Homework 1

1. (10 points) Let $S$ and $T$ be sets and let $f : S \rightarrow T$ be a function. Prove the following statements:
   
   (a) If $U$ is a set and $g : T \rightarrow U$ is a function such that $g \circ f$ is injective, the $f$ is also injective.
   
   (b) If $R$ is a set and $h : R \rightarrow S$ is a function such that $f \circ h$ is surjective then $f$ is also surjective.
   
   (c) Show that if $g, h : T \rightarrow S$ are functions satisfying $g \circ f = \text{id}_S$ and $f \circ h = \text{id}_T$, then $f$ is bijective and $g = h = f^{-1}$. Hint: use parts (a) and (b).

2. (5 points) Let $S$ be a set with 3 elements. How many functions are there from $S$ to $S$? How many bijections are there from $S$ to $S$? What if $S$ has 4 elements? What if $S$ has $n$ elements ($n \in \mathbb{N}$)? Explain.

3. (10 points) Equivalence relations. Let $\sim$ be an equivalence relation on a set $S$. For $x \in S$, $[x] = \{y \in S : x \sim y\}$ denotes the equivalence class containing $x$.
   
   (a) Prove that for all $x \in S$, $[x] \neq \emptyset$.
   
   (b) Prove that $\bigcup_{x \in S} [x] = S$.
   
   (c) Prove that if $y \in [x]$, then $[y] = [x]$ and if $y \notin [x]$, then $[x] \cap [y] = \emptyset$.
   
   (d) A partition of a set $S$ is a collection of subsets of $S$: $\{S_i \subset S : i \in I\}$, where $I$ is an indexing set, such that $\bigcup_{i \in I} S_i = S$ and for all $i \neq j$, $S_i \cap S_j = \emptyset$. Explain why the equivalence classes of $\sim$ partition $S$.
   
   (e) Prove that every partition of a set $S$ can be realized as the equivalence classes of a unique equivalence relation.

4. (5 points) Show by way of an example that the binary operation given by subtraction on $\mathbb{Z}$:

   $$- : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$$
   
   $$(a, b) \mapsto a - b$$

   is not associative.

5. (10 points) Let $f : \mathbb{Q} \rightarrow \mathbb{Q}$ be defined by $f(x) = 3x - 1$.
   
   (a) Show that $f$ is bijective and compute $f^{-1}$.
   
   (b) Find a binary operation $*$ on $\mathbb{Q}$ such that $f : (\mathbb{Q}, +) \rightarrow (\mathbb{Q}, *)$ is an isomorphism.
   
   (c) Find a binary operation $*$ on $\mathbb{Q}$ such that $f : (\mathbb{Q}, *) \rightarrow (\mathbb{Q}, +)$ is an isomorphism.
6. (5 points) Let \((G, \cdot)\) be a group. Prove that for all \(a \in G\), \((a^{-1})^{-1} = a\).

7. (10 points) Let
\[
G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{R}) : a, b, c \in \mathbb{R}, ad \neq 0 \right\}
\]
and consider \(G\) with the binary operation given by matrix multiplication. Prove that \(G\) with this binary operation is a group. Is \(G\) abelian? If so prove it. If not, give an example of two elements of \(G\) that do not commute.

8. (5 points) Prove that the special linear group,
\[
\text{SL}_2(\mathbb{R}) = \{ A \in M_{2 \times 2}(\mathbb{R}) : \det(A) = 1 \}
\]
is a subgroup of the general linear group, \(\text{GL}_2(\mathbb{R}) = \{ A \in M_{2 \times 2}(\mathbb{R}) : \det(A) \neq 0 \}\).

9. (10 points) Let \(f : G \to H\) be a group homomorphism. Show that the image of \(f\) is a subgroup of \(H\). That is, show that the subset
\[
\text{im}(f) = \{ h \in H : \exists g \in G \text{ with } f(g) = h \} \subset H
\]
is a subgroup of \(H\).

10. (10 points) Write down all the subgroups of \(\text{Sym}(3)\). What are their sizes?

11. (extra credit) (10 points) For which \(n\) is the set of nonzero equivalence classes modulo \(n\) with binary operation given by multiplication, \((\mathbb{Z}/n\mathbb{Z} - \{0 + n\mathbb{Z}\}, \cdot)\), a group? Prove that your answer is correct.

12. Question: How difficult was this homework? Feel free to elaborate on the amount of time spent, the harder questions, and the easier questions.