nor equivalent across settings. This label is likely to be used as a proxy for demographic labels rather than as an accurate portrayal of students who are learning English (Gándara and Contreras 2009). Second, language proficiency is a complex construct that can reflect proficiency in multiple contexts, modes, and academic disciplines. Current measures of language proficiency may not accurately reflect an individual’s language competence. In particular, we do not have measures or assessments for language proficiency related to competence in mathematics for different ages or mathematical topics. These two issues—the label “English learner” and the complexities of language proficiency—can bring confusion into any discussion of mathematics instruction for students who are learning English. Thus, instructional decisions should not be made solely on the basis of the label “English learner.” I will sometimes use the phrase “students who are learning English” to highlight that these students may or may not be labeled as English learners. I will also sometimes use the phrase “students who are bilingual (and may be learning English)” to emphasize that students who are learning English are also bilingual.

Generalizing about the instructional needs of all students who are learning English is difficult. Specific information about students’ previous instructional experiences in mathematics is crucial for understanding how bilingual learners communicate in mathematics classrooms. Knowledge of students’ experiences with mathematics instruction, language history, and educational background should guide classroom instruction. In addition to knowledge of the details of students’ experiences, research suggests that high-quality instruction for English learners that supports student achievement has two general characteristics: (1) a view of language as a resource rather than a deficiency and (2) an emphasis on academic achievement, not only on learning English (Gándara and Contreras 2009).

Research provides general guidelines for instruction for this student population. Overall, English learners, students who are learning English, and bilingual students are from nondominant communities. They need access to curricula, instruction, and teachers that have proven to be effective in supporting the academic success of these students. The general characteristics of such environments are that curricula furnish “abundant and diverse opportunities for speaking, listening, reading, and writing” and that instruction “encourage students to take risks, construct meaning, and seek reinterpretations of knowledge within compatible social contexts” (Garcia and Gonzalez 1995, p. 424). Some characteristics of teachers who have succeeded with students from nondominant communities are (a) a high commitment to students’ academic success and to student–home communication, (b) high expectations for all students, (c) the autonomy to change curriculum and instruction to meet the specific needs of students, and (d) a rejection of models of their students as intellectually disadvantaged. Curriculum policies for English learners in mathematics should follow the guidelines for traditionally underserved students (American Educational Research Association 2006), such as instituting systems that broaden course-taking options and avoiding systems of tracking students that limit their opportunities to learn and delay their exposure to college-preparatory mathematics coursework.

Mathematics instruction for English learners should follow the general recommendations for high-quality mathematics instruction: (a) students should focus on mathematical concepts and connections among those concepts, and (b) teachers should reinforce high cognitive demand and maintain the rigor of mathematical tasks—for example, by encouraging students to explain their problem solving and reasoning (American Educational Research Association 2006; Stein, Grover, and Henningsen 1996). Research in mathematics education ascribes two central features to teaching that promote conceptual development: (1) teachers and students attend explicitly to concepts, and (2) students wrestle with important mathematics (Hiebert and Grouws 2007).

In particular, recommendations for supporting English learners in developing literacy (American
Educational Research Association 2004) include providing (a) structured academic conversation built around text and subject-matter activities to develop vocabulary and comprehension and (b) several years of intensive, high-quality instruction to help students master the vocabulary, comprehension, and oral language skills that will make them fully fluent in speaking, reading, and writing English.

Assessment is also important to consider for English learners, because this student population has a history of being inadequately assessed. LaCelle-Peterson and Rivera (1994, p. 56) write that English learners “historically have suffered from disproportionate assignment to lower curriculum tracks on the basis of inappropriate assessment and as a result, from over-referral to special education (Cummins 1984; Durán 1989).” Previous work in assessment has described practices that can improve the accuracy of assessment for this population. Assessment activities should match the language of assessment with the language of instruction and include measures of content knowledge assessed through the medium of the language(s) in which the material was taught (LaCelle-Peterson and Rivera 1994). Assessments should be flexible in modes (oral and written) and length of time for completing tasks.

Assessments should track content learning through oral reports and other presentations rather than relying only on written or one-time assessments. When students are first learning a second language, they can display content knowledge more easily by showing and telling rather than through reading text or choosing from verbal options on a multiple-choice test. Therefore, discussions with a student or observations of hands-on work will yield more accurate assessment data than written assessments. Finally, evaluation should be clear on the degree to which it measures fluency of expression, as distinct from substantive content. This last recommendation raises an interesting question for assessing English learners’ mathematical proficiency. For classroom assessments that are based on mathematical discussions, how can we evaluate content knowledge as distinct from fluency of expression in English? The first example in the following shows how instruction and assessment during classroom discussions can focus on mathematical content and reasoning rather than on fluency of expression in English. (Two previous publications [Moschkovich 1999, 2007a] also offer examples of focusing assessment and instruction on mathematical content and reasoning.)

Overall, research on language and mathematics education for this student population offers a few guidelines for instructional practices for teaching mathematics to English learners:

- Treat language as a resource, not a deficit (Gándara and Contreras 2009; Moschkovich 2000).
- Draw on multiple resources available in classrooms—such as objects, drawings, graphs, and gestures—as well as home languages and experiences outside school.

English language learners, even as they are learning English, can participate in discussions where they grapple with important mathematical content (Moschkovich [1999] and Khisty [1995] offer example lessons). Instruction for this population should not emphasize low-level language skills over opportunities to actively communicate about mathematical ideas. One goal of mathematics instruction for students learning English should be to support all students, regardless of their proficiency in English, in participating in discussions that focus on important mathematical concepts and reasoning rather than on pronunciation, vocabulary, or low-level linguistic skills. By learning to recognize how English learners express their mathematical ideas as they are learning English, teachers can maintain a focus on mathematical reasoning as well as on language development.

Research also describes how mathematical communication is more than vocabulary. Although vocabulary is necessary, it is not sufficient. Learning to communicate mathematically is not merely or primarily a matter of learning vocabulary. During discussions in mathematics classrooms, students are also learning to describe patterns, generalize, and use representations to support their claims. The
question is not whether students who are English learners should learn vocabulary but rather how instruction can best support students as they learn both vocabulary and mathematics. Vocabulary drill and practice is not the most effective instructional practice for learning either vocabulary or mathematics. Instead, experts in vocabulary and second-language acquisition find that vocabulary acquisition in a first or second language is most successful in instructional contexts that are language rich, actively involve students in using language, require both receptive and expressive understanding, and require students to use words in multiple ways over extended periods (Blachowicz and Fisher 2000; Pressley 2000). To develop written and oral communication skills, students need to participate in negotiating meaning (Savignon 1991) and in tasks that require output (Swain 2001). In sum, instruction should furnish opportunities for students to actively use mathematical language to communicate about and negotiate meaning for mathematical situations.

I will use classroom examples to address the following questions:

- How can teachers create opportunities for mathematical reasoning and sense making for high school students who are learning English?
- How can teachers use strategies that develop English learners’ reasoning and sense-making skills?
- How can teachers help students communicate their reasoning effectively in multiple ways?

The following recommendations show three ways to guide mathematics instruction for English learners:

1. Focus on students’ mathematical reasoning, not proficiency in English.
2. Treat home language and everyday experiences as resources, not obstacles.
3. Build on student reasoning and connect student reasoning to mathematical concepts.

First, all the examples in this chapter are of Spanish-speaking students; Latino/a mathematics learners are the focus of my research projects (this is not a coincidence, since Spanish is my own first language). All the examples come from classrooms with many Latino/a students (also not a coincidence since I conducted my research in such classrooms). Because most U.S. English learners are Latinos/as (Planty et al. 2008; Tafoya 2002), these examples are highly relevant to practices in classrooms with Latino/a English learners. However, the examples’ general principles and, more important, the recommendations are not specific to Spanish or to Latino/a students. The three recommendations are relevant to all English learners and come from the research literature on English learners.

Second, all three examples involve students with some proficiency in English and who may be working in a bilingual mode. Thus, these are not examples of newcomers. Newcomers—students who are beginning English learners and are working in a monolingual mode in their first language—require instruction in their home language, by either a bilingual teacher or a bilingual aide. Without use of the home language, newcomers cannot benefit from mathematics instruction. The challenge of offering such bilingual instruction for newcomers is the responsibility of the school system, not of individual teachers, who for the most part are likely to be monolingual English speakers.

Finally, all the examples are of middle school or ninth-grade topics and focus on algebra readiness rather than algebra or more advanced mathematics. Again, this is a result of the focus of my research projects. Understanding students’ experiences with algebra is of practical importance because algebra can be a significant barrier to educational progress, and the transition from arithmetic to algebraic thinking is a watershed where many students have trouble with or give up on mathematics. However, do not conclude from these three examples that mathematics instruction for English learners should in any way be less rigorous than that for other students or that the content for English learners should be limited to algebra readiness. In fact, schools must assess English learners in mathematics in their home language to determine the appropriate level of mathematics instruction. As mentioned, English learners have a history of inadequate assessment (LaCelle-Peterson and Rivera
1994), and these students have suffered from disproportionate assignment to lower curriculum tracks on the basis of inappropriate assessment. With proper assessment, some English learners could be placed in more advanced mathematics courses. All English learners should have access to courses that focus on rigorous mathematics content.

**Focus on Students’ Mathematical Reasoning, Not Proficiency in English: Describing a Pattern (Example 1)**

The first example (Moschkovich 2002, 2007a) is from a classroom of grade 6–8 students in a summer mathematics class. The students constructed rectangles with the same area but different perimeters and looked for a pattern to relate the dimensions and the perimeters of their rectangles. Here is a problem similar to the one students were working on:

1. Look for all the rectangles with area 36 (length and width are integers), and write down the dimensions.
2. Calculate the perimeter for each rectangle.
3. Describe a pattern relating the perimeter and the dimensions.

This classroom had one bilingual teacher and one monolingual teacher. Four students were videotaped as they talked with each other and with the bilingual teacher (primarily in Spanish). They attempted to describe the pattern in their group and searched for the Spanish word for “rectangle.” The students produced several suggestions, including ángulo [angle], triángulo [triangle], rángulos, and ranguenos. Although these students attempted to find a term to refer to the rectangles, neither the teacher nor the other students supplied the correct Spanish word, rectángulo [rectangle].

After the small-group discussion, a second teacher (monolingual English speaker) asked several questions from the front of the class. In response, a student in this small group, Alicia, described a relationship between the length of the sides of a rectangle and its perimeter. (Transcript annotations are in brackets. Translations are in italics).

**Teacher B:** [Speaking from the front of the class] Somebody describe what they saw as a comparison between what the picture looked like and what the perimeter was.

**Alicia:** The longer the ah . . . the longer [traces the shape of a long rectangle with her hands several times] the ah . . . the longer the rángulo [rangle], you know, the more the perimeter, the higher the perimeter is.

Focusing only on this student’s failed attempt to use the right word in English, “rectangle,” would cause an observer to miss this student’s mathematical reasoning. If we were to focus only on Alicia’s inaccurate use of the term “rángulo” (although the word does not exist in Spanish, it might be best translated as “rangle,” perhaps a shortening of the word “rectángulo”), we would miss how she used resources from the situation and how her statement reflects mathematical reasoning and sense making. Alicia’s mathematical reasoning and sense making become visible only if we include gestures and objects as resources. This move is important for instruction because it shifts the focus from a perceived deficiency in the student that needs to be corrected—not using the word “rectangle”—to a competency that can be refined through instruction: using gestures and objects to reason mathematically. This move also shifts our attention from using the correct word in English to mathematical reasoning, as expressed not only through words but also using other modes. This shift is particularly important to uncover the mathematical reasoning in contributions from students who are learning English.

Alicia used gestures to illustrate what she meant, and she referred to the concrete objects in front of her, the drawings of rectangles, to clarify her description. Alicia also used her first language as a
resource. She interjected an invented Spanish word into her statement. In this way, a gesture, objects in the situation, and the student’s first language served as resources for describing a pattern. Even though the word that she used for “rectangle” does not exist in either Spanish or English, Alicia was referring to a rectangle. Her gestures showed that, even though she did not use the words “length” or “width,” she was referring to the length of the side of a rectangle parallel to the floor.

Alicia’s description reflects correct mathematical reasoning. The rectangle with area 36 that has the greatest perimeter (74) is the rectangle with the longest possible length, 36, and shortest possible width, 1. As the length gets longer—say, when one compares a rectangle of length 12, width 3, and perimeter 30 with a rectangle of perimeter 74—the perimeter does in fact become greater. This example illustrates how important it is to focus on a student’s mathematical reasoning and not only on his or her proficiency in English. It also illustrates how gestures and objects offer resources for mathematical reasoning and sense making.

Certainly, Alicia needs to learn the word for “rectangle” (ideally in both English and Spanish), but instruction should not stop there. Rather than only correcting her use of “rángulo” or recommending that she learn vocabulary, instruction should also build on Alicia’s use of gestures, objects, and description of a pattern. If instruction focuses only on what mathematical vocabulary English learners know or don’t know, they will always seem deficient, because they are, in fact, learning a second language. If teachers perceive these students as deficient and only correct their vocabulary use, little room remains for addressing their mathematical reasoning, building on this reasoning, and connecting their reasoning to the discipline. English learners thus risk getting caught in a cycle of remedial instruction that does not focus on mathematical reasoning.

Treat Home Language and Everyday Experiences as Resources, Not Obstacles: Clarifying a Description (Example 2)

Whereas the first example fits the expectation that English learners struggle with vocabulary, the second illustrates how both languages—the language of the home and the language of instruction—can offer resources for mathematical reasoning. In the following discussion, students used both languages to clarify the mathematical meaning of a description. The example is from an after-school interview with two ninth-grade students. The students had been in mainstream English-only mathematics classrooms for several years. One student, Marcela, had some previous mathematics instruction in Spanish. These two students were working on the problem in figure 2.1 (see p. 23). They had graphed the line $y = -0.6x$ on paper (fig. 2.2) and were discussing whether this line was steeper than the line $y = x$.

In a conversation preceding this excerpt, Giselda had first proposed that the line was steeper and then that it was less steep. Marcela had repeatedly asked Giselda whether she was sure. In the following discussion, after Marcela proposed that the line was less steep, we see how she explained her reasoning to Giselda. (Transcript annotations are in brackets. Translations are in italics after the phrases in question.)

1 Marcela: No, it’s less steeper . . .

2 Giselda: Why?

3 Marcela: See, it’s closer to the x-axis . . . [looks at Giselda] . . . isn’t it?

4 Giselda: Oh, so if it’s right here . . . it’s steeper, right?

5 Marcela: Porque fijate, digamos que este es el suelo. [Because look, let’s say that this is the ground.] Entonces, si se acerca más, pues es menos steep. [Then, if it gets closer, then it’s less steep.] . . . ‘cause see this one [referring to the line $y = x$] . . . is . . . está entre el medio de la x y de la y [is between the x and the y]. Right?
8a. If you change the equation $y = x$ to $y = -0.6x$, how would the line change?

A. The steepness would change. [NO] [YES] Steeper
   Why or why not?

Fig. 2.1. Problem for example 2

Fig. 2.2. Lines that Marcela and Giselda drew

6 Giselda: [Nods in agreement.]

7 Marcela: This one [referring to the line $y = -0.6x$] is closer to the $x$ than to the $y$, so this one [referring to the line $y = -0.6x$] is less steep.

In this discussion, the two students were negotiating and clarifying the meanings of “steeper” and “less steep.” Marcela used her first language, code switching (the practice of using two languages during one conversation or within one sentence), mathematical artifacts—the graph, the line $y = x$, and the axes—and everyday experiences as resources to reason mathematically. The premise that
meanings from everyday experiences are obstacles for mathematical reasoning does not hold here. In fact, Marcela used her everyday experiences and the metaphor that the \( x \)-axis is the ground ("Porque fíjate, digamos que este es el suelo" [Because look, let's say that this is the ground]) as resources for making sense of this problem. Rather than finding everyday meanings as obstacles, she used an everyday situation to clarify her reasoning.

What was Marcela’s mathematical reasoning? Marcela explicitly stated an assumption when she said, “Porque fíjate, digamos que este es el suelo” [Because look, let's say that this is the ground]. She supported her claim by making a connection to mathematical representations. She used the graph, in particular the line \( y = x \) (line 5) and the axes (lines 5 and 7), as a reference to support her claim about the steepness of the line. Marcela was participating in two important aspects of mathematical reasoning: stating assumptions explicitly and connecting claims to mathematical representations.

Research does not support viewing code switching as a linguistic deficit (Valdés-Fallis 1978; Zentella 1981). In fact, the opposite is true. Although code switching has an improvised quality, it is a complex, rule-governed, and systematic language practice reflecting speakers’ understanding of their community’s linguistic norms. The most significant reason for a bilingual student’s language choice is the language choice of the person addressing the student. We should not assume that bilingual students switch into their first language because they are missing English vocabulary or cannot recall a word. Neither should we assume that code switching is evidence of a deficiency in a student’s mathematical reasoning. Code switching can offer resources for communicating mathematically (Moschkovich 2007b, 2009). For example, students sometimes code switch as they describe a mathematical situation, explain a concept, justify an answer, elaborate on an explanation, or repeat a statement.

**Build on Student Reasoning and Connect Student Reasoning to Mathematical Concepts: Describing the Scales on Two Graphs (Example 3)**

The third example shows how one teacher enacted instruction that promotes conceptual development by attending explicitly to concepts and allowing students to wrestle with important mathematics (Hiebert and Grouws 2007). This teacher used a mathematical discussion as an opportunity to make student reasoning visible, build on that student reasoning, and connect student reasoning to a mathematical concept. This teacher also balanced explicit attention to concepts while still allowing students themselves to wrestle with important mathematics. The teacher accomplished this not by setting a problem but by transforming student questions into a problem that makes connections.

A focus on discussions that make mathematical reasoning visible is particularly important for classrooms with students who are bilingual or learning English. On the one hand, we could imagine that mathematical discussions would be difficult to create and maintain in such classrooms. After all, if all we can see is that these students are struggling with language, we might be concerned that an emphasis on mathematical discussions will make these students look less competent than would traditional computational work. The following discussion is a counterexample to the imagined difficulties that students who are bilingual or learning English might face in discussing their mathematical reasoning. This mathematical discussion shows that bilingual students can, in fact, participate in discussions of their mathematical reasoning. The question is not whether bilingual students can engage in such discussions but how to support bilingual students in participating in discussions, making sense of mathematical concepts, and learning to communicate their reasoning. Too often, descriptions of bilingual students focus on the obstacles they face in understanding text or utterances in English, and these misunderstandings are invariably ascribed to their lack of proficiency in their
second language. In contrast, the following discussion shows that we can see multiple ways of reasoning as reflecting how a student is reasoning about an important mathematical concept rather than as the result of language difficulties.

**Background**

Carlos and David are students in an eighth-grade bilingual class in an urban area in the United States. They are both bilingual native Spanish speakers. In this classroom, teachers and students use both languages depending on the setting and participants. The class was conducted mostly in English, with some discussions and explanations in Spanish. Carlos and David arrived in the United States from Central America as young children and have both been in a bilingual program since the early grades in elementary school. They report sometimes speaking Spanish at home, and in the classroom they seem to switch easily and fluidly between monolingual and bilingual modes (Grosjean 1999).

When discussing a mathematics problem together, they sometimes used words, phrases, or extended talk in Spanish. When talking to the teacher, they used mostly English. Thus, they represent an important and significant segment of the U.S. student population: those students who would not be labeled as Spanish dominant but may still be learning academic English.

Although both students are bilingual, I selected a discussion that transpired in only one language, English, on purpose and for several reasons. First, their discussion reminds us that many conversations in bilingual classrooms take place in only one language, the language of instruction. Second, and perhaps more important, their discussion highlights how multiple ways of reasoning were not caused by using more than one language but were connected to the negotiation of mathematical meanings. I use this discussion to illustrate how the teacher built on students’ mathematical reasoning and connected their reasoning to a mathematical concept. However, first I will need to show the reasoning that was taking place.

The transcript comes from a larger set of data that I collected from this classroom. (Classroom observations and videotaping were conducted during two curriculum units from Connected Mathematics [Lappan et al. 1998], “Variables and Patterns” and “Moving Straight Ahead.” Data collected included videotapes of whole-class discussions and at least one student group for every lesson, as well as videotaped problem-solving sessions in pairs. Moschkovich [2008] analyzes this transcript in more detail, examining the multiple meanings students used for statements of the form “I went by,” describing how multiple meanings and competing claims arose, showing how the students coordinated meanings with views of graph scales, and exploring the teacher’s role during this discussion.)

This discussion occurred during the unit “Moving Straight Ahead” from Connected Mathematics (Lappan et al. 1998) toward the beginning of a classroom period. The teacher usually started the ninety-minute class with a brief whole-class discussion about a mathematics problem (the problem for that day, a problem from a previous lesson, or a homework problem). Students then worked in groups of two to four, discussing the problem at their tables. The teacher moved from group to group, asking and answering questions. Toward the end of the class period, each group usually reported or gave presentations, and the teacher led whole-class discussions. On the day of this discussion, students expected that each group would go to the front of the classroom to explain their graphs, describe how and why they solved a problem as they had, and answer questions from other students and the teacher.

The class had been working on several problems about a five-day bicycle tour. In the story, whereas some riders rode bicycles, others rode in a van and recorded the total distance from the starting point for the van every half-hour. The problem in figure 2.3 refers to the second day of the tour:
On the second day of their bicycle trip, the group left Atlantic City and rode five hours south to Cape May, New Jersey. This time, Sidney and Sarah rode in the van. From Cape May, they took a ferry across the Delaware Bay to Lewes, Delaware. Sarah recorded the following data about the distance traveled until they reached the ferry.

<table>
<thead>
<tr>
<th>Time (hours)</th>
<th>Distance (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>8</td>
</tr>
<tr>
<td>1.0</td>
<td>15</td>
</tr>
<tr>
<td>1.5</td>
<td>19</td>
</tr>
<tr>
<td>2.0</td>
<td>25</td>
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<td>2.5</td>
<td>27</td>
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<td>3.0</td>
<td>34</td>
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<td>3.5</td>
<td>40</td>
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<td>4.0</td>
<td>40</td>
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<tr>
<td>4.5</td>
<td>40</td>
</tr>
<tr>
<td>5.0</td>
<td>45</td>
</tr>
</tbody>
</table>

1. Make a coordinate graph of the (time, distance) data given in the table.

2. Sidney wants to write a report describing day 2 of the tour. Using information from the table and the graph, what would she write about the day’s travel? Be sure to consider the following questions:
   A. How far did the group travel in the day? How much time did it take them?
   B. During which interval(s) did the riders make the most progress? The least progress?
   C. Did the riders go farther in the first half or the second half of the day’s ride?

3. By analyzing the table, how can you find the time intervals when the riders made the most progress? The least progress? How can you find these intervals by analyzing the graph?

Fig. 2.3. Problem: from Atlantic City to Lewes

Carlos and David often worked together in a small group. We join their discussion in progress (episodes 2–4 and with numbered lines) as they compare their answers to this problem and review the graphs that each had created independently for homework. As the discussion began, David and Carlos looked at their graphs and noticed that these graphs looked different (figs. 2.4 and 2.5).
Carlos and David began the discussion of their solutions to the homework problem by considering whether their graphs were the same. When the teacher joined their group, Carlos asked her how they were supposed to do the graph. She responded first by asking them to compare their axes. The teacher asked the students to consider what was the same and what was different about the two graphs. She first considered whether David and Carlos had the same variable assigned to each axis and concluded that they both had put time on the x-axis. Next she considered what was different about the two graphs and suggested that it was how David and Carlos had “placed their numbers.”

Next, the teacher and the students described the scales on their graphs. Carlos began to describe how he labeled the axes of his graph (fig. 2.4), making tick marks every two segments. In contrast, David had labeled the axes of his graph (fig. 2.5) with tick marks on every segment. (When reading the transcript, continue to refer to the graphs to understand the students’ ways of making sense of the marks on the scales. Comments and gestures are in brackets.)
Episode 2: Describing the shape of the curve

42  **Carlos:** Oh, that's true, 'cause I went by twos, I went 1, 2, ... and then I put that one ... he went by one.

43  **Teacher:** Aha, you skipped one [referring to a segment on the scale on Carlos's graph]. So how does that change how it looks?

45  **Carlos:** Cause it doesn't go up as far, it only goes, it's more steeper. It looks more steeper.

46  **Teacher:** Remember. Similar to the difference between this one and ... and this one here. Right?

48  **Teacher:** [The teacher makes a sign with right thumb and index finger of her hand to show interval differences on their papers. Then she points to a graph on the blackboard. Next she points to a second graph. The first and second graphs have different scales on the x-axis so that the second graph is compressed along the x direction.]

49  **David:** That one.

50  **Carlos:** Yeah.

51  **Teacher:** Here the numbers are closer together so it looks steeper. Other than that, are they the same graph?

52  **Teacher:** [The teacher makes a sign with thumb and index finger. Then she gestures upward with her right hand.]

53  **Carlos:** No, also here in the x-axis.

54  **Carlos:** [Carlos point to the x-axis on his paper.]

55  **David:** [David mutters and points to the axis on his paper.]

56  **David:** I went by twos.

57  **Carlos:** This is the x-axis. Right?

58  **Carlos:** [Carlos points to the axis.]

59  **Teacher:** This is the y-axis ... this is the x-axis.

60  **Teacher:** [The teacher sweeps her pencil vertically to represent the y-axis and then horizontally to represent the x-axis.]

During episode 2, Carlos introduced the phrase "I went by twos" (line 42) to describe his own y-axis scale and the phrase "he went by one" (line 42) to describe David's scale. Carlos continued to use these phrases during episodes 2 and 3 as he described how he had labeled the axes of his own graph. Turning to the graphs (figs. 2.4 and 2.5), we can see that Carlos had labeled his axes by making a tick mark at every two-grid segment. In contrast, David had labeled the axes of his graph, making tick marks on every grid segment.

In episode 3, the teacher and Carlos clarified the meanings for "I went by . . . " (When reading episode 3, focus on the reasoning evident in each participant’s descriptions of the scales.)
Episode 3: Using and clarifying “I went by . . .”

61  David:  I went by twos.
62  Teacher:  You went by twos and you went by [0.2 sec.].
63  Carlos:  I went by twos. You [didn’t] . . . you went by ones! What are you talking about?
64  Teacher:  No, here on the y-axis.
65  [The teacher points to the axis.]
66  Carlos:  Oh, I went by fives.
67  Teacher:  You went by fives. . . . No, actually you didn’t go by fives. You actually went by two and a halves because you’d . . . you did every 2 spaces as 5.
68  [The teacher points to Carlos’s paper while she explains.]  
69  Carlos:  Then he only went by one.
70  [Carlos points to David’s paper.]
71  Teacher:  Every one space was two of his. You see, they’re almost the same. If you look at the next two [she puts down her notebook and points to the graphs]—
72  Carlos:  Wait! But I don’t get what you’re saying.
73  Teacher:  OK.
74  Carlos:  ’Cause I went by fives. [David stands up.]

During episode 3, although David and Carlos had labeled their axes differently, David initially claimed that he also “went by twos” (line 61). The teacher first accepted this claim and proceeded to describe Carlos’s scale. Carlos disagreed with David, insisting that while he “went by twos,” David “went by ones” (line 63). At this point Carlos seemed to notice that they might not all be talking about the same thing, saying, “What are you talking about?” (line 63). Carlos then changed the description of his own scale, saying, “Oh, I went by fives” (line 66). The teacher first agreed with this claim, saying, “You went by fives” (line 67), but then, after a short pause, she disagreed, proposing that Carlos had not gone by fives but rather had gone by “two and a halves” (line 67). In response, Carlos proposed that, if that were the case, then David “went by one” (line 69). The teacher explained that on David’s scale, each space had a value of two units: “Every one space was two of his” (line 71). At this point, Carlos said that he did not understand the teacher’s explanation and returned to claiming that he “went by fives” (line 74).

At the end of episode 3, the teacher focused on a detailed comparison of the two scales. By looking at the two graphs, pointing to the axes on each graph, touching the papers, and orienting the two graphs so that they were facing her, she called on the students to focus their attention on the two y-axis scales. During episode 4, the teacher responded to Carlos’s claim that he “went by fives.” In this next episode, consider these questions: How did the teacher build on students’ own mathematical reasoning? How did she connect the students’ mathematical reasoning to a mathematical concept?
Episode 4: Teacher responds to Carlos's claim that he “went by fives”

75 Teacher: OK, your numbers, right, the numbers you have are by five . . . OK . . . If you look at one line here, what number is he at?

76 [The teacher takes David’s paper and places it next to Carlos’s paper and then points to David’s graph.]

77 Carlos: Two.

78 Teacher: What number would you be at if you had a number here?

79 [The teacher points to Carlos’s graph.]

80 Carlos: Three.

81 Teacher: Almost, two and a half.

82 Carlos: Yeah.

83 Teacher: Because that’d be halfway to five. OK . . . At this point, after 1, 2, 3, he’s got 6. For you after three, 1, 2, 3, you’d be at 7 and a half.

84 [The teacher counts the squares with her pencil.]

85 Carlos: OK.

86 Teacher: See what I mean? So it’s actually two and a half. The numbers you wrote are by fives, but since you skipped a line in between, each one is two and a half.

87 [The teacher raises her hand in the air and uses her thumb and index finger to show the interval.]

These two students were making sense of the scales on their graphs in several ways. The statement “I went by twos” can be interpreted as describing the action taken to construct the scale and where the number labels were placed on the scale, so that “I went by twos” means “I went by two segments.” (See fig. 2.6, Carlos’s graph.)

![Carlos's graph]

Fig. 2.6. Carlos describes his scale as “I went by twos,” describing number of segments.

One could also interpret the phrase “I went by twos” as describing the quantity that the chunk created between two tick marks represented, as in “I made tick marks at every segment, and each segment represents two units.” That is, the statement connects an account of constructing the graph to a quantitative relationship. For David’s scale this would mean “I made tick marks at every two units.” (See fig. 2.7.)
Carlos seemed to be referring to how he labeled the axes of his graph (fig. 2.4), so that tick marks appear every two segments, and to how David labeled the axes of his graph (fig. 2.5), so that tick marks appear every one segment (line 42). Carlos was thus using "went by" to refer to the value between the tick marks on the y-axis of his graph, five units (line 66). In contrast, the teacher was referring to the value of one segment or space in Carlos's graph, two and a halves (line 67). (See fig. 2.8)

Table 2.1 lists three ways of using "I [or you] went by" to describe these two graphs. One meaning refers to the value of the interval between tick marks; the second, to the number of segments between tick marks; and the third, to the value of each segment between tick marks. During episodes 2 and 3, Carlos used the first and second meaning, David used the first and third meaning, and the teacher used the third meaning.
Table 2.1  
*Multiple meanings for “went by”*

<table>
<thead>
<tr>
<th>Utterance</th>
<th>Meaning</th>
<th>Students’ work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carlos: “I went by fives”</td>
<td>Value of the interval between labeled tick marks</td>
<td>“I went by fives”</td>
</tr>
<tr>
<td>David: “I went by twos”</td>
<td></td>
<td>Carlos’s graph: 10 8 6 4 2 0 5 3 1</td>
</tr>
<tr>
<td>Carlos: “I went by twos”</td>
<td>Number of segments between labeled tick marks</td>
<td>“I went by twos”</td>
</tr>
<tr>
<td>Carlos: “He [David] went by one”</td>
<td></td>
<td>Carlos’s graph: 10 8 6 4 2 0 5 3 1</td>
</tr>
<tr>
<td>David: “I went by twos”</td>
<td>Number of units in the interval between tick marks</td>
<td>“You went by two and a halves”</td>
</tr>
<tr>
<td>Teacher: “You [David] went by twos…”</td>
<td></td>
<td>Carlos’s graph: 10 8 6 4 2 0 5 3 1</td>
</tr>
<tr>
<td>Teacher: “No, actually you [Carlos] didn’t go by fives, you actually went by two and a halves, because you did every two spaces as five.”</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The teacher’s role in the discussion

During this discussion, the phrase “I went by” reflected different ways of reasoning. This phrase was used with different—sometimes ambiguous, sometimes shifting—meanings that corresponded to different ways of viewing the scales. This section shows how the teacher responded to these multiple ways of reasoning and ambiguous meanings; she built on the students’ reasoning and connected student reasoning to a mathematical concept: unitizing.

How did the teacher participate in this discussion? The teacher began by asking students to make sense of their graphs. She engaged the students in a discussion detailing and connecting different ways of reasoning. The teacher described how she made sense of the two graphs, sharing her reasoning as she described to the students how she saw the scale and tick marks on each graph. However, reasoning and sense making were up to not only the teacher; the students also engaged in these. The discussion revolved around what each segment or tick mark represented for each student. Thus, the teacher built her explanation on the students’ own reasoning.

The teacher built on student reasoning, in part, by using the students’ own language and referring to the scales as “you went by” (lines 62 and 67). She later used a different phrase, saying “you
did every 2 spaces as 5” (line 67), focusing on the quantitative relationship between the spaces and the units on Carlos’s graph. She also compared the two graphs, saying, “Every one space was two of his” (line 71). Last, she described the quantitative relationship between the numbers on the scale in one graph, saying, “The numbers you have are by five” (line 75). Her descriptions started out by building on students’ own ways of talking and later describing students’ reasoning by using a mathematical concept, unitizing.

What the teacher did and what she did not do are both important. The teacher did not produce another graph but instead based the discussion on the students’ own graphs and reasoning. She did not evaluate the graphs or correct the scales. Instead, she treated both graphs as correct and accepted that multiple ways of reasoning exist. Although the teacher contested Carlos’s description of his scale (“No, actually you didn’t go by fives”), she also accepted his reasoning, saying, “OK, your numbers, right, the numbers you have are by five.” The teacher did not explicitly define what “went by” meant or address the multiple meanings of that phrase. Instead, she made sense of what quantity each segment or tick mark represented to each student.

The teacher did not supply the correct interpretation of the scales or make an explicit contrast between the student reasoning and the right answer. Instead, she clarified and connected different ways of reasoning. She described her own reasoning to the students—how she interpreted the scales and tick marks on both graphs. For part of her descriptions, the teacher connected student reasoning to the concept of unitizing. Her descriptions focused on comparing quantities. In her descriptions, she made a distinction among labels, quantities, and measures. The teacher distinguished between the labels that go “by fives” and the value of the grid segments of spaces as a unit, saying, “You actually went by two and a half fives” (line 67). In the second case, the phrase “you went by” refers to the unit value of one grid segment and is an instance of unitizing. The teacher also compared the grid segments or spaces on the two scales (line 71), again an instance of unitizing (Lamon 1994, 1996).

Finally, the teacher gave the students an opportunity to use a unitized view of the marks on the scales. She set a new problem, determining the value of the y-coordinate on each graph after moving up one grid segment on the y-axis (lines 75–82) and after moving up three grid segments on the y-axis (line 83). As she and Carlos jointly estimated the y-coordinates on the two graphs, she actively engaged him in reasoning about the scales from a unitized point of view.

Overall, the teacher used several strategies to support student reasoning: she used student-generated products; she used gestures and objects to clarify meanings; she accepted and built on students’ responses; and she connected student reasoning to an important mathematical concept, unitizing. She took student reasoning seriously, ensured available time for describing and taking different points of view, and allowed room for clarification. The teacher supported this mathematical discussion by, rather than evaluating student work, describing how she understood each student’s descriptions. Discussions such as the preceding, which make multiple ways of reasoning explicit and compare different meanings, can become important opportunities for students to participate in sense making and develop mathematical reasoning.

This discussion shows that multiple meanings need not be obstacles but can serve as resources for connecting to important mathematical concepts such as unitizing. This positive perspective on multiple meanings shifts the emphasis from asking what difficulties bilingual students encounter to how instruction can support bilingual students in participating in discussions. This teacher supported the mathematical discussion by using multiple interpretations, building on students’ own reasoning, and connecting student reasoning to a mathematical concept. These strategies can serve as a model for engaging bilingual students in discussions that simultaneously build on student reasoning and keep the discussion connected to mathematical concepts.

This example, which transpired in only one language, English, is a model for monolingual teachers who work with bilingual students. How can this example be relevant to English learners? Most classroom discussions with English learners are likely to take place only in English because English is the language of instruction—not only because most teachers are monolingual but also because most textbooks and assessments are in English. This example also shows that teachers who
work with English learners need not imagine that they must develop a new set of skills to work with English learners; they can draw on the skills they have developed for teaching native English speakers. If a teacher develops skills in supporting mathematical discussions by building on student reasoning and by connecting to concepts, he or she should also use those skills to support mathematical discussions with English learners.

References


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