Normal subgps and factor groups

Motivation: Let \( G \) be a gp and \( H \leq G \) a subgp of \( G \). We have the left cosets of \( H \) in \( G \):

\[ G/H = \{ aH : a \in G \} \]

Question: Can we define a binary operation \( \cdot \) on \( G/H \) by the rule

\[ aH \cdot cH = acH \, ? \]

One necessary condition for this binary operation to be well defined is that if \( aH = bH \), then \( acH = bcH \). We know:

\[ aH = bH \text{ iff } a^{-1}b \in H \]

\[ acH = bcH \text{ iff } (ac)^{-1}bc = c^{-1}a^{-1}bc \in H \]

If \( H \) has the property that \( \forall g \in G, \, h \in H, \, ghg^{-1} \in H \), then if \( aH = bH \), then \( acH = bcH \) because \( \bar{c} \bar{a}^{-1}b \in H \) since \( \bar{a}b \in H \) (take \( q = \bar{c}, \, h = \bar{a}b \)).

Def: Let \( H \leq G \) be a subgp of a gp \( G \). For \( g \in G \), define the subset of \( G \)

\[ gHg^{-1} = \{ ghg^{-1} : h \in H \} \]

We say that \( H \) is a normal subgp of \( G \) if \( \forall g \in G \),

\[ gHg^{-1} = H \]

or equivalently if \( \forall g \in G, \, h \in H, \, ghg^{-1} \in H \). We denote \( H \)

being a normal subgp of \( G \) by \( H \triangleleft G \) or \( H \lhd G \).

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Remarks: 1. Observe that this is not saying that given \( g \in G, h \in H \),
\( ghg^{-1} = h \). It is merely saying that \( ghg^{-1} \in H \).

2. If \( G \) is abelian then every subgroup is normal since \( \forall g, h \in G, ghg^{-1} = gh = h \).

3. Let \( G = \text{Sym}(3), H = \{e, (12), (13)\} \). Then \( (13)(12)(13) = (23) \notin H \), so
\( H \) is not normal. Therefore, not all subgroups of a group are normal.

Prop: Let \( H \leq G \) be a normal subgroup. Then the binary operation
\[
G/H \times G/H \rightarrow G/H
\]
\[
(xH, yH) \mapsto (xyH)
\]
is well-defined.

Proof: We need to show that if \( aH = xH \) and \( bH = yH \), then
\( abH = xyH \).
Assume \( aH = xH \), so \( a^{-1}x \in H \).
\( bH = yH \), so \( b^{-1}y \in H \).

We need to show that \( (ab)^{-1}xy \in H \). We have
\[
(ab)^{-1}xy = b^{-1}a^{-1}xy
\]
\[
= b^{-1}y a^{-1}x y
\]
\[
= b^{-1}y (a^{-1}xy)
\]
\( H \) is a normal subgroup implies \( a^{-1}xy \in H \) since \( a^{-1}x \in H \). Therefore,
\( (ab)^{-1}xy \in H \) since \( (ab)^{-1}xy = b^{-1}y (a^{-1}xy) \) and \( b^{-1}y \in H \).
Prop: Let $H \leq G$ be a normal subgroup of a group. Then $G/H$ is a group under the above binary operation. We call this group the quotient group of $G$ by $H$.

Proof: 1. (Identity) We claim that $H \leq G/H$ is an identity element:

For $g \in G$ we have

$$H \cdot gH = 1 \cdot gH = gH = g \cdot 1H = gH \cdot H$$

2. (Associativity) Let $xH, yH, zH \in G/H$. Then

$$(xH \cdot yH) \cdot zH = xyH \cdot zH$$

$$= xyzH \quad \text{because } G \text{ is associative}$$

$$= xH \cdot yzH$$

$$= xH \cdot (yH \cdot zH)$$

3. (Inverses) We claim that $x^{-1}H = (xH)^{-1}$ for all $xH \in G/H$.

$$x^{-1}H \cdot xH = xH \cdot x^{-1}H = 1H = H = xx^{-1}H = xH \cdot x^{-1}H. \quad \blacksquare$$

Prop: The map $\pi: G \rightarrow G/H$ is a group homomorphism if $H$ is a normal subgroup of $G$.

$\ker(\pi) = H$.

Proof: Let $xy \in G$. Then $\pi(xy) = xyH = xHyH = \pi(x) \pi(y)$, so $\pi$ is a group homomorphism.

$\ker(\pi) = \{ x \in G : \pi(x) = H \}$.

Now $\pi(x) = xH = H$ iff $x \in H$, so $\ker(\pi) = H$. $\blacksquare$

Remark. The previous prop shows that any normal subgroup can be realized as the kernel of a group homomorphism.
Examples of quotient groups when $G$ is abelian.

1. $\mathbb{Z} \cong \mathbb{Z}$ normal since $\mathbb{Z}$ is abelian.
   $\mathbb{Z}/n\mathbb{Z}$ - quotient gp

2. $G = \mathbb{Z} \times \mathbb{Z}$, $H = \{(x,0) : x \in \mathbb{Z}\}$, $H \triangleleft \mathbb{Z} \times \mathbb{Z}$ subgp.
   Claim: $\varphi : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}/H$ is a gp isomorphism.
   $\varphi(x) = (0, y) + H$

   Proof: $\varphi$ is a gp hom. Let $y, z \in \mathbb{Z}$, then
   $\varphi(y + z) = (0, y + z) + H = (0, y) + (0, z) + H$
   $= (0, y) + H + (0, z) + H = \varphi(y) + \varphi(z)$

   Injective: Say $\varphi(y) = \varphi(z)$. This means $(0, y) + H = (0, z) + H$.
   $(0, y) + H = (0, z) + H$ iff $(0, y) - (0, z) = (0, y - z) \in H$
   iff $y - z = 0$
   iff $y = z$

   Surjective: Let $(x, y) + H \in \mathbb{Z} \times \mathbb{Z}/H$. Then $(x, y) + H = (0, y) + H$.
   Hence $\varphi(x) = (0, y) + H = (x, y) + H$. □

3. $G = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, $H = \{(x,0) : x \in \mathbb{R}\}$ is the $x$-axis.
   Cosets of $H$ in $\mathbb{R}^2$ are lines parallel to the $x$-axis.
   $\mathbb{R}^2/\mathbb{R} \cong \mathbb{R}_y$, choose coset reps to be the $y$-axis.