Homework 3

1. (20 points) Let $G$ be a group and $H \subset G$ a subgroup. Given $g \in G$, let

$$gHg^{-1} = \{ghg^{-1} : h \in H\}$$

Prove that $gHg^{-1}$ is a subgroup of $G$. Let $\text{Sub}(G)$ denote the set of all subgroups of $G$. Prove that this construction defines an action of $G$ on $\text{Sub}(G)$.

2. Let $M$ and $N$ be normal subgroups of a group $G$.

(a) (5 points) Show that $N \cap M$ is a normal subgroup of $G$.

(b) (15 points) Assume that $G/M$ and $G/N$ are abelian. Show that $G/(M \cap N)$ is abelian. Hint: Show that $f : G \to G/M \times G/N$, $f(g) = (gM, gN)$ is a group homomorphism and use the first isomorphism theorem.

3. Let

$$\text{GL}^+_2 = \{A \in \text{GL}_2(\mathbb{R}) : \det(A) > 0\} \subset \text{GL}_2(\mathbb{R})$$

and let

$$H = \left\{ A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}, a \neq 0 \right\} \subset \text{GL}_2(\mathbb{R})$$

(a) (10 points) Prove that $\text{GL}^+_2(\mathbb{R})$ is a normal subgroup of $\text{GL}_2(\mathbb{R})$. What group is the quotient group, $\text{GL}_2(\mathbb{R})/\text{GL}^+_2(\mathbb{R})$? You must justify your answer to the second part. Hint for the quotient $\text{GL}_2(\mathbb{R})/\text{GL}^+_2(\mathbb{R})$: How many left cosets of $\text{GL}^+_2(\mathbb{R})$ are there in $\text{GL}_2(\mathbb{R})$?

(b) (10 points) Prove that $H$ is a normal subgroup of $\text{GL}^+_2(\mathbb{R})$ and show that $\text{GL}^+_2(\mathbb{R})/H \cong \text{SL}_2(\mathbb{R})$. Hint: Consider the function

$$\varphi : \text{GL}^+_2(\mathbb{R}) \longrightarrow \text{SL}_2(\mathbb{R})$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{1}{\sqrt{\det(A)}} A = \begin{pmatrix} a/\sqrt{ad-bc} & b/\sqrt{ad-bc} \\ c/\sqrt{ad-bc} & d/\sqrt{ad-bc} \end{pmatrix}$$

where $\sqrt{\det(A)} = \sqrt{ad-bc}$ is the positive square root of the determinant of $A$. Show that $\varphi$ is a group homomorphism and then use the first isomorphism theorem.

4. Let $G$ be a group and let $H$ be a subgroup of $G$. Define the map $*$ from $G \times G/H$ to $G/H$ by

$$* : G \times G/H \longrightarrow G/H$$

$$(a, bH) \longmapsto abH$$
(a) (10 points) Show that $*$ is a well defined function, and that $*$ defines a group action of $G$ on $G/H$.

(b) (10 points) Let $\rho : G \to \text{Sym}(G/H)$ be the homomorphism corresponding to $\ast$. Show that $\ker(\rho) \subset H$ and that $\ker(\rho) = H$ if and only if $H$ is a normal subgroup of $G$.

5. Extra credit (10 points): It may be helpful to do this problem in conjunction with problem 1. Determine all possible subgroups of $\text{Sym}(3)$, as in homework 1. Give justification for why your list of subgroups is all the subgroups of $\text{Sym}(3)$ and why there is no overlap in your list. This should be easier than it was on homework 1 because we have more tools (theorems) at our disposal.

Now let $\text{Sym}(3)$ act on $\text{Sub}(\text{Sym}(3))$ as in question 1. Describe the orbits of this action. Calculate the stabiliser subgroups of $\langle (12) \rangle$ and $\langle (123) \rangle$.

6. Extra credit (10 points): Let $D_8$ be the dihedral group with 8 elements, so

$$D_8 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \sigma \tau = \tau \sigma^3 \rangle = \{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma \tau, \sigma^2 \tau, \sigma^3 \tau \}$$

Prove that $\langle \tau \rangle$ is a normal subgroup of $\langle \sigma^2, \tau \rangle$, and $\langle \sigma^2, \tau \rangle$ is a normal subgroup of $D_8$, but that $\langle \tau \rangle$ is not a normal subgroup of $D_8$. Conclude that if $H$ is a normal subgroup of $K$ and $K$ is a normal subgroup of $G$, that does not imply that $H$ is a normal subgroup of $G$.

7. Extra credit (10 points): Let $D_{2n}$ be the dihedral group of order $2n$, so $D_{2n}$ is the group of symmetries of a regular $n$-gon centered at the origin. Recall that $D_{2n}$ has group presentation

$$D_{2n} = \langle \sigma, \tau | \sigma^n = 1, \tau^2 = 1, \sigma \tau = \tau \sigma^{n-1} \rangle$$

where $\sigma$ is rotation by $360/n$ degrees counterclockwise and $\tau$ is any reflection that sends the $n$-gon to itself.

Determine the conjugacy classes of $D_{2n}$. How many are there?