# The Numerical Delta Method \*

Han Hong

hanhong@stanford.edu Department of Economics Stanford University 579 Serra Mall Stanford, CA 94305

Jessie Li

jessieqli@gmail.com Department of Economics University of California, Santa Cruz 1156 High St Santa Cruz, CA 95064

May 22, 2018

#### Abstract

This paper provides a numerical derivative based Delta method that complements the recent work by Fang and Santos (2014) and also generalizes a previous insight by Song (2014). We show that for an appropriately chosen sequence of step sizes, the numerical derivative based Delta method provides consistent inference for functions of parameters that are only directionally differentiable. Additionally, it provides uniformly valid inference for certain convex and Lipschitz functions which include all the examples mentioned in Fang and Santos (2014). We extend our results to the second order Delta method and illustrate its applicability to inference for moment inequality models.

Keywords: Delta Method, Numerical Differentiation, Directional Differentiability

JEL Classification: C12; C13; C50

<sup>\*</sup>Corresponding Author: Han Hong (hanhong@stanford.edu, Tel: 650 723 9766, Fax: 650 725 5702). We thank Andres Santos, Joe Romano, Xiaohong Chen, Zheng Fang, Bruce Hansen, David Kaplan, Adam McCloskey, Frank Wolak, and participants at the Montreal Econometric Society World Congress and various conferences and seminars for helpful comments. In particular, we appreciate the detailed comments by the coeditor, the associate editor, and two anonymous referees that greatly helped improve the paper. We also acknowledge support by the National Science Foundation (SES 1658950), and both the IRiSS and the B.F.Haley and E.S.Shaw SIEPR dissertation fellowships.

## 1 Introduction

Inference on possibly nonsmooth functions of parameters has received much attention in the econometrics literature, as in Woutersen and Ham (2013) and Hirano and Porter (2012). In particular, a recent insightful paper by Fang and Santos (2014) studies inference for functions of the parameters that are only Hadamard directionally differentiable and not necessarily differentiable. Fang and Santos (2014) show that while the asymptotic distribution obtained using the bootstrap is invalid unless the target function of the parameter is differentiable, asymptotic inference using a consistent estimate of the first order directional derivative is valid as long as the target function is Hadamard directionally differentiable. In each of their examples studied, Fang and Santos (2014) constructed consistent analytical estimates of the directional derivative that are tailored to each particular case.

As an alternative to using analytical estimates, we show that numerical differentiation provides a comprehensive approach to estimating the directional derivative. The main advantage of using the numerical directional derivative is its computational simplicity and ease of implementation. In order to compute an estimate of the directional derivative, the user only needs to specify one tuning parameter (the stepsize), and she does not need to perform any additional calculations beyond evaluating the target function twice for each random draw from an approximation of the limiting distribution of the parameter estimates.

Dümbgen (1993) developed a rescaled bootstrap that was implemented for the specific problem of matrix eigenvalues. However, his Proposition 1 essentially provides pointwise consistency of the numerical delta method under directional differentiability. We build on and go beyond these initial contributions by demonstrating how to perform uniformly valid inference under convexity. We also generalize to the second order directional delta method and study its application to a wider range of problems.

The results of this paper also complement Woutersen and Ham (2013), who provide a general inference method for functions of parameters that can be nondifferentiable and even discontinuous. In contrast, our numerical differentiation method only applies to directionally differentiable functions but can be easier to implement. We also contribute to the understanding of the statistical properties of numerical differentiation, which was analyzed in Hong et al. (2010) for different purposes. Most importantly, this paper follows up and complements the insights in Fang and Santos

(2014), as well as the extensive analytic derivations in Amemiya (1985).

In some applications, the first order directional derivative may vanish on a set of parameters, which motivates the use of the second order numerical directional delta method. For example, the test statistics for moment inequality models often use the negative square test function, which has the property that the first order directional derivative is exactly zero over the null set. We demonstrate the pointwise consistency of the second order numerical directional derivative and demonstrate that it can be used to approximate the limiting distribution for the second order directional delta method.

The rest of this paper is organized as follows. Section 2 describes the setup of the model that is mostly based on summarizing Fang and Santos (2014), and describes inference based on numerical differentiation. Section 3 first discusses pointwise validity of the numerical directional delta method for all Hadamard directionally differentiable functions and then demonstrates the uniform asymptotic validity of the numerical directional delta method for convex and Lipschitz functions. Convexity and Lipschitz continuity are satisfied in all the examples provided in Fang and Santos (2014) as well as for test statistics used in certain moment inequality models. Extensions of the uniform asymptotic validity results to statistics containing nuisance parameters is discussed in section **3.3.** Section 4 describes the second order numerical directional delta method, and an application to partially identified models such as those studied in Bugni et al. (2015) is illustrated in subsection A.4 of the appendix. Section 5 reports Monte Carlo simulation results on the coverage frequencies of various types of confidence intervals obtained using the first order numerical directional delta method as well as the rejection frequencies for a moment inequalities test based on critical values obtained using the second order numerical directional delta method. Section 6 proposes a multiple point first order numerical directional derivative that could be used to reduce bias, and section 7 concludes. The appendix contains a list of commonly used symbols, verification of convexity and Lipschitz continuity for several examples, proofs, and other technical material.

## 2 Numerical Directional Delta Method

Fang and Santos (2014) study inference on a nondifferentiable mapping  $\phi(\theta)$  of the parameter  $\theta \in \Theta$ , where  $\theta$  can be either finite or infinite dimensional, under the requirement that  $\theta \in \mathbb{D}_{\phi}$  and  $\phi: \mathbb{D}_{\phi} \subset \mathbb{D} \to \mathbb{E}$  for  $\mathbb{D}$  endowed with norm  $|| \cdot ||_{\mathbb{D}}$  and  $\mathbb{E}$  endowed with norm  $|| \cdot ||_{\mathbb{E}}$ . The domain of

 $\phi$  is  $\mathbb{D}_{\phi}$ .

The true parameter is denoted  $\theta_0$ , for which a consistent estimator  $\hat{\theta}_n$  is available which converges in distribution at a suitable rate  $r_n \to \infty$ :  $r_n \left(\hat{\theta}_n - \theta_0\right) \rightsquigarrow \mathbb{G}_0$  in the sense of equation (2.8) of Kosorok (2007)<sup>1</sup>, where the limit distribution  $\mathbb{G}_0$  is tight and is supported on  $\mathbb{D}_0 \subset \mathbb{D}$ . Examples of nondifferentiable  $\phi(\cdot)$  functions arise in a variety of econometric applications such as moment inequalities models (Andrews and Shi (2013),Ponomareva (2010)) and threshold regression models (Hansen (2017)). Using the notation of Fang and Santos (2014), we describe each of these examples in more detail below.

Generalization of Fang and Santos (2014) Example 2.1 Define  $\phi(\theta) = a\theta^+ + b\theta^-$ , where  $\theta^+ = \max\{\theta, 0\}$  and  $\theta^- = -\min\{\theta, 0\}$ . Let  $X \in \mathbb{R}$ ,  $\theta_0 = E[X]$ , and  $\mathbb{D} = \mathbb{E} = \mathbb{R}$ .

Generalization of Fang and Santos (2014) Example 2.2  $\theta = (\theta_1, \ldots, \theta_K)$  for  $\theta_k \in \mathbb{R}^d$ ,  $\phi(\theta) = \max(\theta_1, \ldots, \theta_K)$ .  $\mathbb{D} = \mathbb{R}^d \times \mathbb{R}^d \times \ldots \times \mathbb{R}^d$  and  $\mathbb{E} = \mathbb{R}$ .

Fang and Santos (2014) Example 2.3 Define  $\phi(\theta_0) = \sup_{f \in \mathcal{F}} E[Yf(Z)]$  as in Andrews and Shi (2013). Here,  $Y \in \mathbb{R}$ ,  $Z \in \mathbb{R}^d$ , and  $\theta_0 \in \ell^{\infty}(\mathcal{F})$ .  $\mathcal{F} \subset \ell^{\infty}(\mathbb{R}^d)$  is a set of functions satisfying  $\theta_0(f) \equiv E[Yf(Z)]$  for all  $f \in \mathcal{F}$ .  $\mathbb{D} = \ell^{\infty}(\mathcal{F})$  and  $\mathbb{E} = \mathbb{R}$ .

**Ponomareva (2010) Example** In theorem 3.5, inference is performed on  $\phi(\theta_0) = \max_{x \in \mathcal{X}} E[m(Z_i) | X_i = x]$ where  $\theta_0(x) = E[m(Z_i) | X_i = x]$  is the conditional expectation function,  $\mathbb{D} = \ell^{\infty}(\mathbb{R}^d)$  and  $\mathbb{E} = \mathbb{R}$ .

The goal of subsequent analysis is to approximate the distribution of  $\phi\left(\hat{\theta}_n\right)$ , or with proper scaling and centering, that of  $r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right)$ , for statistical inference concerning  $\phi\left(\theta_0\right)$ . The asymptotic distribution bootstrap (ADB) method (coined by Woutersen and Ham (2013) and further illustrated in theorems 3 and 4 in Chernozhukov and Hong (2003)) uses the empirical distribution formed by repeated draws from

$$r_n\left(\phi\left(\hat{\theta}_n + \frac{\mathbb{Z}_n^*}{r_n}\right) - \phi\left(\hat{\theta}_n\right)\right). \tag{1}$$

In the above,  $\mathbb{Z}_n^*$  is a function of the data and additional randomness, and its distribution given the data converges to  $\mathbb{G}_1$  in probability, denoted  $\mathbb{Z}_n^* \xrightarrow{\mathbb{P}} \mathbb{G}_1$  in the sense of section 2.2.3 of Kosorok

 $<sup>\</sup>overline{{}^{1}X_{n} \rightsquigarrow X_{n}}$  in the metric space  $(\mathbb{D}, d)$  if and only if  $\sup_{f \in BL_{1}} |E^{*}f(X_{n}) - Ef(X)| \to 0$  where  $BL_{1}$  is the space of functions  $f : \mathbb{D} \mapsto \mathbb{R}$  with Lipschitz norm bounded by 1.

(2007). Here,  $\mathbb{G}_1$  is an identical copy of  $\mathbb{G}_0$ , the random variable whose distribution is the limiting distribution of  $r_n(\hat{\theta}_n - \theta_0)$ . Examples of  $\mathbb{Z}_n^*$  include the following:

- 1. Bootstrap: here  $\mathbb{Z}_n^* = r_n \left(\hat{\theta}_n^* \hat{\theta}_n\right)$ , where  $\hat{\theta}_n^*$  are parameter estimates obtained using multinomial, wild, or other commonly used bootstrap implementations. The bootstrap sample size can also be different from the observed sample size. For example, we can take  $\mathbb{Z}_n^* = r_{m_n} \left(\hat{\theta}_{m_n}^* - \hat{\theta}_n\right)$ , where  $m_n \to \infty$  as  $n \to \infty$ , and  $\hat{\theta}_{m_n}^*$  is computed from a multinomial bootstrap sample of size  $m_n$  that are i.i.d draws from the empirical distribution. Similar modifications apply to the next few methods.
- 2. When  $\theta$  is a finite dimensional parameter, typically  $r_n = \sqrt{n}$  and  $\mathbb{G}_0 = N(0, \Sigma)$  for some variance covariance matrix  $\Sigma$ . Using a consistent estimate  $\hat{\Sigma}$  of  $\Sigma$ ,  $\mathbb{Z}_n^*$  can be a random vector whose distribution given the data is given by  $N(0, \hat{\Sigma})$ .
- 3. For correctly specified parametric models, one can use  $\mathbb{Z}_n^* = r_n \left(\hat{\theta}_n^* \hat{\theta}_n\right)$ , where  $\hat{\theta}_n^*$  are MCMC draws from the (pseudo) posterior distribution based on the likelihood or other objective functions (Chernozhukov and Hong (2003)).
- 4. In Hong and Li (2014), we propose a technique called the numerical bootstrap, which produces estimates  $\theta(\mathbb{Z}_n^*)$  based on the numerical bootstrap empirical measure  $\mathbb{Z}_n^* \equiv P_n + \epsilon_n \sqrt{n} (P_n^* - P_n)$ , where  $P_n$  is the empirical measure,  $P_n^*$  is the bootstrap empirical measure,  $\epsilon_n$  is a positive scalar step size parameter that satisfies  $\epsilon_n \to 0$ , and  $\sqrt{n}\epsilon_n \to \infty$ . We show that the finite sample distribution of  $\mathbb{Z}_n^* = \epsilon_n^{-2\gamma} (\theta(\mathbb{Z}_n^*) - \theta(P_n))$  converges to the same limiting distribution as that of  $n^{\gamma} (\hat{\theta}_n - \theta_0)$  for a class of estimators that converge at rate  $n^{\gamma}$  for some  $\gamma \in [\frac{1}{4}, 1)$ .

Intuitively, ADB approximates the distribution of  $\phi\left(\hat{\theta}_{n}\right)$  around  $\phi\left(\theta_{0}\right)$  with that of  $\phi\left(\hat{\theta}_{n}^{*}\right)$  around  $\phi\left(\hat{\theta}_{n}\right)$ , where  $\hat{\theta}_{n}^{*}$  is a suitable version of the bootstrap in case (1); a draw from a consistent estimate of the asymptotic distribution  $N\left(\hat{\theta}_{n}, \frac{1}{r_{n}^{2}}\hat{\Sigma}\right)$  in case (2); a draw from the MCMC chain in in case (3); and a draw from  $\hat{\theta}_{n} + r_{n}^{-1}\mathbb{Z}_{n}^{*}$  in case (4).

Fang and Santos (2014) showed that the ADB is asymptotically valid only if  $\phi(\theta)$  is Hadamard differentiable. The delta method, however, is applicable more generally even when ADB fails, as

long as  $\phi(\theta)$  is Hadamard directionally differentiable even if it is not Hadamard differentiable. Fang and Santos (2014) make use of the following definition:

**Definition 2.1** The map  $\phi$  is said to be Hadamard directionally differentiable at  $\theta \in \mathbb{D}_{\phi}$  tangentially to a set  $\mathbb{D}_0 \subset \mathbb{D}$  if there is a continuous map  $\phi'_{\theta} : \mathbb{D}_0 \to \mathbb{E}$  such that:

$$\lim_{n \to \infty} \left\| \frac{\phi\left(\theta + t_n h_n\right) - \phi\left(\theta\right)}{t_n} - \phi_{\theta}'\left(h\right) \right\|_{\mathbb{E}} = 0,$$
(2)

for all  $\{h_n\} \subset \mathbb{D}$  and  $\{t_n\} \subset \mathbb{R}_+$  such that  $t_n \downarrow 0, h_n \to h \in \mathbb{D}_0$  as  $n \to \infty$  and  $\theta + t_n h_n \in \mathbb{D}_{\phi}$ .

When  $\phi(\cdot)$  is directionally differentiable in the sense defined above and when the support of the limiting distribution  $\mathbb{G}_0$  is contained in  $\mathbb{D}_0$ , Fang and Santos (2014) showed that under suitable regularity conditions,  $r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right) \rightsquigarrow \phi'_{\theta_0}(\mathbb{G}_0)$ . Based on this result, Fang and Santos (2014) suggested that this limiting distribution can be consistently estimated by  $\hat{\phi}'_n(\mathbb{Z}_n^*)$ , where  $\mathbb{Z}_n^*$ is a consistent estimate of  $\mathbb{G}_0$  (such as the bootstrap, MCMC or asymptotic normal approximation), and in particular  $\hat{\phi}'_n(\cdot)$  is a consistent estimate of  $\phi'_{\theta_0}(\cdot)$  in a sense that is precisely defined in their Assumption 3.3.

**FS Assumption 3.3** For each fixed  $\theta_0$ , each compact set  $K \subseteq \mathbb{D}$ , and for any sequence  $\delta_n \downarrow 0$ ,

$$d_{\delta,K}\left(\hat{\phi}'_{n}\left(\cdot\right),\phi'_{\theta_{0}}\left(\cdot\right)\right) \equiv \sup_{h\in K^{\delta}} \left\|\hat{\phi}'_{n}\left(h\right)-\phi'_{\theta_{0}}\left(h\right)\right\|_{\mathbb{E}} = o_{p}\left(1\right) \qquad \text{as } n\to\infty.$$

$$(3)$$

In the above  $K^{\delta}$  denotes the  $\delta$ -enlargement of a set K:  $K^{\delta} \equiv \{a \in \mathbb{D} : \inf_{b \in K} ||a - b||_{\mathbb{D}} < \delta\}$ . We show that the one-sided numerical derivative provides a  $\hat{\phi}'_n(\cdot)$  for which this assumption holds whenever  $\phi(\cdot)$  is Lipschitz. In particular, Definition 2 motivates the following estimate  $\hat{\phi}'_n(\cdot)$  based on a one-sided finite difference formula. For  $\epsilon_n \to 0$  slowly (in the sense that  $r_n \epsilon_n \to \infty$ , where  $r_n$ is the convergence rate of  $\hat{\theta}_n$  to  $\theta_0$ ), define

$$\hat{\phi}_{n}^{\prime}(h) \equiv \frac{\phi\left(\hat{\theta}_{n} + \epsilon_{n}h\right) - \phi\left(\hat{\theta}_{n}\right)}{\epsilon_{n}} \tag{4}$$

as the numerical directional derivative of  $\phi$  in the direction of  $h \in \mathbb{D}_0$ . The rate requirement on the step size  $\epsilon_n$  is needed to separate numerical differentiation error from the estimation error in  $\hat{\theta}_n$ , and serves the dual purposes of model selection and numerical differentiation.

For functions that are not Lipschitz, section 3.1 shows that the one-sided numerical derivative will continue to consistently estimate the directional derivative as long as the function is Hadamard

directionally differentiable.

The Numerical Directional Delta Method Given the definition in (4), the numerical directional delta method estimates the limiting distribution of  $r_n \left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right)$  using the distribution of the random variable:

$$\hat{\phi}_{n}'(\mathbb{Z}_{n}^{*}) \equiv \frac{\phi\left(\hat{\theta}_{n} + \epsilon_{n}\mathbb{Z}_{n}^{*}\right) - \phi\left(\hat{\theta}_{n}\right)}{\epsilon_{n}}$$
(5)

which can be approximated by the following:

- 1. Draw  $\mathbb{Z}_s$  from the distribution of  $\mathbb{Z}_n^*$  for  $s = 1, \ldots, S$ .
- 2. For the given  $\epsilon_n$ , evaluate for each s:

$$\hat{\phi}_{n}'(\mathbb{Z}_{s}) \equiv \frac{\phi\left(\hat{\theta}_{n} + \epsilon_{n}\mathbb{Z}_{s}\right) - \phi\left(\hat{\theta}_{n}\right)}{\epsilon_{n}}.$$
(6)

The empirical distribution of  $\hat{\phi}'_n(\mathbb{Z}_s)$ ,  $s = 1, \ldots, S$  can then be used for confidence interval construction, hypothesis testing, or variance estimation. Consider the case when  $\phi(\cdot) \in \mathbb{R}$  is a scalar. For example, a  $1 - \tau$  two-sided equal-tailed confidence interval for  $\phi(\theta_0)$  can be formed by

$$\left[\phi(\hat{\theta}) - \frac{1}{r_n}c_{1-\tau/2}, \phi(\hat{\theta}) - \frac{1}{r_n}c_{\tau/2}\right]$$

where  $c_{\tau/2}$  and  $c_{1-\tau/2}$  are the  $\tau/2$  and  $1-\tau/2$  empirical percentiles of  $\hat{\phi}'_n(\mathbb{Z}_s)$ . Symmetric confidence intervals can be formed by, where  $d_{1-\tau}$  is the  $1-\tau$  percentile of  $|\hat{\phi}'_n(\mathbb{Z}_n^*)|$ ,

$$\left[\phi(\hat{\theta}) - \frac{1}{r_n} d_{1-\tau}, \phi(\hat{\theta}) + \frac{1}{r_n} d_{1-\tau}\right]$$

Note that the random variable  $\hat{\phi}'_n(\mathbb{Z}_s)$  only requires two evaluations of the  $\phi(\cdot)$  function for each draw of  $\mathbb{Z}_s$ . The computational simplicity of the numerical derivative is one of its main advantages. In equation (5),  $\mathbb{Z}_n^*$  can be any of the four choices discussed in the ADB method after equation (1). In particular, Fang and Santos (2014) recommended the bootstrap  $\mathbb{Z}_n^* = r_n\left(\hat{\theta}_n^* - \hat{\theta}_n\right)$ . Following the tradition of the literature (except Andrews and Buchinsky (2000)), we take  $S = \infty$  in analyzing  $\hat{\phi}'_n(\mathbb{Z}_n^*)$ . Subsampling is also a special case of (5) when  $\mathbb{Z}_n^*$  is the  $\binom{n}{b}$  point discrete distribution of  $r_b\left(\hat{\theta}_{n,b,i} - \hat{\theta}_n\right)$  (equation (2.1) page 42 of Politis et al. (1999)) and when  $\epsilon_n = 1/\sqrt{b}$ . When all  $\binom{n}{b}$  are used in subsampling, no simulation error is involved ( $S = \infty$ ). Simulating  $\mathbb{Z}_s$  from  $\mathbb{Z}_n$  is only relevant when one randomly draws from the  $\binom{n}{b}$  blocks.

We now give the form of  $\hat{\phi}'_n(\mathbb{Z}^*_n)$  in examples 2.1 and 2.3 of Fang and Santos (2014).

**Fang and Santos (2014) Example 2.1** With  $\mathbb{Z}_n^* \sim N(0, \hat{\sigma}_n^2)$  and  $\hat{\sigma}_n^2$  the usual sample variance:

$$\hat{\phi}_{n}^{\prime}\left(\mathbb{Z}_{n}^{*}\right) \equiv \frac{a\left(\hat{\theta}_{n}+\epsilon_{n}\mathbb{Z}_{n}^{*}\right)^{+}+b\left(\hat{\theta}_{n}+\epsilon_{n}\mathbb{Z}_{n}^{*}\right)^{-}-a\hat{\theta}_{n}^{+}+b\hat{\theta}_{n}^{-}}{\epsilon_{n}}$$

Fang and Santos (2014) Example 2.3 Note that  $\hat{\theta}_n(f) \equiv \theta(P_n)(f) \equiv \frac{1}{n} \sum_{i=1}^n y_i f(z_i)$ . Its multinomial bootstrap version is given by  $\hat{\theta}_n^*(f) \equiv \theta(P_n^*)(f) \equiv \frac{1}{n} \sum_{i=1}^n y_i^* f(z_i^*)$ . Alternatively the multiplier bootstrap can be used:  $\theta(P_n^*)(f) \equiv \frac{1}{n} \sum_{i=1}^n \xi_i^* y_i f(z_i)$  for positive random variables  $\xi_i^*$  with  $E\xi_i^* = 1$ . In this case  $\hat{\theta}_n = \theta(P_n), \mathbb{Z}_n^* = \sqrt{n} (\theta(P_n^*) - \theta(P_n))$ , so that with the multinomial bootstrap,

$$\hat{\phi}_{n}'\left(\mathbb{Z}_{n}^{*}\right) \equiv \frac{\sup_{f\in\mathcal{F}}\theta\left(P_{n}+\epsilon_{n}\sqrt{n}\left(P_{n}^{*}-P_{n}\right)\right)\left(f\right)-\sup_{f\in\mathcal{F}}\theta\left(P_{n}\right)\left(f\right)}{\epsilon_{n}}$$
$$=\frac{\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^{n}\left(y_{i}f\left(z_{i}\right)+\epsilon_{n}\sqrt{n}\left(y_{i}^{*}f\left(z_{i}^{*}\right)-y_{i}f\left(z_{i}\right)\right)\right)-\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^{n}y_{i}f\left(z_{i}\right)}{\epsilon_{n}},$$

or with multiplier bootstrap

$$\hat{\phi}_{n}'(\mathbb{Z}_{n}^{*}) = \frac{\sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \left( y_{i} f\left(z_{i}\right) + \epsilon_{n} \sqrt{n} \left(\xi_{i}^{*} y_{i} f\left(z_{i}\right) - y_{i} f\left(z_{i}\right)\right) \right) - \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} y_{i} f\left(z_{i}\right)}{\epsilon_{n}}.$$

A similar procedure can be applied to each of the examples in Fang and Santos (2014).

In the context of a matrix eigenvalue application and a minimum distance application, Dümbgen (1993) presented a "rescaled bootstrap method" which corresponds essentially to the numerical delta method, where the rescaling sample size is inversely related to the step size in numerical differentiation. Dümbgen (1993) showed pointwise consistency which is essentially Theorem 3.1 in section 3.1, but did not present uniformity results. The idea of using numerical differentiation for directionally differentiable parameters also appeared in Song (2014), although Song (2014) only considered finite dimensional  $\theta \in \mathbb{R}^d$  and scalar functions  $\phi(\cdot) \in \mathbb{R}$  that are (1) translation equivalent:  $\phi(\theta + c) = \phi(\theta) + c$  for  $c \in \mathbb{R}$ ; and (2) scale equivalent:  $\phi(\alpha\theta) = \alpha\phi(\theta)$  for  $\alpha \ge 0$ . Under these conditions Song (2014) gives the following more specialized form of the numerical derivative formula  $\hat{\phi}'_n(\mathbb{Z}^*_n) \equiv \phi(\mathbb{Z}^*_n + \epsilon_n^{-1}(\hat{\theta}_n - \phi(\hat{\theta}_n)))$ . If  $\phi(\cdot)$  is only scale equivalent as in an  $\mathcal{L}_1$  version of Andrews and Soares (2010) and Bugni et al. (2015) discussed in subsection 3.3, then

equivalently,  $\hat{\phi}'_n(\mathbb{Z}_n^*) \equiv \phi\left(\mathbb{Z}_n^* + \epsilon_n^{-1}\hat{\theta}_n\right) - \phi\left(\epsilon_n^{-1}\hat{\theta}_n\right).$ 

## 3 Asymptotic validity

This section shows that the numerical directional delta method provides consistent inference under general conditions. We first verify pointwise consistency and then discuss uniform validity.

#### 3.1 Pointwise asymptotic distribution

In this subsection we show pointwise consistency of the numerical delta method using the definition of Hadamard directional differentiability and (a bootstrap version of) the extended continuous mapping theorem. The first part of the following theorem is a directional delta method due to Dümbgen (1993), Fang and Santos (2014), and references therein. The second part of the theorem shows consistency of the numerical delta method. Let  $BL_1$  be the space of Lipschitz functions  $f: \mathbb{D} \mapsto \mathbb{R}$  with Lipschitz norm bounded by 1. For random variables  $F_1$  and  $F_2$ , let  $\rho_{BL_1}(F_1, F_2) =$  $\sup_{f \in BL_1} |Ef(F_1) - Ef(F_2)|$  metrize weak convergence. As in Kosorok (2007) (pages 19-20), we use  $\stackrel{\mathbb{P}}{\rightarrow}$  to denote weak convergence in probability conditional on the data. <sup>2</sup>

**Theorem 3.1** Suppose  $\mathbb{D}$  and  $\mathbb{E}$  are Banach Spaces and  $\phi : \mathbb{D}_{\phi} \subseteq \mathbb{D} \mapsto \mathbb{E}$  is Hadamard directionally differentiable at  $\theta_0$  tangentially to  $\mathbb{D}_0$ . Let  $\hat{\theta}_n : \{X_i\}_{i=1}^n \mapsto \mathbb{D}_{\phi}$  be such that for some  $r_n \uparrow \infty$ ,  $r_n\{\hat{\theta}_n - \theta_0\} \rightsquigarrow \mathbb{G}_0$  in  $\mathbb{D}$ , where  $\mathbb{G}_0$  is tight and its support is included in  $\mathbb{D}_0$ . Then  $r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right) \rightsquigarrow \phi'_{\theta_0}(\mathbb{G}_0)$ . Let  $\mathbb{Z}_n^* \stackrel{\mathbb{P}}{\longrightarrow} \mathbb{G}_0$  satisfy certain measurability assumptions stated in the appendix. Then for  $\epsilon_n \to 0, r_n \epsilon_n \to \infty$ ,

$$\hat{\phi}_{n}^{\prime}\left(\mathbb{Z}_{n}^{*}\right) \equiv \frac{\phi\left(\hat{\theta}_{n}+\epsilon_{n}\mathbb{Z}_{n}^{*}\right)-\phi\left(\hat{\theta}_{n}\right)}{\epsilon_{n}} \stackrel{\mathbb{P}}{\rightsquigarrow} \phi_{\theta_{0}}^{\prime}\left(\mathbb{G}_{0}\right).$$

An alternative approach to showing consistency is to use remark 3.6 and Lemma A.6 in Fang and Santos (2014), which place Lipschitz and Hölder continuity requirements on  $\hat{\phi}'_n(\cdot)$ , a consistent estimate of the directional derivative function. These results in Fang and Santos (2014) apply more generally to  $\hat{\phi}'_n(\cdot)$  constructed using alternative methods other than numerical differentiation. The particular structure of the numerical delta method allows us to invoke the bootstrap extended continuous mapping theorem directly without having to rely on these intermediate conditions.

 $<sup>{}^{2}\</sup>hat{X}_{n} \xrightarrow{\mathbb{P}} X$  means that  $\hat{X}_{n}$  is a random function of the data and  $\sup_{f \in BL_{1}} \left| E\left[f(\hat{X}_{n})|\mathcal{X}_{n}\right] - Ef(X) \right| \xrightarrow{p} 0$  (where  $\mathcal{X}_{n}$  denotes the data).

However, establishing these conditions turns out to be important for uniform validity considerations in the next section, and are thus presented here.

Lemma 3.1 (Fang and Santos (2014) Remark 3.6 and Lemma A.6) If the directional derivative estimate is Hölder continuous in the direction arguments, namely, if there exists some  $\kappa > 0$ and fixed constant  $C_0 < \infty$  such that for all  $h_1, h_2 \in \mathbb{D}_0$  and all  $n \ge 1$ ,

$$\|\hat{\phi}_{n}'(h_{1}) - \hat{\phi}_{n}'(h_{2})\|_{\mathbb{D}} \leq C_{0}\|h_{1} - h_{2}\|_{\mathbb{D}}^{\kappa}$$
(7)

then Fang and Santos (2014) assumption 3.3 holds as long as pointwise for each  $h \in \mathbb{D}_0$ ,

$$\left\|\hat{\phi}_{n}'(h) - \phi_{\theta_{0}}'(h)\right\|_{\mathbb{E}} = o_{p}(1).$$
(8)

Our first result provides the simple finding that whenever the function  $\phi(\cdot)$  is Lipschitz ( $\kappa = 1$ ), so is the one-sided numerical directional derivative.

**Theorem 3.2** If  $\phi : \mathbb{D}_{\phi} \to \mathbb{E}$  is Lipschitz, satisfying  $\|\phi(h_1) - \phi(h_2)\|_{\mathbb{E}} \leq C\|h_1 - h_2\|_{\mathbb{D}}$  for all  $h_1, h_2 \in \mathbb{D}$ , and for Lipschitz constant C that does not depend on n, then so is  $\hat{\phi}'_n(h) \equiv \frac{\phi(\hat{\theta}_n + \epsilon_n h) - \phi(\hat{\theta}_n)}{\epsilon_n}$  in h for all  $\epsilon_n > 0$ .

Note also that  $\phi'_{\theta}(h)$  is Lipschitz in h for all  $\theta$  whenever  $\phi(\theta)$  is Lipschitz:

$$\|\phi_{\theta}'(h_1) - \phi_{\theta}'(h_2)\|_{\mathbb{E}} \le \lim_{t \downarrow 0} \left\|\frac{\phi\left(\theta + th_1\right)}{t} - \frac{\phi\left(\theta + th_1\right)}{t}\right\|_{\mathbb{E}} \le C\|h_1 - h_2\|_{\mathbb{D}}.$$
(9)

Theorem 3.2 and Lemma 3.1 imply that whenever the function  $\phi(\cdot)$  is Lipschitz, it suffices to verify the pointwise consistency condition in (8).

**Theorem 3.3** Let the conditions in Theorem 3.1 hold for  $\phi(\cdot)$  and  $\hat{\theta}_n$ . If  $\epsilon_n \downarrow 0$  and  $r_n \epsilon_n \to \infty$ , then for  $\hat{\phi}_n(\cdot)$  defined in (4) and for any  $h \in \mathbb{D}_0$ ,  $\left\| \hat{\phi}'_n(h) - \phi'_{\theta_0}(h) \right\|_{\mathbb{E}} = o_p(1)$ .

To summarize, we have shown that if the function  $\phi(\cdot)$  is Lipschitz in its argument of the parameter, then so is the numerical directional derivative  $\hat{\phi}'_n(\cdot)$  in its argument of the direction of differentiation, uniformly in the step size  $\epsilon_n$ . Furthermore, we have shown that  $\hat{\phi}'_n(h)$  converges in probability to  $\phi'_{\theta_0}(h)$  for each fixed  $h \in \mathbb{D}_0$ . Whenever  $\phi(\cdot)$  is Lipschitz, we have shown that the numerical directional derivative  $\hat{\phi}'_n(h)$  satisfies Fang and Santos (2014) Lemma A.6, remark 3.6 and in turn Fang and Santos (2014) Assumption 3.3. Consequently, the remaining results in Fang and Santos (2014) imply that inference based on  $\hat{\phi}'_n(\mathbb{Z}_n^*)$  is asymptotically valid, in a formal sense. Intuitively, when  $\epsilon_n$  is much larger than  $\frac{1}{r_n}$ , the estimation error in  $\hat{\theta}_n$  does not obscure the true direction for which the derivative is being calculated. It turns out that whenever  $\phi(\cdot)$  is Lipschitz, Hadamard differentiability is equivalent to Gateaux differentiability as noted in proposition 3.5 of Shapiro (1990)<sup>3</sup>.

Theorem 3.2 depends crucially on the function  $\phi(\cdot)$  being Lipschitz in the parameter argument. This turns out to be a rather weak requirement that is satisfied by all the examples in Fang and Santos (2014). The calculations in the appendix verify that the Lipschitz condition holds for all the functions  $\phi(\cdot)$  in examples 2.1 to 2.5, as well as the convex projection inference problem in Fang and Santos (2014). Consequently, the numerical delta method (4) provides a (pointwise) consistent asymptotic approximation for the distribution of  $r_n \left(\phi(\hat{\theta}_n) - \phi(\theta_0)\right)$  in each of these examples, including the convex projection problem in Fang and Santos (2014).

For example, for  $\phi(\theta) = \inf_{\lambda \in \Lambda} \|\theta - \lambda\|$  which defines the distance between  $\theta$  and its projection onto the convex set  $\Lambda$ , the distribution of  $r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right)$  is accurately approximated by

$$\hat{\phi}_{n}'\left(\mathbb{Z}_{n}^{*}\right) = \frac{1}{\epsilon_{n}} \left( \inf_{\lambda \in \Lambda} \left\| \hat{\theta}_{n} + \epsilon_{n} \mathbb{Z}_{n}^{*} - \lambda \right\| - \inf_{\lambda \in \Lambda} \left\| \hat{\theta}_{n} - \lambda \right\| \right)$$
(10)

for some  $\mathbb{Z}_n^* \xrightarrow{\mathbb{P}} \mathbb{G}_0$  where  $r_n\left(\hat{\theta}_n - \theta_0\right) \rightsquigarrow \mathbb{G}_0$ . Evaluating the distribution of  $\hat{\phi}'_n(\mathbb{Z}_n^*)$  requires solving  $2 \times S$  optimization routines, where S is the number of draws from  $\mathbb{Z}_n^*$ . This is more computationally efficient than the original solutions provided in Fang and Santos (2014), which are based on combining a model selection scheme with analytic knowledge of the function  $\phi(\cdot)$ . To illustrate this difference, consider again Fang and Santos (2014) example 2.1.

Fang and Santos (2014) Example 2.1 Fang and Santos (2014) proposed to estimate  $\phi'_{\theta_0}(h)$ by h if  $\hat{\theta}_n > \kappa_n$ , by -h if  $\hat{\theta}_n < -\kappa_n$ , and by |h| when  $|\hat{\theta}_n| < \kappa_n$ , where the selection parameter  $\kappa_n$ satisfies the same rate condition as the step size parameter  $\epsilon_n$ :  $\kappa_n \to 0$  but  $\kappa_n \sqrt{n} \to 0$ . In other words, for  $\phi(\theta_0) = |\theta_0|$ ,  $\hat{\phi}'_n(h)$  is set to h if  $\hat{\theta}_n$  is sufficiently positive, to -h if  $\hat{\theta}_n$  is sufficiently negative, and to |h| if  $\hat{\theta}_n$  is sufficiently close to zero.

Instead, we use the numerical directional derivative in (4):

$$\hat{\phi}_{n}'(h) \equiv \frac{\phi\left(\hat{\theta}_{n} + \epsilon_{n}h\right) - \phi\left(\hat{\theta}_{n}\right)}{\epsilon_{n}} = \frac{|\hat{\theta}_{n} + \epsilon_{n}h| - |\hat{\theta}_{n}|}{\epsilon_{n}},\tag{11}$$

is never exactly equal to h, -h, or |h|. Instead, under the condition that  $\epsilon_n \to 0$  and  $\sqrt{n}\epsilon_n \to \infty$ ,  $\hat{\phi}'_n(h)$  converges in probability to h when  $\theta_0 > 0$ , converges to -h when  $\theta_0 < 0$ , and converges to

 $<sup>^{3}\</sup>mathrm{We}$  thank a referee for pointing this out.

|h| when  $\theta_0 = 0$ . Consistent inference follows then from Slutsky's lemma.

The Lipschitz assumption can be relaxed to Hölder continuity and Fang and Santos (2014) Assumption 3.3 can still be satisfied under a stronger condition on the step size parameter, as the following theorem shows.

**Theorem 3.4** If  $\phi(\cdot)$  is Holder continuous with exponent  $\kappa$  and  $r_n^{\kappa} \epsilon_n \to \infty$ , then for all compact  $K \subset \mathbb{D} = \mathbb{R}^d$ ,  $\sup_{h \in K} \left\| \hat{\phi}'_n(h) - \phi'_{\theta_0}(h) \right\|_{\mathbb{E}} = o_p(1).$ 

In finite dimension situations, K can be replaced by  $K^{\delta} \equiv \{a \in \mathbb{D} : \inf_{b \in K} ||a - b||_{\mathbb{D}} < \delta\}$ . In general, as in Fang and Santos (2014), Frechet directional differentiability might be needed to allow for replacement of K by  $K^{\delta}$ .

#### 3.2 Uniform Inference

Uniform asymptotic validity over a class of distributions can be a desirable feature to establish for an inference procedure (Romano and Shaikh 2008; 2012). The Lipschitz and convexity properties of  $\phi(\cdot)$  are key to establishing uniform size control in the test of  $H_0: \phi(\theta_0) \leq 0$  versus  $H_1: \phi(\theta_0) > 0$ .

As we show in the Appendix, the  $\phi(\cdot)$  functionals considered in the examples in Fang and Santos (2014) are not only Lipschitz but also convex, so that for  $\lambda \in [0, 1]$ ,

$$\phi \left(\lambda \theta_1 + (1 - \lambda) \theta_2\right) \le \lambda \phi \left(\theta_1\right) + (1 - \lambda) \phi \left(\theta_2\right).$$

We first note that convexity of the functional  $\phi(\cdot)$  implies subadditivity of the directional derivative  $\phi'_{\theta_0}$ , which then implies sublinearity since the directional derivative is positively homogeneous of degree 1.

**Lemma 3.2** When  $\phi(\cdot)$  is convex and Hadamard directionally differentiable at  $\theta_0$  and  $\mathbb{D}_0$  is a convex set, then  $\forall 0 \leq \lambda \leq 1$ ,

$$\phi_{\theta_0}'(h_1 + h_2) \le \phi_{\theta_0}'(h_1) + \phi_{\theta_0}'(h_2), \quad \phi_{\theta_0}'(\lambda h_1 + (1 - \lambda)h_2) \le \lambda \phi_{\theta_0}'(h_1) + (1 - \lambda)\phi_{\theta_0}'(h_2).$$
(12)

Fang and Santos (2014) use the statistic  $r_n \phi\left(\hat{\theta}_n\right)$  to test:

$$H_0: \phi(\theta_0) \le 0 \quad \text{against} \quad H_1: \phi(\theta_0) > 0. \tag{13}$$

and suggested rejecting  $H_0$  whenever  $r_n \phi\left(\hat{\theta}_n\right) \geq \hat{c}_{1-\tau}$ , where  $\hat{c}_{1-\tau}$  is the  $1-\tau$  quantile of  $\hat{\phi}'_n(\mathbb{Z}_n^*)$  or its simulated version in (6). This is related to the one-sided confidence interval in Part (i) of

Theorem 2.1 in Romano and Shaikh (2012):

$$P\left(r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right) \le \hat{c}_{1-\tau}\right),\tag{14}$$

Whenever  $\phi(\theta)$  is convex and Lipschitz in  $\theta$ , using the  $1 - \tau$  percentile of  $\hat{\phi}'_n(\mathbb{Z}_n^*)$  as  $\hat{c}_{1-\tau}$  provides uniform size control for both (13) and (14) under the condition that  $r_n \epsilon_n \to \infty$  without requiring  $\epsilon_n \to 0$ . Intuitively, convexity implies for  $\epsilon_n > \frac{1}{r_n}$  and for any realization z from  $\mathbb{G}_0$ ,

$$r_n\left(\phi\left(\theta_0 + \frac{z}{r_n}\right) - \phi\left(\theta_0\right)\right) \le \frac{1}{\epsilon_n}\left(\phi\left(\theta_0 + \epsilon_n z\right) - \phi\left(\theta_0\right)\right),\tag{15}$$

so that  $\frac{1}{\epsilon_n} \left( \phi \left( \theta_0 + \epsilon_n \mathbb{G}_0 \right) - \phi \left( \theta_0 \right) \right)$  first order stochastically dominates  $r_n \left( \phi \left( \theta_0 + \frac{\mathbb{G}_0}{r_n} \right) - \phi \left( \theta_0 \right) \right)$ . <sup>4</sup> If we denote, using notations from Romano and Shaikh (2012), the distribution functions of the two sides of (15) by  $J_n(x, \mathbb{G}_0)$  and  $J_{\epsilon_n}(x, \mathbb{G}_0)$ , then equation (15) immediately implies that

$$\sup_{n} \sup_{x \in \mathbb{R}} \{ J_{\epsilon_n} \left( x, \mathbb{G}_0 \right) - J_n \left( x, \mathbb{G}_0 \right) \} \le 0.$$
(16)

Next,  $\phi(\theta)$  being Lipschitz ensures that  $r_n\left(\phi\left(\theta_0 + \frac{\mathbb{G}_0}{r_n}\right) - \phi(\theta_0)\right)$  is close to  $r_n\left(\phi\left(\hat{\theta}_n\right) - \phi(\theta_0)\right)$ , whose distribution function is denoted  $J_n(x, P)$ , while  $\frac{1}{\epsilon_n}\left(\phi\left(\theta_0 + \epsilon_n\mathbb{G}_0\right) - \phi\left(\theta_0\right)\right)$  is close to  $\hat{\phi}'_n(\mathbb{Z}_n^*)$ , whose conditional distribution function given the data is  $J_{\epsilon_n}(x, P)$ , so that  $J_n(x, \mathbb{G}_0)$  and  $J_{\epsilon_n}(x, \mathbb{G}_0)$ in (16) can be replaced by their feasible sample versions.

Uniformity statements in line with those in Romano and Shaikh (2012) are possible under the following assumptions. We focus on the finite dimensional case  $\mathbb{D} = \mathbb{R}^d$  and  $\mathbb{E} = \mathbb{R}$ .

**Assumption 3.1** Let  $\mathcal{P}$  be a class of distributions such that

(i) 
$$\lim_{n\to\infty} \sup_{P\in\mathcal{P}} \rho_{BL_1}\left(r_n\left(\hat{\theta}_n - \theta\left(P\right)\right), \mathbb{G}_0\right) = 0, \lim_{M\to\infty} \sup_{P\in\mathcal{P}} P\left(|\mathbb{G}_0| \ge M\right) = 0;$$

(ii) for each  $\epsilon > 0$ ,  $\lim_{n \to \infty} \sup_{P \in \mathcal{P}} P(\rho_{BL_1}(\mathbb{Z}_n^*, \mathbb{G}_0) \ge \epsilon) = 0$ .

Primitive conditions for Assumption 3.1 can be found for example in the uniform central limit theorems of Romano and Shaikh (2008).

Assumption 3.2 Define for each  $x, a, d, C_{a,d,x} = \{g : \phi(d + \frac{g}{a}) \leq x\}$ . Then

$$\sup_{P \in \mathcal{P}} P\left(\mathbb{G}_0 \in \partial \mathcal{C}_{a,d,x}\right) = 0 \quad for \ all \ x, a, d,$$

where  $\partial C_{a,d,x}$  denotes the boundary of  $C_{a,d,x}$ .

<sup>4</sup>Equation (15) follows from rewriting it as, for  $r_n \epsilon_n > 1$ ,  $\phi\left(\theta_0 + \frac{z}{r_n}\right) \le \frac{1}{r_n \epsilon_n} \phi\left(\theta_0 + \epsilon_n z\right) + \left(1 - \frac{1}{r_n \epsilon_n}\right) \phi\left(\theta_0\right)$ .

Assumption 3.2 is mainly used to invoke versions of Theorem 2.11 of Bhattacharya and Rao (1986), as in Example 3.2 of Romano and Shaikh (2012). If  $\phi(\cdot)$  is scale equivariant, then it is sufficient to check all  $C_{d,x} \equiv \{g : \phi(d+g) \leq x\}$ . Convexity is crucial in the following.

**Theorem 3.5** Define  $\mathcal{P}$  to be a class of DGPs such that  $r_n\left(\hat{\theta}_n - \theta\left(P\right)\right)$  is asymptotically tight uniformly over  $P \in \mathcal{P}$ , and Assumptions 3.1, 3.2 both hold. If  $r_n\epsilon_n \to \infty$ ,  $\epsilon_n \to 0$ , and  $\phi(\cdot)$  is Lipschitz and convex, then  $\forall \epsilon > 0$ ,

$$\lim_{n \to \infty} \sup_{P \in \mathcal{P}} P\left(\sup_{x \in A} J_{\epsilon_n}(x, P) - J_n(x, P) \le \epsilon\right) \to 1$$
$$\lim_{n \to \infty} \sup_{P \in \mathcal{P}} P\left(r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta\left(P\right)\right)\right) \ge \hat{c}_{1-\tau}\right) \le \tau.$$

where A is any set for which  $\lim_{\lambda\to 0} \sup_{P\in\mathcal{P}} \sup_{x\in A} P(J_{\epsilon_n}(\cdot,\mathbb{G}_0)\in(x,x+\lambda)) = o(1)$  and contains a neighborhood of both  $J_{\epsilon_n}^{-1}(1-\tau,\mathbb{G}_0)$  and  $J_n^{-1}(1-\tau,P)$  for all large n. We have used  $J_{\epsilon_n}(\cdot,\mathbb{G}_0)$ to denote the random variable defined by the right hand side of (15).

According to Theorem 3.5, whenever  $\phi(\cdot)$  is convex, the lower one-sided confidence interval  $\left[\phi\left(\hat{\theta}_n\right) - \frac{\hat{c}_{1-\tau}}{r_n}, \infty\right)$ will have uniformly asymptotically valid coverage. Similarly, if  $\phi(\cdot)$  is instead a concave function, then the same arguments will establish that the upper one-sided confidence interval of the form of  $\left(-\infty, \phi\left(\hat{\theta}_n\right) - \frac{\hat{c}_{\tau}}{r_n}\right)$  has uniformly asymptotically valid coverage. Furthermore, if it is known that  $\phi(\cdot) \geq 0$  (e.g. Andrews (2000)), we can use  $\epsilon_n^{-1}\phi\left(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*\right)$  in place of  $\hat{\phi}'_n(\mathbb{Z}_n^*)$  at the cost of being more conservative. Furthermore, if the least favorable null distribution is desired in hypothesis testing, then  $\hat{\theta}_n$  can also be replaced by the least favorable null value  $\theta_0$  if  $\theta_0$  is known. In this case,  $\hat{\phi}'_n(\mathbb{Z}_n^*) = \frac{1}{t_n} \left( \phi \left( \theta_0 + t_n \mathbb{Z}_n^* \right) - \phi \left( \theta_0 \right) \right)$  consistently estimates the null distribution for any  $t_n \to 0$  by the extended continuous mapping theorem. If we take  $t_n = r_n^{-1}$  and use the bootstrap distribution  $\mathbb{Z}_{n}^{*} = r_{n} \left(\hat{\theta}_{n}^{*} - \hat{\theta}_{n}\right)$ , a modified bootstrap uses  $r_{n} \left(\phi \left(\theta_{0} + \hat{\theta}_{n}^{*} - \hat{\theta}_{n}\right) - \phi \left(\theta_{0}\right)\right)$  to approximate the null distribution of  $r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right)$ . However, it does not provide moment selection to improve the power of the test and does not offer uniform size control for  $r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right)$  under drifting sequences of  $\theta_n$ . In some cases, if only  $\phi(\theta) = \phi_0$  but not  $\theta_0$  is known under the null,  $\hat{\theta}_n$  can be either the constrained or unconstrained estimate. Note also that the only use of convexity of  $\phi(\cdot)$  is the stochastic dominance condition in (15) and (16). Therefore the convexity requirement of  $\phi(\cdot)$ can be replaced by the following stochastic dominance condition:

**Assumption 3.3** For all  $\theta_0$ , and for all t > 0,  $\frac{\phi(\theta_0 + t\mathbb{G}_0) - \phi(\theta_0)}{t}$  is nondecreasing in t.

Even if  $\phi(\theta)$  is not convex and does not satisfy Assumption 3.3, it is still possible to establish uniform size control over  $\theta_0$  under sufficient conditions for the limiting distribution of the numerical directional derivative to stochastically dominate the analytic limiting distribution over all  $\theta_0$  that lie in the null set.

**Assumption 3.4** For any  $\theta_0$ , for all  $\eta$  sufficiently close to zero and for all t > 0,  $\frac{\phi'_{\theta_0}(\eta + t\mathbb{G}_0) - \phi'_{\theta_0}(\eta)}{t}$  is nondecreasing in t.

Clearly Assumption 3.3 (which in turn is implied by  $\phi(\cdot)$  being convex) is a sufficient condition for Assumption 3.4. Assumption 3.4 is also satisfied if  $\phi'_{\theta_0}(h)$  is convex in h (which in turn follows from convexity of  $\phi(\cdot)$ ), since for  $t_2 > t_1 > 0$  and any realization z from  $\mathbb{G}_0$ ,  $\frac{\phi'_{\theta_0}(\eta+t_1z)-\phi'_{\theta_0}(\eta)}{t_1} \leq \frac{\phi'_{\theta_0}(\eta+t_2z)-\phi'_{\theta_0}(\eta)}{t_2}$  follows from rewriting  $\phi'_{\theta_0}(\eta+t_1z) \leq \left(1-\frac{t_1}{t_2}\right)\phi'_{\theta_0}(\eta) + \left(\frac{t_1}{t_2}\right)\phi'_{\theta_0}(\eta+t_2z)$ . Assumption 3.4 plays a similar role to (15) and (16) and implies for  $\epsilon_n r_n > 1$  and any realization zfrom  $\mathbb{G}_0$ ,

$$r_n\left(\phi_{\theta_0}'\left(\eta + \frac{z}{r_n}\right) - \phi_{\theta_0}'(\eta)\right) \le \frac{\phi_{\theta_0}'(\eta + \epsilon_n z) - \phi_{\theta_0}'(\eta)}{\epsilon_n} = \phi_{\theta_0}'\left(\frac{\eta}{\epsilon_n} + z\right) - \phi_{\theta_0}'\left(\frac{\eta}{\epsilon_n}\right)$$
(17)

In order for  $r_n\left(\phi'_{\theta_0}\left(\eta + \frac{\mathbb{G}_0}{r_n}\right) - \phi'_{\theta_0}(\eta)\right)$  to provide a good approximation to  $r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right)$ and for  $\phi'_{\theta_0}\left(\frac{\eta}{\epsilon_n} + \mathbb{G}_0\right) - \phi'_{\theta_0}\left(\frac{\eta}{\epsilon_n}\right)$  to provide a good approximation to  $\hat{\phi}'_n\left(\mathbb{Z}_n^*\right)$ , we require the following additional assumption.

**Assumption 3.5** Suppose  $\mathbb{D}_0$  is convex. For any  $t_n \downarrow 0$ ,  $\eta_n \to \infty$ , and any given  $\theta_0$ :

$$\lim_{t_n \downarrow 0, \eta_n \to \infty} \left| \frac{1}{t_n} \left( \phi \left( \theta_0 + \eta_n + t_n h \right) - \phi \left( \theta_0 + \eta_n \right) \right) - \left( \phi_{\theta_0}' \left( \frac{\eta_n}{t_n} + h \right) - \phi_{\theta_0}' \left( \frac{\eta_n}{t_n} \right) \right) \right| = 0.$$

We now state a uniformity result similar to Andrews and Soares (2010) without relying on convexity.

**Theorem 3.6** Let  $\phi(\cdot)$  be Lipschitz,  $r_n \epsilon_n \to \infty$ , and  $\epsilon_n \to 0$ . Define  $\mathcal{P}$  to be a class of DGPs such that  $r_n\left(\hat{\theta}_n - \theta\left(P\right)\right)$  is asymptotically tight uniformly over  $P \in \mathcal{P}$ , Assumptions 3.1 and 3.2 hold, and for which  $\phi(\cdot)$  satisfies either Assumption 3.3 or Assumptions 3.4 and 3.5. Then,  $\forall \epsilon, \delta > 0$  and  $x = J_n^{-1}(1 - \tau - \epsilon, P)$ ,  $\sup_{P \in \mathcal{P}} (J_{\epsilon_n}(x, P) \leq J_n(x, P) + \epsilon) \geq 1 - \delta$ . Consequently,  $\limsup_{n \to \infty} \sup_{P \in \mathcal{P}} P\left(r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta\left(P\right)\right)\right) \geq \hat{c}_{1-\tau}\right) \leq \tau$ .

It turns out that the following additional condition is also satisfied in most of the examples in Fang and Santos (2014) and in Andrews and Soares (2010): For all  $v_n \to v$ , |v| = 1, and all  $|a_n| \to 0$ ,  $\phi'_{\theta_0,v}(\cdot) = \lim_{n\to\infty} \phi'_{\theta_0+|a_n|v_n}(\cdot)$ , which is the limit of the directional derivative along direction v, is well defined. It is not required for results in this section, and its only additional implication is that the asymptotic size is exact along local parameter sequences drifting sufficiently slowly: for  $\epsilon_n/|\theta_0| \to 0$ ,  $\lim_{n\to\infty} P\left(r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right)\right) \ge \hat{c}_{1-\tau}\right) = \tau$ .

#### **3.3** Dealing with Nuisance Parameters

Unlike conventional derivatives, directional derivatives are not generally linearly separable in different subsets of parameters unless more assumptions are made. Consider now  $\phi(\theta, \alpha)$  where  $\alpha$  are a set of nuisance parameters. In addition to requiring that  $\phi(\cdot, \cdot)$  be jointly Hadamard directionally differentiable in  $\theta, \alpha$  tangentially to  $\mathbb{D}_0 = (\mathbb{D}_{0,\theta}, \mathbb{D}_{0,\alpha})$ , we impose the following assumption of separability and partial linearity in  $\alpha$ :

# Assumption 3.6 Suppose $\mathbb{D}_{0,\alpha}$ is convex and $\phi'_{\theta,\alpha}\left(h_{\theta},h^{1}_{\alpha}+h^{2}_{\alpha}\right)=\phi'_{\theta,\alpha}\left(h_{\theta},h^{1}_{\alpha}\right)+\phi'_{\theta,\alpha}\left(0,h^{2}_{\alpha}\right)$ .

This assumption holds for example in Hansen (2017) when  $\theta$  is the threshold parameter and  $\alpha$  are the regression coefficients. Under Assumption 3.6, while (5) can be used to estimate  $\phi'_{\theta,\alpha}(h_{\theta}, h_{\alpha})$  jointly in  $\theta, \alpha$ , it is also possible to estimate  $\phi'_{\theta,\alpha}(h_{\theta}, 0)$  and  $\phi'_{\theta,\alpha}(0, h_{\alpha})$  separately, using the numerical delta method and the bootstrap respectively. For  $r_n \epsilon_n \to \infty$ ,

$$\hat{\phi}_{n}'(h_{\theta},0) = \frac{\phi\left(\hat{\theta}_{n} + \epsilon_{n}h_{\theta},\hat{\alpha}_{n}\right) - \phi\left(\hat{\theta}_{n},\hat{\alpha}_{n}\right)}{\epsilon_{n}}$$

$$\hat{\phi}_{n}'(0,h_{\alpha}) = r_{n}\left(\phi\left(\hat{\theta}_{n},\hat{\alpha}_{n} + r_{n}^{-1}h_{\alpha}\right) - \phi\left(\hat{\theta}_{n},\hat{\alpha}_{n}\right)\right).$$
(18)

Then (5) can be replaced by, with  $\mathbb{Z}_{n}^{*} = \left(\mathbb{Z}_{n,\theta}^{*}, \mathbb{Z}_{n,\alpha}^{*}\right), \hat{\phi}_{n}'(\mathbb{Z}_{n}^{*}) \equiv \hat{\phi}_{n}'\left(\mathbb{Z}_{n,\theta}^{*}, 0\right) + \hat{\phi}_{n}'\left(0, \mathbb{Z}_{n,\alpha}^{*}\right)$ . In particular, when  $\mathbb{Z}_{n,\theta}^{*} = r_{n}\left(\hat{\theta}_{n}^{*} - \theta_{0}\right)$  and  $\mathbb{Z}_{n,\alpha}^{*} = r_{n}\left(\hat{\alpha}_{n}^{*} - \alpha_{0}\right)$ , the distribution of  $r_{n}\left(\phi\left(\hat{\theta}_{n}, \hat{\alpha}_{n}\right) - \phi\left(\theta_{0}, \alpha_{0}\right)\right)$  is approximated by  $\frac{1}{\epsilon_{n}}\left(\phi\left(\hat{\theta}_{n} + \epsilon_{n}r_{n}\left(\hat{\theta}_{n}^{*} - \hat{\theta}_{n}\right), \hat{\alpha}_{n}\right) - \phi\left(\hat{\theta}_{n}, \hat{\alpha}_{n}\right)\right) + r_{n}\left(\phi\left(\hat{\theta}_{n}, \hat{\alpha}_{n}^{*}\right) - \phi\left(\hat{\theta}_{n}, \hat{\alpha}_{n}\right)\right)$ .

The Fang and Santos (2014) assumptions (2.1, 2.2, 2.3, 3.1, 3.2 and 3.3) are implicitly understood to hold jointly in  $\theta, \alpha$  in the rest of this section.

**Theorem 3.7** The result of Theorem 3.3 holds with (18) under Assumption 3.6.

A special case of Assumption 3.6 is when estimating  $\alpha$  does not affect the asymptotic distribution, as in for example the weighting matrix in moment inequality models (e.g., Andrews and Soares (2010)).

Assumption 3.7  $\phi'_{\theta,\alpha}(h_{\theta},h_{\alpha}) = \phi'_{\theta,\alpha}(h_{\theta},0)$  for all  $h = (h_{\theta},h_{\alpha})$ .

Under A3.7, it is natural to estimate  $\phi'_{\theta,\alpha}(h)$  by  $\hat{\phi}'_n(h_{\theta},0)$ , and replace  $\hat{\phi}'_n(\mathbb{Z}^*_n)$  in (5) with

$$\hat{\phi}_{n}'\left(\mathbb{Z}_{n,\theta}^{*},0\right) = \frac{\phi\left(\hat{\theta}_{n}+\epsilon_{n}\mathbb{Z}_{n,\theta}^{*},\hat{\alpha}_{n}\right)-\phi\left(\hat{\theta}_{n},\hat{\alpha}_{n}\right)}{\epsilon_{n}}$$

Pointwise consistency of  $\hat{\phi}'_n(h_{\theta}, 0)$  for  $\phi'_{\theta,\alpha}(h_{\theta}, 0)$  follows directly from Theorem 3.3 with  $h = (h_{\theta}, 0)$ . Furthermore,  $\hat{\phi}'_n(h_{\theta}, 0)$  is Lipschitz in  $h_{\theta}$  as long as  $\phi(\theta, \alpha)$  is Lipschitz in  $\theta$  uniformly in  $\alpha$ :  $\left\|\hat{\phi}'_n(h_1, 0) - \hat{\phi}'_n(h_2, 0)\right\|_{\mathbb{E}} = \left\|\frac{\phi(\hat{\theta}_n + \epsilon_n h_1, \hat{\alpha}_n) - \phi(\hat{\theta}_n + \epsilon_n h_2, \hat{\alpha}_n)}{\epsilon_n}\right\|_{\mathbb{E}} \leq C \|h_1 - h_2\|_{\mathbb{D}}.$ 

Under Assumption 3.7, we also obtain uniform size control with  $\phi(\theta, \alpha)$  for (13) and (14), whenever  $\phi(\theta, \alpha)$  is convex in  $\theta$  for each  $\alpha$ . In this case, analogous to (15), for any realization zfrom  $\mathbb{G}_{0,\theta}$ , where  $\mathbb{Z}_{n,\theta}^* \xrightarrow{\mathbb{P}} \mathbb{G}_{0,\theta}$ ,

$$r_n\left(\phi\left(\theta_0 + \frac{z}{r_n}, \alpha_0\right) - \phi\left(\theta_0, \alpha_0\right)\right) \le \frac{1}{\epsilon_n}\left(\phi\left(\theta_0 + \epsilon_n z, \alpha_0\right) - \phi\left(\theta_0, \alpha_0\right)\right),\tag{19}$$

so that  $\frac{1}{\epsilon_n} \left( \phi \left( \theta_0 + \epsilon_n \mathbb{G}_{0,\theta}, \alpha_0 \right) - \phi \left( \theta_0, \alpha_0 \right) \right)$  stochastically dominates  $r_n \left( \phi \left( \theta_0 + \frac{\mathbb{G}_{0,\theta}}{r_n}, \alpha_0 \right) - \phi \left( \theta_0, \alpha_0 \right) \right)$ . Directional differentiability and Assumption 3.7 ensure that  $r_n \left( \phi \left( \theta_0 + \frac{\mathbb{G}_{0,\theta}}{r_n}, \alpha_0 \right) - \phi \left( \theta_0, \alpha_0 \right) \right)$  is close to  $r_n \left( \phi \left( \hat{\theta}_n, \hat{\alpha}_n \right) - \phi \left( \theta_0, \alpha_0 \right) \right)$  while  $\frac{1}{\epsilon_n} \left( \phi \left( \theta_0 + \epsilon_n \mathbb{G}_{0,\theta}, \alpha_0 \right) - \phi \left( \theta_0, \alpha_0 \right) \right)$  is close to  $\hat{\phi}'_n \left( \mathbb{Z}^*_{n,\theta}, 0 \right)$ . Formally, under Assumption 3.7, Assumptions 3.3 and 3.4 are only required to hold in  $\theta_0$ :

**Assumption 3.8** For all  $\theta_0$ ,  $\alpha_0$ , and t > 0,  $\frac{\phi(\theta_0 + t\mathbb{G}_{0,\theta}, \alpha_0) - \phi(\theta_0, \alpha_0)}{t}$  is nondecreasing in t.

**Assumption 3.9** Suppose  $\mathbb{D}_{0,\theta}$  is convex. For any  $\theta_0$  and  $\alpha_0$ , for all  $\eta$  and  $\nu$  sufficiently close to zero, and for all t > 0,  $\frac{\phi'_{\theta_0,\alpha_0}(\eta + t\mathbb{G}_{0,\theta},\nu) - \phi'_{\theta_0,\alpha_0}(\eta,\nu)}{t}$  is nondecreasing in t. Furthermore, Assumption 3.5 holds with  $\theta_0, \alpha_0$  and for any  $h = (h_{\theta}, h_{\alpha}) = o(1)$ .

Then we can state the following theorem.

**Theorem 3.8** The conclusions of Theorem 3.5 hold under its stated conditions and Assumption 3.7, where we now call  $J_{\epsilon_n}(x_n, P_n)$  the distribution function of  $\hat{\phi}'_n\left(\mathbb{Z}^*_{n,\theta}, 0\right)$ , and  $J_n(x_n, P_n)$  that of  $r_n\left(\phi\left(\hat{\theta}_n, \hat{\alpha}_n\right) - \phi\left(\theta_0, \alpha_0\right)\right)$ . Furthermore, the conclusions of Theorem 3.6 hold under its stated conditions and Assumption 3.7, when  $\hat{c}_{1-\tau}$  refers to the  $(1-\tau)$ th percentile of the conditional distribution of  $\hat{\phi}'_n\left(\mathbb{Z}^*_{n,\theta}, 0\right)$  given the data, and if for any  $\theta_0 \in \Theta$ , either Assumption 3.8 or Assumption 3.9 holds.

While we have required  $r_n(\hat{\alpha}_n - \alpha_0) = O_p(1)$ , in many applications the weaker condition  $\hat{\alpha}_n \xrightarrow{p} \alpha_0$  suffices, such as for the variance in a t-statistic and the weighting matrix for moment conditions. However, in these problems  $r_n(\hat{\alpha}_n - \alpha_0) = O_p(1)$  always holds under stronger regularity conditions.

When  $\phi(\cdot, \cdot)$  is fully Hadamard differentiable, Assumption 3.6 holds with

$$\phi_{\theta,\alpha}'(h_{\theta},h_{\alpha}) = \frac{\partial}{\partial\theta}\phi_{\theta,\alpha}(h_{\theta},0) + \frac{\partial}{\partial\alpha}\phi_{\theta,\alpha}(0,h_{\alpha}).$$

In this case the bootstrap can approximate the distribution of  $r_n \left( \phi \left( \hat{\theta}_n, \hat{\alpha}_n \right) - \phi \left( \theta_0, \alpha_0 \right) \right)$  by that of  $r_n \left( \phi \left( \hat{\theta}_n + r_n^{-1} \mathbb{Z}_{n,\theta}^*, \hat{\alpha}_n + r_n^{-1} \mathbb{Z}_{n,\alpha}^* \right) - \phi \left( \hat{\theta}_n, \hat{\alpha}_n \right) \right)$ , or by that of  $r_n \left( \phi \left( \hat{\theta}_n + r_n^{-1} \mathbb{Z}_{n,\theta}^*, \hat{\alpha}_n \right) - \phi \left( \hat{\theta}_n, \hat{\alpha}_n \right) + \phi \left( \hat{\theta}_n, \hat{\alpha}_n + r_n^{-1} \mathbb{Z}_{n,\alpha}^* \right) - \phi \left( \hat{\theta}_n, \hat{\alpha}_n \right) \right).$ 

In particular, if  $\phi(\cdot)$  is a model parameter itself (now denoted  $\theta$ ), and if  $\theta$  denotes the underlying distribution (now denoted P), then the distribution of  $\hat{\theta}_n - \theta_0 = \theta(P_n, \hat{\alpha}_n) - \theta(P, \alpha_0)$  can be approximated by  $\theta(P_n^*, \hat{\alpha}_n^*) - \theta(P_n, \hat{\alpha}_n)$ , where  $P_n^*$  is the bootstrap data set and  $\hat{\alpha}_n^*$  is computed on the same bootstrap data set. In some situations, if  $\alpha$  is computed from an independent data set such that  $\hat{\alpha}_n \sim N(\alpha, \hat{\Omega})$ , then  $\hat{\alpha}_n^*$  can be draws from  $N(\hat{\alpha}_n, \hat{\Omega})$ . In this case an alternative approximation is  $\theta(P_n^*, \hat{\alpha}_n) - \theta(P_n, \hat{\alpha}_n) + \theta(P_n, \hat{\alpha}_n^*) - \theta(P_n, \hat{\alpha}_n)$  where  $\theta(P_n^*, \hat{\alpha}_n) - \theta(P_n, \hat{\alpha}_n)$  can also be replaced by any approximate distribution of  $\hat{\theta}_n$  treating  $\hat{\alpha}_n$  as known.

## 3.4 Application to Partially Identified Models: The $\mathcal{L}_1$ version

As an application, we relate the numerical delta method to a  $\mathcal{L}_1$  version of the partially identified model studied by Andrews and Soares (2010). While the current partial identification literature chooses to work with  $S(x, \Sigma) = \sum_{k=1}^{K} (x_k^-)^2$ , an alternative is to choose  $S(\cdot)$  to be a  $L_p$  norm. For example, we may choose  $S(x) = \min_{h \in \Lambda = R_+^k} ||x - h||_p = \left(\sum_{i=1}^k (x_i^-)^p\right)^{1/p}$ . For p = 2 and when a weighting matrix W is employed,

$$S(x, W) = \min_{h \in \Lambda = R_{+}^{k}} \sqrt{(x-h)' W(x-h)}.$$

A consistent estimate  $\hat{W}$  of the weighting matrix W is often available, and can be treated as a nuisance parameter that does not affect the asymptotic distribution in the sense of assumption 3.7.

If such a  $L_p$  norm is used instead in Andrews and Soares (2010), then  $S(\cdot)$  is convex and theorem 3.5 can be applied. On the one hand, whether to take the 1/p root makes no difference in a point identified model since optimization is invariant to monotonic transformations. On the other hand, it implies a different directional derivative, and does make a difference in set identified models and GMS methods.

Suppose we are testing  $H_0: \theta_0 \ge 0$  using the sample mean  $\hat{\theta}_n$ . Let's consider the case of p = 2 and a single moment equality. If we do not take the square root, we reject whenever  $n\left(\hat{\theta}_n^-\right)^2$ 

is greater than the  $(1 - \alpha)^{th}$  percentile of  $\left(\left(\frac{\hat{\theta}_n}{\epsilon_n} + \mathbb{Z}_n^*\right)^-\right)^2 - \left(\left(\frac{\hat{\theta}_n}{\epsilon_n}\right)^-\right)^2$ , where  $\mathbb{Z}_n^*$  is a normal random variable. However, if we take the square root, we reject whenever  $\sqrt{n} \left(\hat{\theta}_n^-\right)$  is greater than the  $(1 - \alpha)^{th}$  percentile of  $\left(\frac{\hat{\theta}_n}{\epsilon_n} + \mathbb{Z}_n^*\right)^- - \left(\frac{\hat{\theta}_n}{\epsilon_n}\right)^-$ . The transformation for the critical values is not the same as the transformation for the test statistic, and therefore the resulting rejection areas will be different.

## 4 Second Order Numerical Directional Delta Method

In situations in which the first order delta method limiting distribution is degenerate, the second (or higher) order delta method may provide the necessary nondegenerate large sample approximation. For example, Andrews and Soares (2010) conducts inference using  $\phi(\theta) = \sum_{k=1}^{K} (\theta_k^-)^2$ , which has a first order directional derivative of  $\phi'_{\theta}(h) = -\sum_{k=1}^{K} 2\theta_k^- h_k$ . Under the null hypothesis of  $\inf_{k=1...K} \theta_k \ge 0, \ \phi'_{\theta}(h) = 0$ , which leads to a degenerate first order delta method limiting distribution.

We will maintain the assumption that  $\phi(\cdot)$  is first order Hadamard differentiable at  $\theta_0$ . The second order Hadamard directional derivative at  $\theta_0$  in the direction h tangential to  $\mathbb{D}_0 \subseteq \mathbb{D}$  is defined as

$$\phi_{\theta_0}''(h) \equiv \lim_{t_n \downarrow 0, h_n \to h \in \mathbb{D}_0} \frac{\phi(\theta_0 + t_n h_n) - \phi(\theta_0) - t_n \phi_{\theta_0}'(h_n)}{\frac{1}{2} t_n^2}$$
(20)

Sufficient conditions for the existence of  $\phi_{\theta_0}''(h)$  are that  $\phi(\theta)$  is Hadamard differentiable uniformly in  $\theta$  around some neighborhood of  $\theta_0$  and that  $\phi_{\theta}'(h)$  is directionally differentiable in  $\theta$  at  $\theta_0$ . Although the definition of the second order directional derivative contains only one direction h, in principle we can use different directions  $h_1$  and  $h_2$ . For  $g(t_n, h_n^1, h_n^2) = t_n^{-1} \left( \phi_{\theta_0+t_n h_n^1}'(h_n^2) - \phi_{\theta_0}'(h_n^2) \right)$ ,  $\lim_{t_n \downarrow 0, (h_n^1, h_n^2) \to (h_1, h_2)} g(t_n, h_n^1, h_n^2) = \phi_{\theta_0}''(h_1, h_2)$  for  $h_1 \in \mathbb{D}_0, h_2 \in \mathbb{D}_0$ . In this paper, if there is only one argument in the  $\phi_{\theta_0}'(\cdot)$  function, then we are assuming that  $h_1 = h_2$ .

Note that  $\phi_{\theta_0}''(h)$  is continuous with respect to  $h \in \mathbb{D}_0$ , and it is also positively homogeneous of degree 2:  $\phi_{\theta_0}''(ch) = c^2 \phi_{\theta_0}''(h)$  for all  $c \ge 0$  and  $h \in \mathbb{D}_0$ . A simple illustrative example is  $\phi(\theta) = (\theta^-)^2$ . For this function, the first order directional derivative is  $\phi_{\theta}'(h) = -2\theta^-h$ , which is identically zero for  $\theta \ge 0$ . The second order directional derivative is  $\phi_{\theta_0}''(h) = 2(h^-)^2 \mathbf{1}$  ( $\theta_0 = 0$ ) +  $2h^2\mathbf{1}$  ( $\theta_0 < 0$ ).

The first part of the following theorem is due to Römisch (2005) and Shapiro (2000); in the second part we incorporate the numerical directional derivative.  $^{5}$ 

<sup>&</sup>lt;sup>5</sup>Recent independent work by Chen and Fang (2015) also studies inference under first order degeneracy

**Theorem 4.1** (Second Order Directional Delta Method): Suppose  $\mathbb{D}$  and  $\mathbb{E}$  are Banach Spaces and  $\phi : \mathbb{D}_{\phi} \subseteq \mathbb{D} \mapsto \mathbb{E}$  is second order Hadamard directionally differentiable at  $\theta_0$  tangentially to  $\mathbb{D}_0$ . Let  $\hat{\theta}_n : \{X_i\}_{i=1}^n \mapsto \mathbb{D}_{\phi}$  be such that for some  $r_n \uparrow \infty$ ,  $r_n\{\hat{\theta}_n - \theta_0\} \rightsquigarrow \mathbb{G}_0$  in  $\mathbb{D}$  and assume the support of  $\mathbb{G}_0$  is included in  $\mathbb{D}_0$ . Then,

$$r_n^2 \left[ \phi(\hat{\theta}_n) - \phi(\theta_0) - \phi'_{\theta_0}(\hat{\theta}_n - \theta_0) \right] \rightsquigarrow \mathcal{J} \equiv \frac{1}{2} \phi''_{\theta_0}(\mathbb{G}_0)$$
(21)

Let  $\epsilon_n \to 0$ ,  $r_n \epsilon_n \to \infty$ , and  $\mathbb{Z}_n^* \stackrel{\mathbb{P}}{\rightsquigarrow} \mathbb{G}_0$ . Then if  $\phi'_{\theta_0}(h) \equiv 0 \ \forall h \in \mathbb{D}_0$ ,

$$\frac{\phi\left(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*\right) - \phi\left(\hat{\theta}_n\right)}{\epsilon_n^2} \xrightarrow{\mathbb{P}} \mathcal{J} \equiv \frac{1}{2}\phi_{\theta_0}''(\mathbb{G}_0).$$
(22)

Pointwise asymptotic validity of the numerical directional delta method is justified by (22). There are several alternatives for approximating  $\frac{1}{2}\phi_{\theta_0}''(\mathbb{G}_0)$ . First, the left hand side of (22) can be replaced by  $\hat{\phi}_n''(\mathbb{Z}_n^*)$  where the second order directional derivative can be estimated by

$$\hat{\phi}_n''(h) \equiv \frac{\phi(\hat{\theta}_n + 2\epsilon_n h) - 2\phi(\hat{\theta}_n + \epsilon_n h) + \phi(\hat{\theta}_n)}{\epsilon_n^2}$$
(23)

**Theorem 4.2** Under convexity of  $\mathbb{D}_0$  and the same conditions as in Theorem 4.1, except without  $\phi'_{\theta_0}(h) \equiv 0$ , for  $\hat{\phi}''_n(h)$  in (23),  $\hat{\phi}''_n(\mathbb{Z}^*_n) \xrightarrow{\mathbb{P}} \phi''_{\theta_0}(\mathbb{G}_0)$ .

If the first derivative  $\phi'_{\theta}(h)$  is analytically known, as in Andrews and Soares (2010), another alternative is to estimate the second order directional derivative (21) by

$$\bar{\phi}_n''(h_1, h_2) \equiv \frac{\phi_{\hat{\theta}_n + \epsilon_n h_1}'(h_2) - \phi_{\hat{\theta}_n}'(h_2)}{\epsilon_n}$$
(24)

**Theorem 4.3** For  $\bar{\phi}''_n(h,h)$  defined in (24),  $\bar{\phi}''_n(\mathbb{Z}^*_n,\mathbb{Z}^*_n) \stackrel{\mathbb{P}}{\leadsto} \phi''_{\theta_0}(\mathbb{G}_0).$ 

We can show that  $\bar{\phi}''_n(h,h) = \frac{\phi'_{\hat{\theta}_n + \epsilon_n h}(h) - \phi'_{\hat{\theta}_n}(h)}{\epsilon_n}$  is Lipschitz whenever  $\phi'_{\theta}(h)$  is. **Theorem 4.4** If  $\phi'_{\theta}(h)$  :  $\mathbb{D}_{\phi} \to \mathbb{E}$  is Lipschitz in  $\theta$  and h, then for all  $\epsilon_n \downarrow 0$ ,  $\bar{\phi}''_n(h,h) = \frac{\phi'_{\hat{\theta}_n + \epsilon_n h}(h) - \phi'_{\hat{\theta}_n}(h)}{\epsilon_n}$  is Lipschitz in h.

Theorem 4.1 applies when  $\phi'_{\theta_0}(h) \equiv 0$ , in which case  $r_n^2 \left( \phi\left(\hat{\theta}_n\right) - \phi\left(\theta_0\right) \right) \rightsquigarrow \mathcal{J}$ . By Theorems 4.1, 4.2 and 4.3,  $\frac{\phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*) - \phi(\hat{\theta}_n)}{\epsilon_n^2}$  in (22),  $\hat{\phi}''_n(\mathbb{Z}_n^*)$  in (23) and  $\bar{\phi}''_n(\mathbb{Z}_n^*, \mathbb{Z}_n^*)$  in (24) converges to the same limiting distribution  $\mathcal{J} = \frac{1}{2}\phi''_{\theta_0}(\mathbb{G}_0)$  under fixed  $\theta_0$  asymptotics and under a local drifting sequence of parameters  $\theta_n$  where  $r_n(\theta_n - \theta_0) \rightarrow c$  for  $||c|| < \infty$ . In the latter case, let  $\mathbb{Z}_n =$  $r_n\left(\hat{\theta}_n - \theta_n\right) \rightsquigarrow \mathbb{G}_0$ . Then  $r_n^2\left(\phi(\hat{\theta}_n) - \phi(\theta_n)\right) - r_n^2\left(\phi\left(\frac{1}{2}\left(r_n(\theta_n - \theta_0)\right)\right) - \phi(\theta_0)\right) \rightsquigarrow \frac{1}{2}\phi''_{\theta_0}(c + \mathbb{G}_0) - \frac{1}{2}\phi''_{\theta_0}(c + \mathbb{G}$ 

$$r_n^2\left(\phi\left(\frac{1}{r_n}\left(r_n\left(\theta_n-\theta_0\right)+\mathbb{Z}_n\right)\right)-\phi(\theta_0)\right)-r_n^2\left(\phi\left(\frac{1}{r_n}\left(r_n\left(\theta_n-\theta_0\right)\right)\right)-\phi(\theta_0)\right) \rightsquigarrow \frac{1}{2}\phi_{\theta_0}''(c+\mathbb{G}_0)-\frac{1}{2}\phi_{\theta_0}''(c).$$

The equalities follow from  $r_n (\theta_n - \theta_0) + \mathbb{Z}_n \rightsquigarrow c + \mathbb{G}_0$ ,  $r_n (\theta_n - \theta_0) \rightsquigarrow c$ , and the definition of the second order delta method.

The behaviors of  $\hat{\phi}_n''(\mathbb{Z}_n^*)$ ,  $\bar{\phi}_n''(\mathbb{Z}_n^*,\mathbb{Z}_n^*)$  and  $\frac{\phi(\hat{\theta}_n+\epsilon_n\mathbb{Z}_n^*)-\phi(\hat{\theta}_n)}{\epsilon_n^2}$  differ under a more distant local drifting sequence of parameters  $\frac{\theta_n-\theta_0}{\epsilon_n} \to c$ , when  $0 < ||c|| < \infty$ , which implies different finite sample behaviors.

On the one hand,  $\frac{1}{\epsilon_n^2} \left( \phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*) - \phi(\hat{\theta}_n) \right) \rightsquigarrow \frac{1}{2} \phi_{\theta_0}''(c + \mathbb{G}_0) - \frac{1}{2} \phi_{\theta_0}''(c)$ . On the other hand, for (23),

$$\frac{1}{2}\hat{\phi}_{n}^{\prime\prime}(\mathbb{Z}_{n}^{*}) = \frac{1}{2}\frac{1}{\epsilon_{n}^{2}}\left[\phi\left(\epsilon_{n}\left(\frac{\theta_{n}-\theta_{0}}{\epsilon_{n}}+\frac{\mathbb{Z}_{n}}{r_{n}\epsilon_{n}}+2\mathbb{Z}_{n}^{*}\right)\right)-\phi(\theta_{0})\right] - \frac{1}{\epsilon_{n}^{2}}\left[\phi\left(\epsilon_{n}\left(\frac{\theta_{n}-\theta_{0}}{\epsilon_{n}}+\frac{\mathbb{Z}_{n}}{r_{n}\epsilon_{n}}+\mathbb{Z}_{n}^{*}\right)\right)-\phi(\theta_{0})\right] + \frac{1}{2}\frac{1}{\epsilon_{n}^{2}}\left[\phi\left(\epsilon_{n}\left(\frac{\theta_{n}-\theta_{0}}{\epsilon_{n}}+\frac{\mathbb{Z}_{n}}{r_{n}\epsilon_{n}}\right)\right)-\phi(\theta_{0})\right] \rightarrow \frac{1}{4}\phi_{\theta_{0}}^{\prime\prime}(c+2\mathbb{G}_{0}) - \frac{1}{2}\phi_{\theta_{0}}^{\prime\prime}(c+\mathbb{G}_{0}) + \frac{1}{4}\phi_{\theta_{0}}^{\prime\prime}(c).$$

It can also be shown that for (24),

$$\frac{1}{2}\bar{\phi}_{n}^{\prime\prime}(\mathbb{Z}_{n}^{*},\mathbb{Z}_{n}^{*}) \equiv \frac{\phi_{\hat{\theta}_{n}+\epsilon_{n}\mathbb{Z}_{n}^{*}}^{\prime}(\mathbb{Z}_{n}^{*})-\phi_{\hat{\theta}_{n}}^{\prime}(\mathbb{Z}_{n}^{*})}{2\epsilon_{n}} \rightsquigarrow \frac{1}{2}\phi_{\theta_{0}}^{\prime\prime}(c+\mathbb{G}_{0},\mathbb{G}_{0})-\frac{1}{2}\phi_{\theta_{0}}^{\prime\prime}(c,\mathbb{G}_{0}).$$

The differences between various methods of estimating the second order derivative when  $\frac{\theta_n - \theta_0}{\epsilon_n} \to c$ can be illustrated using a simple test of  $H_0$ :  $\theta_0 \ge 0$  against  $H_1$ :  $\theta_0 < 0$ , which is converted to  $H_0$ :  $\phi(\theta_0) = 0$  against  $H_1$ :  $\phi(\theta_0) > 0$  using the test function  $\phi(\theta) = (\theta^-)^2$ , which has  $\phi'_{\theta}(h) = -2\theta^- h$  and  $\phi''_{\theta}(h) = 2(h^-)^2 \mathbf{1}(\theta = 0) + 2h^2 \mathbf{1}(\theta < 0)$ . Consider a level  $\alpha$  test with rejection region  $\{r_n^2 \phi(\hat{\theta}_n) \ge d_{1-\alpha}\}$ , where  $d_{1-\alpha}$  is the  $1-\alpha$  percentile of one of the following four distributions:  $(1) \frac{1}{\epsilon_n^2} \phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*); (2) \frac{1}{2} \frac{1}{\epsilon_n} \left(\phi'_{\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*}(\mathbb{Z}_n^*) - \phi'_{\hat{\theta}_n}(\mathbb{Z}_n^*)\right); (3) \frac{1}{\epsilon_n^2} \left(\phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*) - \phi(\hat{\theta}_n)\right); (4) \frac{1}{2} \hat{\phi}''_n(\mathbb{Z}_n^*).$ Let  $\theta_0 = 0$  and  $\frac{\theta_n}{\epsilon_n} \to c$ . The corresponding limiting distributions are

1. 
$$\frac{1}{2}\phi_0''(c+\mathbb{G}_0) = ((\mathbb{G}_0+c)^{-})^2$$

2. 
$$\frac{1}{2} \left( \phi'_{\mathbb{G}_0+c}(\mathbb{G}_0) - \phi'_c(\mathbb{G}_0) \right) = -(\mathbb{G}_0+c)^- \mathbb{G}_0 + c^- \mathbb{G}_0$$

3. 
$$\frac{1}{2}\phi_0''(c+\mathbb{G}_0) - \frac{1}{2}\phi_0''(c) = \left((\mathbb{G}_0+c)^{-}\right)^2 - (c^{-})^2$$

4. 
$$\frac{1}{4}\phi_0''(c+2\mathbb{G}_0) - \frac{1}{2}\phi_0''(c+\mathbb{G}_0) + \frac{1}{4}\phi_0''(c) = \frac{1}{2}\left(\left(2\mathbb{G}_0+c\right)^{-}\right)^2 - \left(\mathbb{G}_0^{-}\right)^2 + \frac{1}{2}\left(c^{-}\right)^2$$

First consider the case of c > 0, which corresponds to size control. In this case it is not difficult to see that (4)  $\succeq$  (2)  $\succeq$  (1) = (3) in descending order of first order stochastic dominance. Furthermore, (1) through (4) all stochastically dominate the distribution of the test statistic under the null of  $\theta_0 > 0$ , which is  $\lim_{h\to\infty} \frac{1}{2}\phi_{\theta_0}''(h + \mathbb{G}_0) - \frac{1}{2}\phi_{\theta_0}''(h) = 0$  because  $r_n(\theta_n - \theta_0) \to \infty$  when  $\frac{\theta_n - \theta_0}{\epsilon_n} \to c$ . By imposing a zero first order derivative under the null, (2) and (4) provide better finite sample size control. However, comparing the finite sample powers of these tests when  $\frac{\theta_n}{\epsilon_n} \to c < 0$  does not give a conclusive ranking. While it is clear that the recentered version (3) is always more powerful than the nonrecentered version (1), there does not seem to be a uniform ranking among (2), (3), and (4). The ranking might depend on the range of the alternative hypothesis.

## 5 Monte Carlo Simulations

In this section we report two finite sample simulations. The first uses a simple parametric example to show consistency of the first order numerical delta method, while the second applies the second order numerical delta method to the moment inequalities setup in Andrews and Soares (2010).

#### 5.1 Confidence intervals in a basic model

Consider a simple set up of i.i.d data  $X_i \stackrel{iid}{\sim} N(\theta_n, 1)$  and  $\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n X_i \equiv \bar{X}$ . The function of interest is  $\phi(\theta) = a\theta^+ + b\theta^-$ , where  $\theta^+ = \max\{\theta, 0\}$  and  $\theta^- = -\min\{\theta, 0\}$ . Functions of this type appear in Hansen (2017)'s continuous threshold regression model and in moment inequality inference models. We approximate the distribution of  $r_n(\phi(\hat{\theta}_n) - \phi(\theta_n))$  using  $\hat{\phi}'_n(\mathbb{Z}_n^*) = \frac{\phi(\hat{\theta} + \epsilon_n \mathbb{Z}_n^*) - \phi(\hat{\theta})}{\epsilon_n}$ . where  $\mathbb{Z}_n^* \stackrel{\mathbb{P}}{\longrightarrow} \mathbb{G}_0$  and  $r_n(\hat{\theta}_n - \theta_n) \rightsquigarrow \mathbb{G}_0$ . We use  $\mathbb{Z}_n^* = N(0, \hat{\sigma})$ , where  $\hat{\sigma} = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2}$ . For  $c_\alpha$  denoting the  $\alpha$  quantile of  $\hat{\phi}'_n(\mathbb{Z}_n^*)$  and  $d_\alpha$  denoting the  $\alpha$  quantile of  $|\hat{\phi}'_n(\mathbb{Z}_n^*)|$ , we report (1) a symmetric two sided interval  $\left[\phi(\hat{\theta}_n) - \frac{1}{r_n}d_{1-\alpha}, \phi(\hat{\theta}_n) + \frac{1}{r_n}d_{1-\alpha}\right]$ ; (2) an equal-tailed two-sided interval  $\left[\phi(\hat{\theta}_n) - \frac{1}{r_n}c_{\alpha/2}\right]$ ; (3) an upper one-sided confidence interval  $\left(-\infty, \phi(\hat{\theta}_n) - \frac{1}{r_n}c_{\alpha}\right]$ ; (4) a lower one-sided confidence interval  $\left[\phi(\hat{\theta}_n) - \frac{1}{r_n}c_{1-\alpha}, \infty\right)$ .

For a > 0, b > 0 or a > 0, b < 0, a > |b| or  $a < 0, b > 0, |a| < b, \phi(\theta)$  is a convex function of  $\theta$ . Then Theorem 3.5 implies that the lower one-sided interval is uniformly valid at least conservatively. Both the upper one-sided interval and as a result the equal-tailed two sided interval are only valid under fixed asymptotics, but can undercover for local drifting parameter sequences between orders of  $1/\sqrt{n}$  and  $\epsilon_n$ .

Analogously, for a < 0, b < 0 or a < 0, b > 0, a < |b| or  $a > 0, b < 0, |a| > b, \phi(\theta)$  is a concave function of  $\theta$ . Then Theorem 3.5 implies that the upper one-sided interval is uniformly valid at least conservatively. Both the lower one-sided interval and as a result the equal-tailed two sided interval are only valid under fixed asymptotics, but can undercover for local drifting parameter sequences between orders of  $1/\sqrt{n}$  and  $\epsilon_n$ .

For the two sided symmetric interval, note that in this model, the directional derivative  $\phi'_{\theta}(h)$ is given by (1) ah if  $\theta > 0$ ; (2) -bh if  $\theta < 0$ ; (3)  $ah^+ + bh^-$  if  $\theta = 0$ . It satisfies the condition that

$$|\phi_{\theta}'(h_1 + h_2) - \phi_{\theta}'(h_2)| \le |\phi_{\theta}'(h_1)|, \tag{25}$$

Note that  $|\phi'_{\theta}(\mathbb{G}_0 + c) - \phi'_{\theta}(c)|$  and  $|\phi'_{\theta}(\mathbb{G}_0)|$  are, respectively, the analytic limit and numerical delta method limit under the Fang and Santos (2014) local sequence  $\theta_n = c/\sqrt{n}$ . Therefore (25) implies that the symmetric two sided interval is at least conservatively valid under the local sequence of  $\theta_n = c/\sqrt{n}$ . The two sided symmetric interval may undercover, however, for the local parameter sequence of  $\theta_n = c\epsilon_n$ . In other words, when  $\sqrt{n\epsilon_n} \to \infty$ , neither the symmetric nor the equal-tailed two sided intervals are uniformly valid, but the symmetric interval is valid for a wider range of local parameter sequences than the equal-tailed interval.

The set of tables titled "Monte Carlo Simulations for the Normal Mean Model" show empirical coverage frequencies for a = 1.5, b = 0.5, which corresponds to convex  $\phi(\theta)$ . Results for concave  $\phi(\theta)$  are analogous and omitted for brevity. Empirical coverage frequencies are computed for four different values of  $\epsilon_n$ :  $n^{-1/6}, n^{-1/3}, n^{-1/2}, n^{-1}$ ; and eleven different values of  $\theta_n$ :  $-2, -n^{-1/6}, -n^{-1/3}, 0, n^{-1}, n^{-1/1.5}, n^{-1/2}, n^{-1/3}, n^{-1/6}, n^{-1/10}$ , and 2. The empirical coverage frequencies for the four different kinds of confidence intervals (symmetric two-sided, equal-tailed two-sided, upper one-sided, and lower one-sided) when  $\epsilon_n = n^{-1/6}, \epsilon_n = n^{-1/3}, \epsilon_n = n^{-1/2}, \text{ and } \epsilon_n = n^{-1}$  are summarized in tables 1 through 4, tables 5 through 8, tables 9 through 12, and tables 13 through 16 respectively. The nominal coverage frequency is 95%.

When  $\sqrt{n}\epsilon_n \to \infty$ , the symmetric two-sided confidence intervals have an empirical coverage frequency close to the nominal frequency in the regions  $\theta_n \in \{0, n^{-1}, n^{-1/1.5}, n^{-1/2}\}$  and  $\frac{\theta_n}{\epsilon_n} \to \pm \infty$ . The empirical coverage frequency is below the nominal frequency when  $\frac{\theta_n}{\epsilon_n} \to c$  for  $0 < c < \infty$ . The equal-tailed two-sided confidence intervals have an empirical coverage frequency close to the nominal frequency in the regions  $\theta_n \in \{0, n^{-1}\}$  and  $\frac{\theta_n}{\epsilon_n} \to \pm \infty$ . In the region where  $\theta_n \sqrt{n} \to c_1$ for  $0 < |c_1| \le \infty$  and  $\frac{\theta_n}{\epsilon_n} \to c_2$  for  $0 \le |c_2| < \infty$ , the empirical coverage frequency is far below the nominal frequency.

When  $\sqrt{n}\epsilon_n \to \infty$ , the lower one-sided confidence intervals provide conservatively valid coverage

for all values of  $\theta_n$ , which is to be expected given the theoretical results. On the other hand, the upper one-sided confidence intervals undercover for values of  $\theta_n$  that satisfy  $\theta_n \sqrt{n} \to c_1$  for  $|c_1| > 0$ and  $\frac{\theta_n}{\epsilon_n} \to c_2$  for  $0 \le |c_2| < \infty$  while providing coverage close to the nominal frequency for the other values of  $\theta_n$ .

#### 5.2 Small step size in the basic example

While the theory in the previous sections is provided for larger step sizes  $(\sqrt{n}\epsilon_n \to \infty)$ , it turns out that in the example above a small step size might also be a possible choice for constructing confidence intervals in some situations. In this section we let  $\sqrt{n}\epsilon_n \to 0$  and examine the consequences for the numerical delta method. Let  $\mathbb{Z}_n = \sqrt{n} \left(\hat{\theta}_n - \theta_n\right)$  so that  $(\mathbb{Z}_n^*, \mathbb{Z}_n) \rightsquigarrow (\mathbb{G}_1, \mathbb{G}_0)$ , where  $\mathbb{G}_1 \sim N(0, 1)$ ,  $\mathbb{G}_0 \sim N(0, 1)$ ,  $\mathbb{G}_1 \perp \mathbb{G}_0$ . Also note that  $\phi(\theta) = a\theta^+ + b\theta^-$  is homogeneous of degree one. We can write down the following heuristic calculations.

$$\hat{\phi}_{n}'(\mathbb{Z}_{n}^{*}) = \phi\left(\frac{\hat{\theta}_{n}}{\epsilon_{n}} + \mathbb{Z}_{n}^{*}\right) - \phi\left(\frac{\hat{\theta}_{n}}{\epsilon_{n}}\right) = \phi\left(\frac{\mathbb{Z}_{n}}{\sqrt{n}\epsilon_{n}} + \frac{\theta_{n}}{\epsilon_{n}} + \mathbb{Z}_{n}^{*}\right) - \phi\left(\frac{\mathbb{Z}_{n}}{\sqrt{n}\epsilon_{n}} + \frac{\theta_{n}}{\epsilon_{n}}\right)$$

Also note that  $\sqrt{n} \left( \phi\left(\hat{\theta}_n\right) - \phi\left(\theta_n\right) \right) = \phi\left(\mathbb{Z}_n + \sqrt{n}\theta_n\right) - \phi\left(\sqrt{n}\theta_n\right)$ . We now consider three regimes separately.

**Case 1:** If 
$$\sqrt{n}\theta_n \to 0$$
, then  $\sqrt{n} \left( \phi \left( \hat{\theta}_n \right) - \phi \left( \theta_n \right) \right) \rightsquigarrow a \mathbb{G}_0^+ + b \mathbb{G}_0^-$ . Also,  
 $\hat{\phi}'_n \left( \mathbb{Z}_n^* \right) \rightsquigarrow W = \begin{cases} a \mathbb{G}_1^+ & \text{with probability} \quad P \left( \mathbb{G}_0 > 0 \right) \\ -b \mathbb{G}_1^- & \text{with probability} \quad P \left( \mathbb{G}_0 < 0 \right) \end{cases}$ 

It can be verified that |W| and  $a\mathbb{G}_0^+ + b\mathbb{G}_0^-$  have the same distribution, so that two sided symmetric intervals are valid. By symmetry, so are the two sided equal-tailed intervals.

**Case 2:** If  $\sqrt{n}\theta_n = a_n \to \pm \infty$ , both two sided intervals are valid since the analytic limit and the numeric limit have the same distribution:

$$\sqrt{n} \left( \phi \left( \hat{\theta}_n \right) - \phi \left( \theta_n \right) \right) \rightsquigarrow \begin{cases} a \mathbb{G}_0 & \text{if } a_n > 0 \\ -b \mathbb{G}_0 & \text{if } a_n < 0 \end{cases} \xrightarrow{\phi'_n} (\mathbb{Z}_n^*) \rightsquigarrow \begin{cases} a \mathbb{G}_1 & \text{if } a_n > 0 \\ -b \mathbb{G}_1 & \text{if } a_n < 0 \end{cases}$$

**Case 3:** If  $\sqrt{n\theta_n} \to c$ , where  $0 < |c| < \infty$ , then the two distributions differ, and two sided intervals are generally invalid since

$$\sqrt{n} \left( \phi \left( \hat{\theta}_n \right) - \phi \left( \theta_n \right) \right) \rightsquigarrow a \left( c + \mathbb{G}_0 \right)^+ + b \left( c + \mathbb{G}_0 \right)^- - ac^+ - bc^-,$$
$$\hat{\phi}'_n \left( \mathbb{Z}_n^* \right) \rightsquigarrow \begin{cases} a \mathbb{G}_1 & \text{with probability} \quad P \left( \mathbb{G}_0 > -c \right) \\ -b \mathbb{G}_1 & \text{with probability} \quad P \left( \mathbb{G}_0 < -c \right) \end{cases}$$

However, in a special case of case 3, when a = b = 1, the analytic limit becomes  $|\mathbb{G}_0 + c| - |c|$ and the numeric limit becomes  $\mathbb{G}_1$ . Since  $|\mathbb{G}_1|$  first order stochastically dominates  $||\mathbb{G}_0 + c| - |c||$ , symmetric two sided intervals are at least conservatively valid.

The knife-edge case of  $\epsilon_n = n^{-1/2}$  corresponds essentially to the bootstrap. With the bootstrap,

$$\hat{\phi}_n'(\mathbb{Z}_n^*) \rightsquigarrow \phi\left(\mathbb{G}_0 + \mathbb{G}_1 + \lim\sqrt{n}\theta_n\right) - \phi\left(\mathbb{G}_0 + \lim\sqrt{n}\theta_n\right)$$

Comparing this to  $\sqrt{n} \left( \phi \left( \hat{\theta}_n \right) - \phi \left( \theta_n \right) \right) = \phi \left( \mathbb{Z}_n + \sqrt{n} \theta_n \right) - \phi \left( \sqrt{n} \theta_n \right) \rightsquigarrow \phi \left( \mathbb{G}_0 + \lim \sqrt{n} \theta_n \right) - \phi \left( \lim \sqrt{n} \theta_n \right)$  shows that when  $\theta_n = 0$ , the analytic limit is  $\phi \left( \mathbb{G}_0 \right)$  and the numerical limit is  $\phi \left( \mathbb{G}_0 + \mathbb{G}_1 \right) - \phi \left( \mathbb{G}_0 \right)$ . Since  $|\phi \left( \mathbb{G}_0 \right)|$  first order stochastically dominates  $|\phi \left( \mathbb{G}_0 + \mathbb{G}_1 \right) - \phi \left( \mathbb{G}_0 \right)|$ , the bootstrap symmetric two-sided interval will undercover. However, when  $\sqrt{n} |\theta_n|$  is larger (e.g. when  $\sqrt{n} \theta_n \to \infty$ ), the bootstrap symmetric two-sided interval will not undercover.

#### 5.3 Second Order Numerical Derivative

The purpose of these Monte Carlo simulations is to investigate the power and size of moment inequality tests of the form  $H_0: \inf_{j=1...J} \theta_{n,j} \ge 0$  and  $H_1: \inf_{j=1...J} \theta_{n,j} < 0$ . Let  $\phi(\theta) = \sum_{j=1}^{J} \left(\theta_j^-\right)^2 = \sum_{j=1}^{J} (-\min\{\theta_j, 0\})^2$  and  $\phi'_{\theta}(h) = -\sum_{j=1}^{J} 2\theta_j^- h_j$ . Data are drawn from  $X_i \stackrel{iid}{\sim} N(\theta_n, I_2)$  and  $\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \equiv \bar{X}$ . We reject when  $r_n^2 \phi(\hat{\theta}_n) > \hat{c}_{1-\alpha}$ , where  $\hat{c}_{1-\alpha}$  is the  $1 - \alpha$  quantile of one of the following four ways of estimating the second order numerical derivative:

- 1. And rews and Soares (2010) with 4th GMS function:  $\frac{1}{\epsilon_n^2}\phi(\hat{\theta}_n + \epsilon_n\mathbb{Z}_n^*)$
- 2. Derivative of Analytic First Order Derivative:  $\frac{1}{2} \frac{1}{\epsilon_n} \left( \phi'_{\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*}(\mathbb{Z}_n^*) \phi'_{\hat{\theta}_n}(\mathbb{Z}_n^*) \right)$
- 3. Numerical Second Order Derivative 1:  $\frac{1}{\epsilon_n^2} \left( \phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*) \phi(\hat{\theta}_n) \right)$
- 4. Numerical Second Order Derivative 2:  $\frac{1}{2}\hat{\phi}_n''(\mathbb{Z}_n^*) = \frac{1}{2}\frac{\phi(\hat{\theta}_n + 2\epsilon_n \mathbb{Z}_n^*) 2\phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*) + \phi(\hat{\theta}_n)}{\epsilon_n^2}$

We take  $\mathbb{Z}_n^* = N(0,\hat{\sigma})$ , where  $\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2}$ . We use four different choices of  $\epsilon_n$ :  $\sqrt{\log(n)}/\sqrt{n}, n^{-1/6}, n^{-1/3}, n^{-1/2}$  and eleven different choices of  $\theta_n$ :  $-n^{-1/6}, -n^{-1/3}, -n^{-1/2}, -n^{-1/1.5}, -n^{-1}, 0, n^{-1}, n^{-1/1.5}, n^{-1/2}, n^{-1/3},$  and  $n^{-1/6}$ . The choice of  $\epsilon_n = \sqrt{\log(n)}/\sqrt{n}$  is the one proposed by Andrews and Soares (2010). The set of tables titled "Monte Carlo Simulations for the Second Order Directional Delta Method" show the empirical rejection frequencies for the four different tests.

We can see that when  $\epsilon_n = \frac{\sqrt{\log(n)}}{\sqrt{n}}$ , the Andrews and Soares (2010) test has lower power than the other three tests for alternatives of the form  $\theta_n \in \{-n^{-1/3}, -n^{-1/2}, -n^{-1/1.5}, -n^{-1}\}$ . The Andrews and Soares (2010) test also has worse size control than all of the other tests except for the numerical second order derivative 1 test. The tests using the derivative of the analytic first order derivative and the numerical second order derivative 2 have the highest power against all alternatives and exhibit good size control.

As we go from  $\epsilon_n = \frac{\sqrt{\log(n)}}{\sqrt{n}}$  to  $\epsilon_n = n^{-1/6}$ , the power of the Andrews and Soares (2010) test increases so that it is approximately equal to the power of the tests using the derivative of the analytic first order derivative and the numerical second order derivative 2 for all alternatives except  $\theta_n = -n^{-1/2}$ , in which case the Andrews and Soares (2010) test has lower power. The Andrews and Soares (2010) test has slightly better size control than the tests using the derivative of the analytic first order derivative and the numerical second order derivative 2 when  $\theta_n \in \{0, n^{-1}\}$ .

As we decrease  $\epsilon_n$  from  $n^{-1/6}$  to  $n^{-1/2}$ , the power of the Andrews and Soares (2010) test for alternatives of the form  $\theta_n \in \{-n^{-1/6}, -n^{-1/3}, -n^{-1/2}\}$  decreases dramatically, and the size for  $\theta_n \in \{n^{-1}, n^{-1/1.5}, n^{-1/2}\}$  increases to above the nominal size. In contrast, for the test using the numerical second order derivative 2, the power for alternatives of the form  $\theta_n \in \{-n^{-1/6}, -n^{-1/3}, -n^{-1/2}, -n^{-1/1.5}\}$ and the size for all nonnegative  $\theta_n$  are not greatly affected. The power of the test using the derivative of the analytic first order derivative is not greatly affected for  $\theta_n \in \{-n^{-1/6}, -n^{-1/3}\}$  but the power does decrease dramatically for alternatives drifting faster to zero. The size of the test using the derivative of the analytic first order derivative decreases to almost 0 when  $\epsilon_n = n^{-1/2}$  while the size of the test using the numerical second order derivative 2 is not greatly affected.

Note that for a given value of  $\epsilon_n$  and any value of  $\theta_n$  in the alternative, the power of the Andrews and Soares (2010) test is always no greater than the power of the test using the numerical second order derivative 1. This is consistent with our prediction at the end of section 4. Moreover, for all values of  $\theta_n$  in the alternative and for  $\epsilon_n \in \{\sqrt{\log(n)}/\sqrt{n}, n^{-1/6}, n^{-1/3}\}$ , the power of the test using the numerical second order derivative 2 is the greatest among the four tests. Only when  $\epsilon_n = n^{-1/2}$  and only for alternatives  $\theta_n \in \{-n^{-1/1.5}, -n^{-1}\}$  drifting very quickly to zero is its power lower than that of the Andrews and Soares (2010) test and the test using the numerical second order derivative 1, while still having higher power than the test using the derivative of the analytic first order derivative.

## 6 Bias reduction

If the functional of interest  $\phi(\theta)$  admits a higher order directional Taylor expansion with a nondegenerate first order derivative, it is possible to modify the first order numerical directional delta method to make use of a higher order multiple point differentiation formula to reduce the bias in approximating the first order directional derivative numerically (Hong et al. (2010)). Estimating the first derivative using multiple point numerical differentiation is akin to the use of (one sided) higher order kernel and local polynomial methods for bias reduction. Specifically, assume that, for  $\phi_{\theta}^{(j)}(h)$  being functionals of h that are homogeneous of degree j, for  $h_n \to h$ ,

$$\phi\left(\theta + th_{n}\right) = \sum_{j=0}^{r} \frac{1}{j!} \phi_{\theta}^{(j)}\left(h_{n}\right) t^{j} + O\left(t^{r+1}\right), \qquad \phi_{\theta}^{(j)}\left(h_{n}\right) - \phi_{\theta}^{(j)}\left(h\right) = O\left(h_{n} - h\right) = o\left(1\right).$$
(26)

Consider a *p*-point operator for estimating the first order directional derivative, with  $p \leq r$ ,

$$\begin{aligned} L_{\theta,p}^{\epsilon_n}\left(h\right) &= \frac{1}{\epsilon_n} \sum_{l=0}^p a_l \phi\left(\theta + \epsilon_n lh\right) = \frac{1}{\epsilon_n} \sum_{l=0}^p a_l \left[\sum_{j=0}^r \frac{1}{j!} \phi_{\theta}^{(j)}\left(h\right) \epsilon_n^j l^j + O\left(\epsilon_n^{r+1}\right)\right] \\ &= \sum_{j=0}^p \phi_{\theta}^{(j)}\left(h\right) \frac{\epsilon_n^{j-1}}{j!} \sum_{l=0}^p a_l l^j + O\left(\epsilon_n^p\right) \end{aligned}$$

The coefficients  $a_l, l = 0, \ldots, p$  are determined by the system of equations:

$$\sum_{l=0}^{p} a_l l^j = \begin{cases} 1 & \text{for} \quad j=1\\ 0 & \text{for} \quad j \neq 1, j \leq p. \end{cases}$$
(27)

Using these choices for  $a_l$  and  $\epsilon_n \to 0$  leads to

$$L_{\theta,p}^{\epsilon_n}(h) = \phi_{\theta}^{(1)}(h) + O(\epsilon_n^p)$$
(28)

The p-point first order numerical derivative is

$$\hat{\phi}_n'(\mathbb{Z}_n^*; p) \equiv L^{\epsilon_n}_{\hat{\theta}, p}(\mathbb{Z}_n^*) \tag{29}$$

For example,  $\hat{\phi}'_n(\mathbb{Z}^*_n) = \frac{\phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}^*_n) - \phi(\hat{\theta}_n)}{\epsilon_n}$  corresponds to  $p = 1, a_0 = -1, a_1 = 1$ . When  $p = 2, a_0 = -\frac{3}{2}, a_1 = 2, a_2 = -\frac{1}{2}$ :

$$\hat{\phi}_{n}^{\prime}(\mathbb{Z}_{n}^{*};2) \equiv \frac{-\frac{1}{2}\phi\left(\hat{\theta}_{n}+2\epsilon_{n}\mathbb{Z}_{n}^{*}\right)+2\phi\left(\hat{\theta}_{n}+\epsilon_{n}\mathbb{Z}_{n}^{*}\right)-\frac{3}{2}\phi\left(\hat{\theta}_{n}\right)}{\epsilon_{n}}.$$
(30)

It is straightforward to generalize Theorem 3.1 to show consistency of (29).

**Theorem 6.1** Let the conditions in Theorem 3.1 and (26) hold. Then  $\hat{\phi}'_n(\mathbb{Z}^*_n;p) \xrightarrow{\mathbb{P}} \phi'_{\theta_0}(\mathbb{G}_0)$ .

## 7 Conclusion

We have proposed using the one-sided finite difference numerical directional derivative as a computationally simple estimator for the directional directive developed in Fang and Santos (2014). We have demonstrated that when the  $\phi(\cdot)$  function is Lipschitz, the numerical directional derivative is a uniformly consistent estimator for the directional derivative. Additionally, we have shown how to conduct uniformly valid inference using the first order delta method when  $\phi(\cdot)$  is a convex and Lipschitz function. Lastly, we have demonstrated how to consistently estimate the second order directional derivative and use it to conduct pointwise valid inference using the second order directional delta method.

## References

Amemiya, Takeshi, Advanced Econometrics, Harvard University Press, 1985. 3

- Andrews, Donald WK, "Inconsistency of the bootstrap when a parameter is on the boundary of the parameter space," *Econometrica*, 2000, 68 (2), 399–405. 14
- \_ and Gustavo Soares, "Inference for parameters defined by moment inequalities using generalized moment selection," *Econometrica*, 2010, 78 (1), 119–157. 8, 15, 16, 18, 19, 20, 22, 25, 26, 27, 34, 36
- and Moshe Buchinsky, "A Three-step Method for Choosing the Number of Bootstrap Repetitions," *Econometrica*, 2000, 68 (1), 23–51.
- and Xiaoxia Shi, "Nonparametric inference based on conditional moment inequalities," Journal of Econometrics, 2013. 4
- Bhattacharya, Rabindra N and Ramaswamy Ranga Rao, Normal approximation and asymptotic expansions, Vol. 64, SIAM, 1986. 14, 39

- Bonnans, J Frederic and Alexander Shapiro, Perturbation analysis of optimization problems, Springer Science & Business Media, 2013. 35
- Bugni, Federico A, Ivan A Canay, and Xiaoxia Shi, "Specification tests for partially identified models defined by moment inequalities," *Journal of Econometrics*, 2015, 185 (1), 259–282. 3, 8, 34
- Bugni, Federico, Ivan Canay, and Xiaoxia Shi, "Inference for functions of partially identified parameters in moment inequality models," *Quantitative Economics*, 2017, 8 (1), 1–38. 34, 36, 37
- Chen, Qihui and Zheng Fang, "Inference on Functionals Under First Order Degeneracy," Available at SSRN, 2015. 19
- Chernozhukov, Victor and Han Hong, "A MCMC Approach to Classical Estimation," Journal of Econometrics, 2003, 115 (2), 293–346. 4, 5
- Dümbgen, Lutz, "On nondifferentiable functions and the bootstrap," Probability Theory and Related Fields, 1993, 95 (1), 125–140. 2, 8, 9
- Fang, Zheng and Andres Santos, "Inference on Directionally Differentiable Functions," arXiv preprint arXiv:1404.3763, 2014. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 23, 28, 30, 31, 32, 37, 38, 42, 43
- Hansen, Bruce E, "Regression kink with an unknown threshold," Journal of Business & Economic Statistics, 2017, 35 (2), 228–240. 4, 16, 22
- Hirano, Keisuke and Jack R Porter, "Impossibility results for nondifferentiable functionals," Econometrica, 2012, 80 (4), 1769–1790. 2
- Hong, Han and Jessie Li, "The Numerical Bootstrap," 2014. working paper, Stanford University. 5
- \_\_, Aprajit Mahajan, and Denis Nekipelov, "Extremum estimation and numerical derivatives," forthcoming, Journal of Econometrics, 2010. 2, 27
- Kosorok, Michael R, Introduction to empirical processes and semiparametric inference, Springer, 2007. 4, 9, 40, 41, 42, 43, 45
- Politis, D., J. Romano, and M. Wolf, Subsampling, Springer Series in Statistics, 1999. 7
- Ponomareva, Maria, "Inference in Models Defined by Conditional Moment Inequalities with Continuous Covariates," 2010. working paper. 4
- Romano, Joseph P and Azeem M Shaikh, "Inference for identifiable parameters in partially identified econometric models," Journal of Statistical Planning and Inference, 2008, 138 (9), 2786–2807. 12, 13, 36
- and \_\_, "On the uniform asymptotic validity of subsampling and the bootstrap," The Annals of Statistics, 2012, 40 (6), 2798–2822. 12, 13, 14, 40, 41
- Römisch, Werner, "Delta method, infinite dimensional," Encyclopedia of Statistical Sciences, 2005. 19, 43
- Shapiro, Alexander, "On concepts of directional differentiability," Journal of optimization theory and applications, 1990, 66 (3), 477–487. 11
- \_\_\_\_, "Statistical inference of stochastic optimization problems," in "Probabilistic Constrained Optimization," Springer, 2000, pp. 282–307. 19
- Song, Kyungchul, "Local asymptotic minimax estimation of nonregular parameters with translation-scale equivariant maps," *Journal of Multivariate Analysis*, 2014, *125*, 136–158. 1, 8
- van der Vaart, AW and Jon Wellner, Weak Convergence and Empirical Processes, Springer, 1996. 37, 38, 43, 44, 45

Woutersen, Tiemen and John C Ham, "Calculating confidence intervals for continuous and discontinuous functions of parameters," cemmap working paper, Centre for Microdata Methods and Practice, 2013. 2, 4

## A Appendix

## A.1 List of Commonly Used Symbols

 $P_n$  empirical measure

 $P_n^*$  bootstrap empirical measure

 $\mathcal{Z}_n^* \qquad P_n + \epsilon_n \sqrt{n} \left( P_n^* - P_n \right)$ 

- $\rightsquigarrow$  weak convergence
- $\stackrel{\mathbb{P}}{\leadsto} \qquad \qquad \text{weak convergence conditional on the data}$

$$\theta^- - min(\theta, 0)$$

 $\theta^+ \qquad max(\theta,0)$ 

 $\rho_{BL_{1}}(F_{1},F_{2}) = \sup_{f \in BL_{1}} |Ef(F_{1}) - Ef(F_{2})|$ 

 $BL_1$  the space of Lipschitz functions  $f: \mathbb{D} \mapsto \mathbb{R}$  with Lipschitz norm bounded by 1

# A.2 Verification of Lipschitz property of $\phi(\cdot)$ in Fang and Santos (2014) examples

Fang and Santos (2014) Example 2.1  $\phi(\theta) = |\theta|, \mathbb{D} = \mathbb{R}, \mathbb{E} = \mathbb{R}.$ 

$$\|\phi(\theta+h) - \phi(\theta)\|_{\mathbb{E}} = |\phi(\theta+h) - \phi(\theta)| = ||\theta+h| - |\theta|| \le |h| \equiv ||h||_{\mathbb{D}}$$

Fang and Santos (2014) Example 2.2  $\phi(\theta) = max\{\theta^{(1)}, \theta^{(2)}\}, \mathbb{D} = \mathbb{R}^2, \mathbb{E} = \mathbb{R}.$ 

$$\begin{split} \|\phi(\theta+h) - \phi(\theta)\|_{\mathbb{E}} &= |\phi(\theta+h) - \phi(\theta)| = \left| \max\{\theta^{(1)} + h^{(1)}, \theta^{(2)} + h^{(2)}\} - \max\{\theta^{(1)}, \theta^{(2)}\} \right| \\ &= \begin{cases} \left|h^{(1)}\right| &, \theta^{(1)} + h^{(1)} \ge \theta^{(2)} + h^{(2)} \text{ and } \theta^{(1)} \ge \theta^{(2)} \\ \left|\theta^{(1)} - \theta^{(2)} + h^{(1)}\right| &, \theta^{(1)} + h^{(1)} \ge \theta^{(2)} + h^{(2)} \text{ and } \theta^{(1)} \le \theta^{(2)} \\ \left|\theta^{(2)} - \theta^{(1)} + h^{(2)}\right| &, \theta^{(1)} + h^{(1)} < \theta^{(2)} + h^{(2)} \text{ and } \theta^{(1)} \ge \theta^{(2)} \\ \left|h^{(2)}\right| &, \theta^{(1)} + h^{(1)} < \theta^{(2)} + h^{(2)} \text{ and } \theta^{(1)} < \theta^{(2)} \\ &\le 2(\left|h^{(1)}\right| + \left|h^{(2)}\right|) \equiv 2 \|h\|_{\mathbf{R}^2} \end{split}$$

Fang and Santos (2014) Example 2.3  $\phi(\theta) = \sup_{f \in \mathcal{F}} \theta(f)$ ,  $\mathbb{D} = \ell^{\infty}(\mathcal{F})$ ,  $\mathbb{E} = \mathbb{R}$ 

$$\begin{split} \left\|\phi(\theta+h) - \phi(\theta)\right\|_{\mathbb{E}} &= \left|\phi(\theta+h) - \phi(\theta)\right| = \left|\sup_{f \in \mathcal{F}} (\theta(f) + h(f)) - \sup_{f \in \mathcal{F}} \theta(f)\right| \\ &\leq \left|\sup_{f \in \mathcal{F}} (h(f))\right| \leq \sup_{f \in \mathcal{F}} |h(f)| \equiv \|h\|_{\ell^{\infty}(\mathcal{F})} \end{split}$$

Fang and Santos (2014) Example 2.4 For any  $\lambda$  in a convex, compact set  $\Lambda \subseteq \mathbf{R}^d$ ,  $\phi(\theta) = \sup_{p \in \mathbb{S}^d} \{ \langle p, \lambda \rangle - \theta(p) \}, \mathbb{D} = \mathcal{C}(\mathbb{S}^d), \mathbb{E} = \mathbf{R}$ 

$$\begin{split} \|\phi(\theta+h) - \phi(\theta)\|_{\mathbb{E}} &= |\phi(\theta+h) - \phi(\theta)| \\ &= \left| \sup_{p \in \mathbb{S}^d} \{\langle p, \lambda \rangle - \theta(p) - h(p)\} - \sup_{p \in \mathbb{S}^d} \{\langle p, \lambda \rangle - \theta(p)\} \right| \leq \sup_{p \in \mathbb{S}^d} |h(p)| \equiv \|h\|_{\mathcal{C}(\mathbb{S}^d)} \end{split}$$

Fang and Santos (2014) Example 2.5  $\phi((\theta^{(1)}, \theta^{(2)})) = \int_{\mathbf{R}} max\{\theta^{(1)}(u) - \theta^{(2)}(u), 0\}w(u)du$ , where  $w : \mathbf{R} \to \mathbf{R}_+$  is a positive, integrable weighting function.  $\mathbb{D} = \ell^{\infty}(\mathbf{R}) \times \ell^{\infty}(\mathbf{R}), \mathbb{E} = \mathbf{R}$ .

$$\begin{split} \|\phi(\theta+h) - \phi(\theta)\|_{\mathbb{E}} &= \left|\phi((\theta^{(1)} + h^{(1)}, \theta^{(2)} + h^{(2)})) - \phi((\theta^{(1)}, \theta^{(2)}))\right| \\ &= \left|\int_{\mathbf{R}} \max\{\theta^{(1)}(u) - \theta^{(2)}(u) + h^{(1)}(u) - h^{(2)}(u), 0\}w(u)du - \int_{\mathbf{R}} \max\{\theta^{(1)}(u) - \theta^{(2)}(u), 0\}w(u)du\right| \\ &= \begin{cases} \left|\int_{\mathbf{R}} (h^{(1)}(u) - h^{(2)}(u))w(u)du\right| & , \theta^{(1)}(u) - \theta^{(2)}(u) + h^{(1)}(u) - h^{(2)}(u) \ge 0 \\ & \text{and } \theta^{(1)}(u) - \theta^{(2)}(u) + h^{(1)}(u) - h^{(2)}(u))w(u)du\right| & , \theta^{(1)}(u) - \theta^{(2)}(u) + h^{(1)}(u) - h^{(2)}(u) \ge 0 \\ & \text{and } \theta^{(1)}(u) - \theta^{(2)}(u) + h^{(1)}(u) - h^{(2)}(u) \ge 0 \\ & \text{and } \theta^{(1)}(u) - \theta^{(2)}(u) + h^{(1)}(u) - h^{(2)}(u) \ge 0 \\ & \text{and } \theta^{(1)}(u) - \theta^{(2)}(u) + h^{(1)}(u) - h^{(2)}(u) \le 0 \\ & \text{and } \theta^{(1)}(u) - \theta^{(2)}(u) + h^{(1)}(u) - h^{(2)}(u) \le 0 \\ & \text{and } \theta^{(1)}(u) - \theta^{(2)}(u) + h^{(1)}(u) - h^{(2)}(u) \le 0 \\ & \text{and } \theta^{(1)}(u) - \theta^{(2)}(u) \ge 0 \\ & \text{otherwise} \end{cases}$$

$$\le C \sup_{u \in \mathbf{R}} \left| h^{(1)}(u) - h^{(2)}(u) \right|, \text{ where } C = \int_{\mathbf{R}} w(u) du$$

$$\le C (\sup_{u \in \mathbf{R}} \left| h^{(1)}(u) \right| + \sup_{u \in \mathbf{R}} \left| h^{(2)}(u) \right|) \equiv C \left\| (h^{(1)}, h^{(2)}) \right\|_{\ell^{\infty}(\mathbf{R}) \times \ell^{\infty}(\mathbf{R})}$$

**Fang and Santos (2014) Subset Projection** Let  $\mathbb{H}$  be a metric space with a norm  $\|\cdot\|_{\mathbb{H}}$  that admits the triangle inequality, so that for any elements  $a, b \in \mathbb{H}$ :

$$- ||a - b||_{\mathbb{H}} \le ||a||_{\mathbb{H}} - ||b||_{\mathbb{H}} \le ||a - b||_{\mathbb{H}}.$$

Let  $\Lambda \subseteq \mathbb{H}$  be a known set (that does not even have to be convex). We now show that  $\phi(\theta) \equiv \inf_{v \in \Lambda} \|\theta - v\|_{\mathbb{H}}$  is Lipschitz:  $|\phi(\theta_1) - \phi(\theta_2)| \leq \|\theta_1 - \theta_2\|_{\mathbb{H}}$ . For this purpose, choose two sequences  $v_{1n} \in \Lambda$  and  $v_{2n} \in \Lambda$  such that  $\phi(\theta_1) = \lim_{n \to \infty} \|\theta_1 - v_{1n}\|_{\mathbb{H}}$ ,  $\phi(\theta_2) = \lim_{n \to \infty} \|\theta_2 - v_{2n}\|_{\mathbb{H}}$ . By definition,  $\lim_{n \to \infty} \|\theta_1 - v_{1n}\|_{\mathbb{H}} \leq \|\theta_1 - v_{2n}\|_{\mathbb{H}}$ ,  $\lim_{n \to \infty} \|\theta_2 - v_{2n}\|_{\mathbb{H}} \leq \|\theta_2 - v_{1n}\|_{\mathbb{H}}$ . Then we can write, using the triangle inequality,

$$\phi(\theta_{1}) - \phi(\theta_{2}) \leq \lim_{n \to \infty} \left[ \|\theta_{1} - v_{2n}\|_{\mathbb{H}} - \|\theta_{2} - v_{2n}\|_{\mathbb{H}} \right] \leq \|\theta_{1} - \theta_{2}\|_{\mathbb{H}}$$
  
$$\phi(\theta_{2}) - \phi(\theta_{1}) \leq \lim_{n \to \infty} \left[ \|\theta_{2} - v_{1n}\|_{\mathbb{H}} - \|\theta_{1} - v_{1n}\|_{\mathbb{H}} \right] \leq \|\theta_{1} - \theta_{2}\|_{\mathbb{H}}.$$

Therefore we have shown that  $|\phi(\theta_2) - \phi(\theta_1)| \le \|\theta_1 - \theta_2\|_{\mathbb{H}}$ .

## A.3 Convexity of $\phi(\cdot)$ in Fang and Santos (2014) examples

**Fang and Santos (2014) Example 2.1** For any  $\lambda \in [0, 1]$  and  $\theta_1, \theta_2 \in \mathbb{R}$ ,

$$\phi(\lambda\theta_1 + (1-\lambda)\theta_2) = |\lambda\theta_1 + (1-\lambda)\theta_2| \le \lambda |\theta_1| + (1-\lambda)|\theta_2| = \lambda\phi(\theta_1) + (1-\lambda)\phi(\theta_2)$$
(31)

**Fang and Santos (2014) Example 2.2** For any  $\lambda \in [0, 1]$  and  $\theta_1, \theta_2 \in \mathbb{D}$ ,

$$\begin{aligned} \phi(\lambda\theta_1 + (1-\lambda)\theta_2) &= \max\{\lambda\theta_1^{(1)} + (1-\lambda)\theta_2^{(1)}, \lambda\theta_1^{(2)} + (1-\lambda)\theta_2^{(2)}\} \\ &\leq \lambda\max\{\theta_1^{(1)}, \theta_1^{(2)}\} + (1-\lambda)\max\{\theta_2^{(1)}, \theta_2^{(2)}\} = \lambda\phi(\theta_1) + (1-\lambda)\phi(\theta_2) \end{aligned}$$

**Fang and Santos (2014) Example 2.3** For any  $\lambda \in [0, 1]$  and  $\theta_1, \theta_2 \in \mathbb{D}$ ,

$$\begin{split} \phi(\lambda\theta_1 + (1-\lambda)\theta_2) &= \sup_{f \in \mathcal{F}} \{\lambda\theta_1(f) + (1-\lambda)\theta_2(f)\} \\ &\leq \lambda \sup_{f \in \mathcal{F}} \theta_1(f) + (1-\lambda) \sup_{f \in \mathcal{F}} \theta_2(f) = \lambda\phi(\theta_1) + (1-\lambda)\phi(\theta_2) \end{split}$$

Fang and Santos (2014) Example 2.5 Note that  $\phi((\theta^{(1)}, \theta^{(2)})) = \int_{\mathbf{R}} max\{\theta^{(1)}(u) - \theta^{(2)}(u), 0\}w(u)du$ , where  $w : \mathbf{R} \to \mathbf{R}_+$ , can be written as  $h(g(\theta))$ , where  $g(\theta) = g(\theta^{(1)}, \theta^{(2)}) = max\{\theta^{(1)} - \theta^{(2)}, 0\}$ and  $h(\gamma) = \int_{\mathbf{R}} \gamma(u)w(u)du$ . We can show that  $g(\theta)$  is convex and  $h(\gamma)$  is linear and nondecreasing , which implies that their composition  $h(g(\theta^{(1)}, \theta^{(2)}))$  is convex.  $g(\theta)$  is convex because for any  $\lambda \in [0, 1]$  and  $\theta_1, \theta_2 \in \mathbb{D}$ ,

$$g(\lambda\theta_1 + (1-\lambda)\theta_2) = max\{\lambda\theta_1^{(1)} + (1-\lambda)\theta_2^{(1)} - (\lambda\theta_1^{(2)} + (1-\lambda)\theta_2^{(2)}), 0\}$$
  
$$\leq \lambda max\{\theta_1^{(1)} - \theta_1^{(2)}, 0\} + (1-\lambda)max\{\theta_2^{(1)} - \theta_2^{(2)}, 0\}$$
  
$$= \lambda g(\theta_1) + (1-\lambda)g(\theta_2).$$

Also  $h(\gamma) = \int_{\mathbf{R}} \gamma(u) w(u) du$  is nondecreasing because w(u) is positive, and it's linear because integration is a linear operator.

**Convex Set Projection** Let  $\mathbb{H}$  be a metric space with a norm  $\|\cdot\|_{\mathbb{H}}$ . Let  $\Lambda \subseteq \mathbb{H}$  be a known convex set.  $\phi(\theta) \equiv \inf_{v \in \Lambda} \|\theta - v\|_{\mathbb{H}}$  can be interpreted as the shortest distance between  $\theta$  and a point in  $\Lambda$ . We will now verify convexity by showing that for any  $\lambda \in [0, 1]$  and  $\theta_1, \theta_2 \in \mathbb{H}$ ,  $\phi(\lambda \theta_1 + (1 - \lambda)\theta_2) \leq \lambda \phi(\theta_1) + (1 - \lambda)\phi(\theta_2)$ .

For any  $\lambda \in [0,1]$  and  $\theta_1, \theta_2 \in \mathbb{H}$ , there exists sequences  $v_{1n}$  and  $v_{2n}$  such that  $\phi(\theta_1) = \lim_{n \to \infty} \|\theta_1 - v_{1n}\|_{\mathbb{H}}$  and  $\phi(\theta_2) = \lim_{n \to \infty} \|\theta_1 - v_{2n}\|_{\mathbb{H}}$ . By convexity of  $\Lambda$ , for each n,  $\lambda v_{1n} + (1-\lambda)v_{2n} \in \Lambda$ . Then for each n,

$$\begin{split} \phi(\lambda\theta_1 + (1-\lambda)\theta_2) &= \inf_{v \in \Lambda} \|\lambda\theta_1 + (1-\lambda)\theta_2 - v\|_{\mathbb{H}} \\ &\leq \|\lambda\theta_1 + (1-\lambda)\theta_2 - (\lambda v_{1n} + (1-\lambda)v_{2n})\|_{\mathbb{H}} \\ &\leq \lambda \|\theta_1 - v_{1n}\|_{\mathbb{H}} + (1-\lambda) \|\theta_2 - v_{2n}\|_{\mathbb{H}} \end{split}$$

By taking  $n \to \infty$ ,

$$\phi(\lambda\theta_1 + (1-\lambda)\theta_2) \le \lim_{n \to \infty} \lambda \|\theta_1 - v_{1n}\|_{\mathbb{H}} + (1-\lambda) \|\theta_2 - v_{2n}\|_{\mathbb{H}} = \lambda\phi(\theta_1) + (1-\lambda)\phi(\theta_2).$$

#### A.4 Application of Directional Delta Method to Partially Identified Models

The directional delta method can be used to perform hypothesis tests, confidence set construction, model specification tests, and subvector inference in moment inequalities models. Let  $\mathcal{B}$  be a nonempty, compact parameter space for a partially identified parameter  $\beta_0$  defined by a set of J moment inequalities  $Pg(\cdot, \beta_0) \ge 0$ , where  $g(\cdot, \beta) = (g_j(\cdot, \beta), j = 1, ..., J)$  and  $Pg(\cdot, \beta)$  are continuous functions of  $\beta$ . We are interested in testing

$$H_0: \sup_{\beta \in \mathbb{B}^{j=1\dots J}} Pg_j(\cdot, \beta) \ge 0 \quad H_1: \sup_{\beta \in \mathbb{B}^{j=1\dots J}} Pg_j(\cdot, \beta) < 0$$

For example, in Bugni et al. (2017),  $\mathbb{B} = \mathbb{B}(\gamma) = \{\beta : f(\beta) = \gamma\}$ . In Bugni et al. (2015),  $\mathbb{B}$  corresponds to the entire parameter space  $\mathcal{B}$ . In Andrews and Soares (2010),  $\mathbb{B} = \beta^*$  corresponds to a singleton parameter value for a pointwise testing procedure. The infinite dimensional parameter  $\theta_0(\cdot)$  is a function of  $\beta \in \mathcal{B}$  defined by  $\theta_0(\beta) = Pg(\cdot, \beta)$ . The hypotheses can be converted into

$$H_{0}: \inf_{\beta \in \mathbb{B}} M(\theta_{0})(\beta) = 0 \qquad H_{1}: \inf_{\beta \in \mathbb{B}} M(\theta_{0})(\beta) > 0$$

where  $M(\cdot)$  is a mapping from the function  $\theta(\beta)$  to another function of  $\beta$ , such that  $M(\theta)(\beta) = S(\theta(\beta))$ , where S(x) is a nonincreasing and continuous function that satisfies  $S(x) \ge 0$  for all x, S(x) = 0 for all  $x \ge 0$ , and  $S(cx) = c^{\rho}S(x)$  for either  $\rho = 1$  or 2. Define

$$\phi(\theta) \equiv \inf_{\beta \in \mathbb{B}} M(\theta)(\beta) = (f \circ M)(\theta), \quad f(m) = \inf_{\beta \in \mathbb{B}} m(\beta).$$

We would like to obtain the limiting distribution of  $\sqrt{n}^{\rho} \left( \phi \left( \hat{\theta}_n \right) - \phi(\theta_0) \right) = \inf_{\beta \in \mathbb{B}} S \left( \sqrt{n} \hat{\theta}_n(\beta) \right) - \inf_{\beta \in \mathbb{B}} S \left( \sqrt{n} \theta_0(\beta) \right)$  under  $H_0$  using the directional delta method. We can compute the first order directional derivative using the chain rule:

$$\phi_{\theta}'(h) = (f \circ M)_{\theta}'(h) = \left(f_{M(\theta)}' \circ M_{\theta}'\right)(h)$$
$$f_{m}'(h) = \inf_{\beta \in \mathbb{B}_{0}(m)} h(\beta)$$
$$\mathbb{B}_{0}(m) = \left\{\beta \in \mathbb{B} : m(\beta) = \inf_{\beta \in \mathbb{B}} m(\beta)\right\}$$
$$\phi_{\theta}'(h) = \inf_{\beta \in \mathbb{B}_{0}(M(\theta))} M_{(\theta)}'(h)(\beta)$$

If  $S(x) = \sum_{j=1}^{J} x^{-}$ , then  $\rho = 1$ ,  $M'_{\theta}(h)(\beta) = \sum_{j=1}^{J} \left\{ -h_{j}(\beta) \mathbf{1}(\theta_{j}(\beta)^{-} < 0) + h_{j}(\beta)^{-} \mathbf{1}(\theta_{j}(\beta)^{-} = 0) \right\}$ . It follows from the first order directional delta method that  $\sqrt{n} \left( \phi\left(\hat{\theta}\right) - \phi(\theta_{0}) \right) \rightsquigarrow \phi'_{\theta_{0}}(\mathbb{G}_{0})$ , where  $\sqrt{n} \left( \hat{\theta}_{n}(\cdot) - \theta_{0}(\cdot) \right) \rightsquigarrow \mathbb{G}_{0}$  and  $\mathbb{G}_{0}$  is a mean zero Gaussian process.

$$\phi_{\theta}'(h) = \inf_{\beta \in \mathbb{B}_{0}(M(\theta))} \sum_{j=1}^{J} \left\{ -h_{j}(\beta) \, 1\left(\theta_{j}(\beta)^{-} < 0\right) + h_{j}(\beta)^{-} \, 1\left(\theta_{j}(\beta)^{-} = 0\right) \right\}.$$

If  $S(x) = \sum_{j=1}^{J} (x^{-})^2$ , then  $\rho = 2$  and

$$M_{\theta}'(h)(\beta) = -2\sum_{j=1}^{J} \theta_j(\beta)^- h_j(\beta), \quad \phi_{\theta}'(h) = \inf_{\beta \in \mathbb{B}_0(M(\theta))} \left\{ -2\sum_{j=1}^{J} \theta_j(\beta)^- h_j(\beta) \right\}$$

The chain rule for second order directional derivatives gives us

$$\phi_{\theta}''(h) = (f \circ M)_{\theta}''(h) = f_{M(\theta)}''(h), M_{\theta}''(h), M_{\theta}''(h)),$$
$$M_{\theta}''(h)(\beta) = \sum_{j=1}^{J} \left\{ 2 \left( h_j(\beta)^- \right)^2 1 \left( \theta_j(\beta) = 0 \right) + 2h_j(\beta)^2 1 \left( \theta_j(\beta) < 0 \right) \right\}.$$

Using equations (4.426), (4.429), and (4.430) of Bonnans and Shapiro (2013), we can obtain the second order directional derivative of f:  $f''_m(\eta, w) = \inf_{\beta \in \mathbb{B}} \{w(\beta) - \tau_{m,\eta}(\beta)\}$ , where

$$\tau_{m,\eta}\left(\beta\right) = \begin{cases} 0, & \text{if } \beta \in interior\left(\mathbb{B}_{0}\left(m\right)\right) \\ \lim_{\beta' \to \beta} & \frac{\left(\left(\inf_{b \in \mathbb{B}_{0}(M)} \eta(b) - \eta(\beta')\right)^{+}\right)^{2}}{2\left(m(\beta') - \inf_{b \in \mathbb{B}} m(b)\right)} & \text{if } \beta \in boundary\left(\mathbb{B}_{0}\left(m\right)\right) \\ -\infty & \text{otherwise} \end{cases}$$

We can equivalently write the second order directional derivative as

$$f_{m}^{\prime\prime}(\eta,w) = \inf_{\substack{\beta \in \mathbb{B}_{1}(m,\eta) \\ \mu(\beta') > \beta \in \mathbb{B}}} \left\{ w(\beta) - \nu_{m,\eta}(\beta) \right\}$$
$$\nu_{m,\eta}(\beta) = \limsup_{\substack{\beta' \to \beta \\ m(\beta') > \inf_{\beta \in \mathbb{B}}} m(\beta)} \frac{\left( \left( \inf_{b \in \mathbb{B}_{0}(m)} \eta(b) - \eta(\beta') \right)^{+} \right)^{2}}{2 \left( m(\beta') - \inf_{\beta \in \mathbb{B}} m(\beta) \right)}$$

where  $\mathbb{B}_{1}(m,\eta) \equiv \left\{ \beta \in \text{boundary}\left(\mathbb{B}_{0}(m)\right) : \eta(\beta) = \inf_{b \in \mathbb{B}_{0}(m)} \eta(b) \right\}$  are the set of values on the boundary of the identified set  $\mathbb{B}_{0}(m)$  that achieve the minimum value of the function  $\eta(\cdot)$ .

Therefore, by the second order directional delta method,  $n\left(\phi\left(\hat{\theta}_n\right) - \phi(\theta_0)\right) \rightsquigarrow \frac{1}{2}\phi_{\theta_0}''(\mathbb{G}_0)$ , where  $\sqrt{n}\left(\hat{\theta}_n\left(\cdot\right) - \theta_0\left(\cdot\right)\right) \rightsquigarrow \mathbb{G}_0$  and  $\mathbb{G}_0$  is a mean zero Gaussian process.

$$\phi_{\theta}^{\prime\prime}(h) = \inf_{\beta \in \mathbb{B}} \left\{ M_{\theta}^{\prime\prime}(h)\left(\beta\right) - \tau_{M(\theta),M_{\theta}^{\prime}(h)}\left(\beta\right) \right\} = \inf_{\beta \in \mathbb{B}_{1}\left(M(\theta),M_{\theta}^{\prime}(h)\right)} \left\{ M_{\theta}^{\prime\prime}(h)(\beta) - \nu_{M(\theta),M_{\theta}^{\prime}(h)}\left(\beta\right) \right\}$$

Although demonstrating validity of the directional delta method requires showing existence of

the directional derivatives, implementing the numerical directional delta method does not require knowledge of the analytic derivatives. For the case of  $\rho = 2$ , the non-recentered level  $\alpha$  one-sided test using the numerical delta method rejects when  $\inf_{\beta \in \mathbb{B}} S(\sqrt{n}P_ng(\cdot;\beta)) > \hat{c}_{1-\alpha}$ , where  $\hat{c}_{1-\alpha}$  is the  $(1-\alpha)$  percentile of one of the following distributions:

- 1. Numerical Second Order Derivative 1:  $\inf_{\beta \in \mathbb{B}} S\left(\frac{1}{\epsilon_n} \mathcal{Z}_n^* g(\cdot; \beta)\right) \inf_{\beta \in \mathbb{B}} S\left(\frac{1}{\epsilon_n} P_n g(\cdot; \beta)\right).$
- 2. Numerical Second Order Derivative 2:

$$\frac{1}{2} \left( \inf_{\beta \in \mathbb{B}} S\left( \frac{1}{\epsilon_n} \mathcal{Z}_{2n}^* g(\cdot; \beta) \right) - 2 \inf_{\beta \in \mathbb{B}} S\left( \frac{1}{\epsilon_n} \mathcal{Z}_n^* g(\cdot; \beta) \right) + \inf_{\beta \in \mathbb{B}} S\left( \frac{1}{\epsilon_n} P_n g(\cdot; \beta) \right) \right)$$
  
where  $\mathcal{Z}_n^* = P_n + \epsilon_n \sqrt{n} \left( P_n - P \right)$  and  $\mathcal{Z}_{2n}^* = P_n + 2\epsilon_n \sqrt{n} \left( P_n - P \right).$ 

For example, we can perform subvector inference by constructing a nominal  $1 - \alpha$  confidence set using test statistic inversion:  $C = \{\gamma : \inf_{\beta \in \mathbb{B}(\gamma)} S(\sqrt{n}P_ng(\cdot;\beta)) \leq \hat{c}_{1-\alpha}\}$ . In empirical work, researchers typically use a recentered form of the test statistic  $\inf_{\beta \in \mathbb{B}(\gamma)} S(\sqrt{n}\hat{\theta}_n(\beta)) - \inf_{\beta \in \mathcal{B}} S(\sqrt{n}\hat{\theta}_n(\beta))$ because it results in a confidence set that is non-empty with probability one. For the recentered

test statistic, we recenter the estimates of the limiting distributions analogously as follows:

1. Numerical Second Order Derivative 1:

$$\inf_{\beta \in \mathbb{B}} S\left(\frac{1}{\epsilon_n} \mathcal{Z}_n^* g(\cdot; \beta)\right) - \inf_{\beta \in \mathbb{B}} S\left(\frac{1}{\epsilon_n} P_n g(\cdot; \beta)\right) - \left(\inf_{\beta \in \mathcal{B}} S\left(\frac{1}{\epsilon_n} \mathcal{Z}_n^* g(\cdot; \beta)\right) - \inf_{\beta \in \mathcal{B}} S\left(\frac{1}{\epsilon_n} P_n g(\cdot; \beta)\right)\right)$$

2. Numerical Second Order Derivative 2:

$$\frac{1}{2} \left( \inf_{\beta \in \mathbb{B}} S\left( \frac{1}{\epsilon_n} \mathcal{Z}_{2n}^* g(\cdot; \beta) \right) - 2 \inf_{\beta \in \mathbb{B}} S\left( \frac{1}{\epsilon_n} \mathcal{Z}_n^* g(\cdot; \beta) \right) + \inf_{\beta \in \mathbb{B}} S\left( \frac{1}{\epsilon_n} P_n g(\cdot; \beta) \right) \right) \\ - \left( \frac{1}{2} \left( \inf_{\beta \in \mathcal{B}} S\left( \frac{1}{\epsilon_n} \mathcal{Z}_{2n}^* g(\cdot; \beta) \right) - 2 \inf_{\beta \in \mathcal{B}} S\left( \frac{1}{\epsilon_n} \mathcal{Z}_n^* g(\cdot; \beta) \right) + \inf_{\beta \in \mathcal{B}} S\left( \frac{1}{\epsilon_n} P_n g(\cdot; \beta) \right) \right) \right)$$

Other approaches in the literature for obtaining critical values include

- 1. Minimum Resampling Test in Bugni et al. (2017):  $\min\left\{\inf_{\beta\in\hat{\mathbb{B}}_{0}(\gamma_{0})}S\left(\frac{1}{\kappa_{n}}\left(\psi\left(P_{n}g\left(\cdot,\beta\right)\right)+\kappa_{n}\hat{\mathcal{G}}_{n}^{*}g\left(\cdot,\beta\right)\right)\right),\inf_{\beta\in\mathbb{B}(\gamma_{0})}S\left(\frac{1}{\kappa_{n}}\left(\left(P_{n}+\kappa_{n}\hat{\mathcal{G}}_{n}^{*}\right)g\left(\cdot,\beta\right)\right)\right)\right\}$
- 2. Subsampling Test in Romano and Shaikh (2008):

$$\inf_{\beta \in \mathbb{B}} S\left(\sqrt{b}P_b g(\cdot;\beta)\right) - \inf_{\beta \in \mathbb{B}(\gamma_0)} S\left(\sqrt{b}P_n g(\cdot;\beta)\right).$$
3. And rews and Soares (2010):  $S\left(\frac{1}{\kappa_n}\left(\psi\left(P_n g\left(\cdot,\beta^*\right)\right) + \kappa_n \hat{\mathcal{G}}_n^* g\left(\cdot,\beta^*\right)\right)\right)$ 

Here,  $\kappa_n = \sqrt{\log n/n}$ ,  $\psi(P_n g(\cdot, \beta))$  is one of the GMS functions in Andrews and Soares (2010), and  $\hat{\mathbb{B}}_0(\gamma_0) = \{\beta : S(\sqrt{n}P_n g(\cdot; \beta)) \leq \inf_{\beta \in \mathbb{B}(\gamma_0)} S(\sqrt{n}P_n g(\cdot; \beta)) + \sqrt{\log(n)}^{1/3}\}$  is an estimate of the identified set. Note that while the above results demonstrate pointwise validity of the directional delta method for deriving the limiting distribution of the test statistics in Bugni et al. (2017), we are unable to apply the uniformity results in subsection 3.2 because the test statistic  $\phi(\cdot)$  is not a convex function of the structural parameters, and the parameters  $\theta(\cdot)$  are infinite-dimensional. For a detailed discussion of uniformity, see Bugni et al. (2017).

#### A.5 Proofs

**Proof of Theorem 3.1** Part 1 is exactly in Theorem 2.1 of Fang and Santos (2014). We make use of Lemma A.1 to show part 2. Lemma A.1 at the end of this appendix provides a bootstrap version of the extended continuous mapping theorem. Assume the following measurability conditions:

- $\mathbb{Z}_n^*$  is asymptotically measurable jointly in the data and the bootstrap weights.
- f(Z<sup>\*</sup><sub>n</sub>) is a measurable function of the bootstrap weights outer almost surely in the data for every bounded, continuous map f : D → R.
- $\mathbb{G}_0$  is Borel measurable and separable.

Define  $g_n(h) = \frac{1}{\epsilon_n} (\phi(\theta_0 + \epsilon_n h) - \phi(\theta_0))$ . By Hadamard directional differentiability, for  $h_n \to h$ ,  $h \in \mathbb{D}_0, \ \theta + \epsilon_n h_n \in \mathbb{D}, \ g_n(h_n) \to g(h) = \phi'_{\theta_0}(h)$ . Then we write

$$\frac{\phi\left(\hat{\theta}_{n}+\epsilon_{n}\mathbb{Z}_{n}^{*}\right)-\phi\left(\hat{\theta}_{n}\right)}{\epsilon_{n}} = \frac{\phi\left(\theta_{0}+\epsilon_{n}\left(\mathbb{Z}_{n}^{*}+\frac{\hat{\theta}_{n}-\theta_{0}}{\epsilon_{n}}\right)\right)-\phi(\theta_{0})}{\epsilon_{n}} - \frac{\phi\left(\theta_{0}+\epsilon_{n}\left(\epsilon_{n}^{-1}\left(\hat{\theta}_{n}-\theta_{0}\right)\right)\right)-\phi(\theta_{0})}{\epsilon_{n}}$$
$$= g_{n}\left(\mathbb{Z}_{n}^{*}+\frac{\hat{\theta}_{n}-\theta_{0}}{\epsilon_{n}}\right) - g_{n}\left(\epsilon_{n}^{-1}\left(\hat{\theta}_{n}-\theta_{0}\right)\right)$$

Since  $\mathbb{Z}_n^* \xrightarrow{\mathbb{P}} \mathbb{G}_0$ ,  $\epsilon_n^{-1} \left( \hat{\theta}_n - \theta_0 \right) = o_P(1)$ ,  $\mathbb{Z}_n^* + \epsilon_n^{-1} \left( \hat{\theta}_n - \theta_0 \right) \xrightarrow{\mathbb{P}} \mathbb{G}_0$  (also  $\rightsquigarrow \mathbb{G}_0$ ). Apply Lemma A.1 to the first term on the right side, the first term  $\xrightarrow{\mathbb{P}} \phi'_{\theta_0}(\mathbb{G}_0)$  (and also  $\rightsquigarrow \phi'_{\theta_0}(\mathbb{G}_0)$  by van der Vaart and Wellner (1996) Theorem 1.11.1). The second term is  $o_P(1)$  by van der Vaart and Wellner (1996) Theorem 1.11.1. Since  $X_n \xrightarrow{\mathbb{P}} X$  and  $Y_n = o_P(1)$  implies  $X_n + Y_n \xrightarrow{\mathbb{P}} X$ , summing the two terms on the right hand side leads to  $\frac{\phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*) - \phi(\hat{\theta}_n)}{\epsilon_n} \xrightarrow{\mathbb{P}} \phi'_{\theta_0}(\mathbb{G}_0)$ .

**Proof of theorem 3.2** We can write

$$\left\| \hat{\phi}_n'(h_1) - \hat{\phi}_n'(h_2) \right\|_{\mathbb{E}} = \left\| \frac{\phi(\hat{\theta}_n + \epsilon_n h_1) - \phi(\hat{\theta}_n + \epsilon_n h_2)}{\epsilon_n} \right\|_{\mathbb{E}} \le C \left\| h_1 - h_2 \right\|_{\mathbb{D}}$$

where C is the Lipschitz constant for  $\phi : \mathbb{D}_{\phi} \to \mathbb{E}$ .

## Proof of Theorem **3.3** <sup>6</sup>

The proof follows from the definition of the directional delta method.

$$\begin{aligned} \left\| \hat{\phi}'_{n}(h) - \phi'_{\theta_{0}}(h) \right\|_{\mathbb{E}} &= \left\| \frac{\phi(\hat{\theta}_{n} + \epsilon_{n}h) - \phi(\hat{\theta}_{n})}{\epsilon_{n}} - \phi'_{\theta_{0}}(h) \right\|_{\mathbb{E}} \\ &\leq \left\| \frac{(\phi(\hat{\theta}_{n} + \epsilon_{n}h) - \phi(\theta_{0}))}{\epsilon_{n}} - \phi'_{\theta_{0}}(h) \right\|_{\mathbb{E}} + \left\| \frac{r_{n}(\phi(\hat{\theta}_{n}) - \phi(\theta_{0}))}{r_{n}\epsilon_{n}} \right\|_{\mathbb{E}} \stackrel{p}{\to} 0 \end{aligned}$$

For the first term,  $\frac{1}{\epsilon_n}((\hat{\theta}_n + \epsilon_n h) - \theta_0) = \frac{1}{r_n \epsilon_n} r_n(\hat{\theta}_n - \theta_0) + h \xrightarrow{d} h$  since  $r_n \epsilon_n \to \infty$ , and  $r_n(\hat{\theta}_n - \theta_0) = O_p(1)$ . Using the directional delta method (Theorem 2.1 Fang and Santos (2014), Theorem 1.11.1 van der Vaart and Wellner (1996)),  $\frac{1}{\epsilon_n}(\phi(\hat{\theta}_n + \epsilon_n h) - \phi(\theta_0)) \xrightarrow{d} \phi'_{\theta_0}(h)$ . Since  $\phi'_{\theta_0}(h)$  is constant for each fixed h,  $\frac{1}{\epsilon_n}(\phi(\hat{\theta}_n + \epsilon_n h) - \phi(\theta_0)) \xrightarrow{p} \phi'_{\theta_0}(h)$ . For the second term,  $r_n(\phi(\hat{\theta}_n) - \phi(\theta_0)) = O_p(1)$ . Consequently,  $\frac{r_n(\phi(\hat{\theta}_n) - \phi(\theta_0))}{r_n \epsilon_n} \xrightarrow{p} 0$ .

**Proof of theorem 3.4** The proof relies on the definitions of Hadamard directional differentiability and Holder continuity.

$$\begin{split} \sup_{h \in K} \left\| \hat{\phi}'_n(h) - \phi'_{\theta_0}(h) \right\|_{\mathbb{E}} &= \sup_{h \in K} \left\| \frac{\phi(\hat{\theta}_n + \epsilon_n h) - \phi(\hat{\theta}_n)}{\epsilon_n} - \phi'_{\theta_0}(h) \right\|_{\mathbb{E}} \\ &\leq \sup_{h \in K} \left\| \frac{(\phi(\hat{\theta}_n + \epsilon_n h) - \phi(\theta_0))}{\epsilon_n} - \phi'_{\theta_0}(h) \right\|_{\mathbb{E}} + \left\| \frac{r_n(\phi(\hat{\theta}_n) - \phi(\theta_0))}{r_n \epsilon_n} \right\|_{\mathbb{E}} \\ &\leq \sup_{h \in K} \left\| \frac{(\phi(\hat{\theta}_n + \epsilon_n h) - \phi(\theta_0 + \epsilon_n h))}{\epsilon_n} \right\|_{\mathbb{E}} + \sup_{h \in K} \left\| \frac{(\phi(\theta_0 + \epsilon_n h) - \phi(\theta_0))}{\epsilon_n} - \phi'_{\theta_0}(h) \right\|_{\mathbb{E}} \\ &+ \left\| \frac{r_n(\phi(\hat{\theta}_n) - \phi(\theta_0))}{r_n \epsilon_n} \right\|_{\mathbb{E}} \\ &\leq \frac{1}{r_n^{\kappa} \epsilon_n} C_0 \left\| r_n(\hat{\theta}_n - \theta_0) \right\|_{\mathbb{D}}^{\kappa} + o_p(1) + o_p(1) = o_p(1). \end{split}$$

**Proof of Lemma 3.2** For  $h_1$  and  $h_2$ , it follows from the convexity of  $\phi(\cdot)$  that

$$\phi\left(\theta_{0} + t\left(h_{1} + h_{2}\right)\right) = \phi\left(\frac{1}{2}\left(\theta_{0} + 2th_{1}\right) + \frac{1}{2}\left(\theta_{0} + 2th_{2}\right)\right) \le \frac{1}{2}\phi\left(\theta_{0} + 2th_{1}\right) + \frac{1}{2}\phi\left(\theta_{0} + 2th_{2}\right)$$

<sup>6</sup>We thank Andres Santos for providing the arguments in this proof and in Theorem 3.4.

Hence 
$$\phi(\theta_0 + t(h_1 + h_2)) - \phi(\theta_0) \le \frac{1}{2} (\phi(\theta_0 + 2th_1) - \phi(\theta_0)) + \frac{1}{2} (\phi(\theta_0 + 2th_2) - \phi(\theta_0))$$
 and

$$\frac{\phi\left(\theta_{0}+t\left(h_{1}+h_{2}\right)\right)-\phi\left(\theta_{0}\right)}{t} \leq \frac{\left(\phi\left(\theta_{0}+2th_{1}\right)-\phi\left(\theta_{0}\right)\right)}{2t} + \frac{\left(\phi\left(\theta_{0}+2th_{2}\right)-\phi\left(\theta_{0}\right)\right)}{2t}.$$

Taking  $t \to 0$  on both sides we conclude that  $\phi'_{\theta_0}(h_1 + h_2) \leq \phi'_{\theta_0}(h_1) + \phi'_{\theta_0}(h_2)$ .

**Proof of Theorem 3.5** We first note that the arguments in the proofs of Theorem 2.11 in Bhattacharya and Rao (1986) can be revised for a convergence in probability version: Let C be a class of convex sets such that  $\sup_{P \in \mathcal{P}} P(\mathbb{G}_0 \in \partial C) = 0$  for all  $C \in C$ . Let  $P(\mathbb{Z}_n^* \in C | \mathcal{X}_n)$  denote the conditional probability of  $\mathbb{Z}_n^*$  given the data (denoted  $\mathcal{X}_n$ ). Then  $\forall \epsilon > 0$ ,

$$\lim_{n \to \infty} \sup_{P \in \mathcal{P}} P\left( \sup_{C \in \mathcal{C}} |P\left(\mathbb{Z}_n^* \in C | \mathcal{X}_n\right) - P\left(\mathbb{G}_0 \in C\right)| \ge \epsilon \right) \to 0,$$
(32)

whenever  $\forall \epsilon > 0$ ,  $\lim_{n \to \infty} \sup_{P \in \mathcal{P}} P(\rho_{BL_1}(\mathbb{Z}_n^*, \mathbb{G}_0) \ge \epsilon) \to 0$ . Under assumption 3.2, the key to invoking (32) is the fact that level sets of convex functions are convex.

Under assumption 3.1 part (i),  $\forall \epsilon > 0$ ,  $\sup_{P \in \mathcal{P}} P\left(\left|\epsilon_n^{-1}\left(\hat{\theta}_n - \theta(P)\right)\right| \ge \epsilon\right) = o(1)$ . It can therefore be combined with part (ii) of assumption 3.1 to show that

$$\sup_{P \in \mathcal{P}} P\left(\rho_{BL_1}\left(\mathbb{Z}_n^* + \frac{\hat{\theta}_n - \theta(P)}{\epsilon_n}, \mathbb{G}_0\right) \ge \epsilon\right) = o\left(1\right).$$
(33)

Next note that the set  $C = \{g : \frac{1}{\epsilon_n} \left( \phi \left( \theta_0 + \epsilon_n g \right) - \phi \left( \hat{\theta}_n \right) \right) \leq x \}$  is convex whenever  $\phi(\cdot)$  is a convex function and is a member of the class specified in assumption 3.2. Then by (33) and (32),

$$\sup_{P \in \mathcal{P}} P\left(\sup_{x} \left| P\left(\hat{\phi}_{n}'\left(\mathbb{Z}_{n}^{*}\right) \leq x | \mathcal{X}_{n}\right) - P\left(\frac{1}{\epsilon_{n}}\left(\phi\left(\theta_{0} + \epsilon_{n}\mathbb{G}_{0}\right) - \phi\left(\hat{\theta}_{n}\right)\right) \leq x\right) \right| \geq \epsilon\right) = o\left(1\right). \quad (34)$$

Finally, we use the last condition in the theorem statement to show that

$$\sup_{P \in \mathcal{P}} \sup_{x \in A} \left| P\left( \frac{1}{\epsilon_n} \left( \phi\left(\theta_0 + \epsilon_n \mathbb{G}_0\right) - \phi\left(\hat{\theta}_n\right) \right) \le x \right) - J_{\epsilon_n}\left(x, \mathbb{G}_0\right) \right| = o\left(1\right).$$
(35)

Note that we do not need the last equation or the last condition of the theorem if we replace  $\hat{\theta}_n$  in  $\hat{\phi}'_n(\mathbb{Z}^*_n)$  by a fixed  $\theta$  in hypothesis testing settings. Similarly, the level set

$$C = \{g : r_n \left(\phi \left(\theta(P) + r_n^{-1}g\right) - \phi \left(\theta(P)\right)\right) \le x\}$$

is also convex. By Theorem 2.11 of Bhattacharya and Rao (1986), part (i) of Assumption 3.1, and

Assumption 3.2,

$$\sup_{P \in \mathcal{P}} \sup_{x \in A} \left| J_n\left(x, P\right) - J_n\left(x, \mathbb{G}_0\right) \right| = o\left(1\right).$$
(36)

The first conclusion of the theorem follows from combining (34), (35), (36) and (16). The second conclusion follows from similar arguments as Lemma A.1(vi) in Romano and Shaikh (2012).

**Proof of Theorem 3.6** Consider any sequence  $\{P_n \in \mathcal{P} : n \ge 1\}$  that determines  $\theta_n = \theta(P_n)$  and the laws of  $r_n\left(\hat{\theta}_n - \theta(P_n)\right)$ ,  $\mathbb{G}_0$ , and  $\mathbb{Z}_n^*$ . Note that assumptions 3.1 and 3.2 imply the following: Assumption A.1 Let the sequence  $\theta_n$ ,  $P_n$  and  $\mathbb{G}_0$  be such that

$$\rho_{BL_1}\left(r_n\left(\hat{\theta}_n-\theta_n\right),\mathbb{G}_0\right)=o\left(1\right)\quad and\quad \rho_{BL_1}\left(\mathbb{Z}_n^*,\mathbb{G}_0\right)=o_{P_n}\left(1\right).$$

**Assumption A.2** For all  $\epsilon$  small enough, and all  $x = J_n^{-1}(1 - \tau - \epsilon, P_n)$ ,  $x_n$  is a sequence of asymptotic equicontinuity points of J(x) being either  $J_{\epsilon_n}(x, \mathbb{G}_0)$  or  $J_n(x, \mathbb{G}_0)$ :

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{|x - x_n| \le \delta} |J(x) - J(x_n)| = 0.$$

First  $\rho_{BL_1}(\mathbb{Z}_n^*, \mathbb{G}_0) = o_{P_n}(1)$  in part (ii) of assumption A.1 implies that  $\rho_{BL_1}(\mathbb{Z}_n^* + o_{P_n}(1), \mathbb{G}_0) = o_{P_n}(1)$ , which follows from

$$\sup_{h \in BL_{1}} |E_{M}h\left(\mathbb{Z}_{n}^{*}+o_{P_{n}}\left(1\right)\right)-Eh\left(\mathbb{G}_{0}\right)| \leq \sup_{h \in BL_{1}} |E_{M}h\left(\mathbb{Z}_{n}^{*}\right)-Eh\left(\mathbb{G}_{0}\right)|+o_{P_{n}}\left(1\right)$$

Since  $r_n \epsilon_n \to \infty$ ,  $\epsilon_n^{-1} \left( \hat{\theta}_n - \theta_n \right) = o_{P_n}(1)$ , so that

$$\rho_{BL_1}\left(\mathbb{Z}_n^* + \frac{1}{\epsilon_n}\left(\hat{\theta}_n - \theta_n\right), \mathbb{G}_0\right) = o_{P_n}\left(1\right).$$

Next note that the functions  $g_n(Z) = \frac{1}{\epsilon_n} (\phi(\theta_n + \epsilon_n Z) - \phi(\theta_n))$  are uniformly Lipschitz in Z with the Lipschitz constant bounded by that of  $\phi(\cdot)$ . The same arguments as in Proposition 10.7 of Kosorok (2007) adapted to a sequence of such functions  $g_n(\cdot)$  show that

$$\rho_{BL_1}\left(g_n\left(\mathbb{Z}_n^* + \frac{1}{\epsilon_n}\left(\hat{\theta}_n - \theta_n\right)\right), g_n\left(\mathbb{G}_0\right)\right) = o_{P_n}\left(1\right).$$

Then we can write

$$\hat{\phi}_{n}'(\mathbb{Z}_{n}^{*}) = g_{n}\left(\mathbb{Z}_{n}^{*} + \frac{1}{\epsilon_{n}}\left(\hat{\theta}_{n} - \theta_{n}\right)\right) - \frac{1}{\epsilon_{n}}\left(\phi\left(\hat{\theta}_{n}\right) - \phi\left(\theta_{n}\right)\right) = g_{n}\left(\mathbb{Z}_{n}^{*} + \frac{1}{\epsilon_{n}}\left(\hat{\theta}_{n} - \theta_{n}\right)\right) + o_{P_{n}}\left(1\right),$$

so that also  $\rho_{BL_1}\left(\hat{\phi}'_n\left(\mathbb{Z}_n^*\right), g_n\left(\mathbb{G}_0\right)\right) = o_{P_n}(1)$ . Then using assumption A.2, similar arguments to those in Lemma 10.11 in Kosorok (2007) can be used to establish

$$J_{\epsilon_n}(x_n, P_n) - J_{\epsilon_n}(x_n, \mathbb{G}_0) = o_{P_n}(1).$$
(37)

Next using part (i) of assumption A.1 and applying a nonstochastic version of the arguments in Proposition 10.7 of Kosorok (2007), it can be shown that  $\rho_{BL_1}(J_n(\cdot, P_n), J_n(\cdot, \mathbb{G}_0)) = o(1)$ . The  $J_n(\cdot, \mathbb{G}_0)$  part of assumption A.2 in combination with modified arguments in Lemma 10.11 in Kosorok (2007) produces that

$$J_n(x_n, P_n) - J_n(x_n, \mathbb{G}_0) = o(1).$$
(38)

When  $\phi(\cdot)$  satisfies assumption 3.3, equations (15),(16), (37), and (38) imply that  $\forall \epsilon, \eta > 0$  and n large enough,  $P_n(J_{\epsilon_n}(x_n, P_n) \leq J_n(x_n, P_n) + \epsilon) \geq 1 - \delta$ . Next we consider arguments similar to Lemma A.1 parts (i) and (vi) and Theorem 2.4 in Romano and Shaikh (2012), If  $J_{\epsilon_n}(x_n, P_n) \leq J_n(x_n, P_n) + \epsilon$  at  $x_n = J_n^{-1}(1 - \alpha - \epsilon, P_n)$ , then  $J_{\epsilon_n}^{-1}(1 - \alpha, P_n) \geq J_n^{-1}(1 - \alpha - \epsilon, P_n)$ . Combining these inequalities,

$$P_{n}\left(r_{n}\left(\phi\left(\hat{\theta}_{n}\right)-\phi\left(\theta_{n}\right)\right)\leq J_{\epsilon_{n}}^{-1}\left(1-\alpha,P_{n}\right)\right)$$

$$\geq P_{n}\left(r_{n}\left(\phi\left(\hat{\theta}_{n}\right)-\phi\left(\theta_{n}\right)\right)\leq J_{\epsilon_{n}}^{-1}\left(1-\alpha,P_{n}\right)\cap J_{\epsilon_{n}}\left(x_{n},P_{n}\right)\leq J_{n}\left(x_{n},P_{n}\right)+\epsilon\right)$$

$$\geq P_{n}\left(r_{n}\left(\phi\left(\hat{\theta}_{n}\right)-\phi\left(\theta_{n}\right)\right)\leq J_{n}^{-1}\left(1-\alpha-\epsilon,P_{n}\right)\cap J_{\epsilon_{n}}\left(x_{n},P_{n}\right)\leq J_{n}\left(x_{n},P_{n}\right)+\epsilon\right)$$

$$\geq P_{n}\left(r_{n}\left(\phi\left(\hat{\theta}_{n}\right)-\phi\left(\theta_{n}\right)\right)\leq J_{n}^{-1}\left(1-\alpha-\epsilon,P_{n}\right)\right)-P_{n}\left(J_{\epsilon_{n}}\left(x_{n},P_{n}\right)>J_{n}\left(x_{n},P_{n}\right)+\epsilon\right)$$

$$\geq 1-\alpha-\epsilon-\delta.$$

Since both  $\epsilon$  and  $\delta$  can be arbitrarily small,  $\limsup_{n\to\infty} P_n\left(r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_n\right)\right) \ge \hat{c}_{1-\tau}\right) \le \tau$ . Now define  $\beta = \limsup_{n\to\infty} \sup_{P\in\mathcal{P}} P\left(r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta(P)\right)\right) \ge \hat{c}_{1-\alpha}\right)$ . Then one can find a sequence of  $P_n \in \mathcal{P}$  such that, for  $\theta_n = \theta_{P_n}$ ,  $\beta = \lim_{n\to\infty} P_n\left(r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_n\right)\right) \ge \hat{c}_{1-\alpha}\right)$ . Find a subsequence  $\mu_n$  of n for which  $\theta_n$  converges, with its limit denoted  $\theta$ . The previous arguments allow us to claim that  $\limsup_{\mu_n\to\infty} P_{\mu_n}\left(r_{\mu_n}\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_{\mu_n}\right)\right) \ge \hat{c}_{1-\alpha}\right) \le \alpha$ . Since  $P_{\mu_n}, \theta_{\mu_n}$  is a subsequence of  $P_n, \theta_n$ , it is also the case that  $\beta = \lim_{\mu_n\to\infty} P_{\mu_n}\left(r_{\mu_n}\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_{\mu_n}\right)\right) \ge \hat{c}_{1-\alpha}\right) \le \alpha$ . Now suppose  $\phi(\cdot)$  does not satisfy assumption 3.3, but assumptions 3.4 and 3.5 hold. For  $t_n = r_n^{-1}, \epsilon_n$ , define  $g_{t_n,n}(h) = \frac{1}{t_n}\left(\phi\left(\theta_n + t_nh\right) - \phi\left(\theta_n\right)\right)$ . We first show that  $\rho_{BL_1}\left(\hat{\phi}'_n\left(\mathbb{Z}_n^*\right), g_{\epsilon_n,n}\left(\mathbb{G}_0\right)\right) = o_{P_n}(1)$ , and

 $\rho_{BL_1}\left(r_n\left(\phi\left(\hat{\theta}_n\right) - \phi\left(\theta_n\right)\right), g_{r_n^{-1}, n}\left(\mathbb{G}_0\right)\right) = o\left(1\right).$ 

Next we can use Assumption 3.5 to show that both

$$\rho_{BL_1}\left(g_{\epsilon_n,n}\left(\mathbb{G}_0\right),\phi_{\theta_0}'\left(\frac{\eta_n}{\epsilon_n}+\mathbb{G}_0\right)-\phi_{\theta_0}'\left(\frac{\eta_n}{\epsilon_n}\right)\right)=o\left(1\right)$$
$$\rho_{BL_1}\left(g_{r_n^{-1},n}\left(\mathbb{G}_0\right),\phi_{\theta_0}'\left(\eta_n r_n+\mathbb{G}_0\right)-\phi_{\theta_0}'\left(\eta_n r_n\right)\right)=o\left(1\right)$$

Define then  $J'_{\epsilon_n}(x, \mathbb{G}_0)$  and  $J'_n(x, \mathbb{G}_0)$ , respectively, as the CDFs of  $\phi'_{\theta_0}\left(\frac{\eta_n}{\epsilon_n} + \mathbb{G}_0\right) - \phi'_{\theta_0}\left(\frac{\eta_n}{\epsilon_n}\right)$  and  $\phi'_{\theta_0}(\eta_n r_n + \mathbb{G}_0) - \phi'_{\theta_0}(\eta_n r_n)$ . Using Assumption A.2 and arguments analogous to Lemma 10.11 in Kosorok (2007) then shows that for each sequence  $x_n$  of asymptotic equicontinuity points,

$$J_{\epsilon_n}(x_n, P_n) - J_{\epsilon_n}(x_n, \mathbb{G}_0) = o_{P_n}(1), \ J_n(x_n, P_n) - J_n(x_n, \mathbb{G}_0) = o(1)$$
$$J_{\epsilon_n}(x_n, \mathbb{G}_0) - J'_{\epsilon_n}(x_n, \mathbb{G}_0) = o(1), \ J_n(x_n, \mathbb{G}_0) - J'_n(x_n, \mathbb{G}_0) = o(1).$$

So that  $J_{\epsilon_n}(x_n, P_n) - J'_{\epsilon_n}(x_n, \mathbb{G}_0) = o_{P_n}(1)$  and  $J_n(x_n, P_n) - J'_n(x_n, \mathbb{G}_0) = o(1)$ . Then by Assumption 3.4,  $J'_{\epsilon_n}(x_n, \mathbb{G}_0) \leq J'_n(x_n, \mathbb{G}_0)$ .

Then for each  $\epsilon > 0$  and *n* sufficiently large,  $J_n(x_n, P_n) \ge J'_n(x_n, \mathbb{G}_0) - \frac{\epsilon}{2}$ , and

$$\lim_{n \to \infty} P_n \left( J_{\epsilon_n} \left( x_n, P_n \right) \le J_n \left( x_n, P_n \right) + \epsilon \right) \ge \lim_{n \to \infty} P_n \left( J_{\epsilon_n} \left( x_n, P_n \right) \le J'_n \left( x_n, \mathbb{G}_0 \right) + \frac{\epsilon}{2} \right) \to 1. \quad \blacksquare$$

**Proof for Theorem 3.8** First consider Assumption 3.8. Since  $\phi(\cdot)$  is Lipschitz in  $\theta$  uniformly in  $\alpha$ , the same arguments as in the proof of Theorem 3.6 show that

$$\rho_{BL_{1}}\left(\frac{1}{\epsilon_{n}}\left(\phi\left(\hat{\theta}_{n}+\epsilon_{n}\mathbb{Z}_{n,\theta}^{*},\hat{\alpha}_{n}\right)-\phi\left(\theta_{n},\hat{\alpha}_{n}\right)\right),\frac{1}{\epsilon_{n}}\left(\phi\left(\theta_{n}+\epsilon_{n}\mathbb{G}_{0,\theta},\hat{\alpha}_{n}\right)-\phi\left(\theta_{n},\hat{\alpha}_{n}\right)\right)\right)=o_{P_{n}}\left(1\right).$$

Next by repeated use of the first part of Theorem 2.1 in Fang and Santos (2014),

 $\epsilon_n^{-1}\left(\phi\left(\hat{\theta}_n,\hat{\alpha}_n\right)-\phi\left(\theta_n,\hat{\alpha}_n\right)\right)=o_{P_n}(1).$  Together they imply that (with  $\mathbb{G}_{0,\theta}$  independent of  $\hat{\alpha}_n$ ):

$$\rho_{BL_1}\left(\hat{\phi}_n'\left(\mathbb{Z}_{n,\theta}^*,0\right),\frac{1}{\epsilon_n}\left(\phi\left(\theta_n+\epsilon_n\mathbb{G}_{0,\theta},\hat{\alpha}_n\right)-\phi\left(\theta_n,\hat{\alpha}_n\right)\right)\right)=o_{P_n}\left(1\right).$$

By two additional applications of Fang and Santos (2014) Theorem 2.1,

$$\rho_{BL_1}\left(r_n\left(\phi\left(\hat{\theta}_n,\hat{\alpha}_n\right)-\phi\left(\theta_n,\hat{\alpha}_n\right)\right),r_n\left(\phi\left(\theta_n+r_n^{-1}\mathbb{G}_{0,\theta},\hat{\alpha}_n\right)-\phi\left(\theta_n,\hat{\alpha}_n\right)\right)\right)=o\left(1\right)$$

Denote, by  $J_{\epsilon_n}(\cdot, P_n)$ ,  $J_{\epsilon_n}(\cdot, \mathbb{G}_0)$ ,  $J_n(\cdot, P_n)$  and  $J_n(\cdot, \mathbb{G}_0)$ , the four distributions in the above two

equations. Then by a suitable version of A.2 and suitable modification of Lemma 10.11 in Kosorok (2007), and by noting that  $J_{\epsilon_n}(x, \mathbb{G}_0) \leq J_n(x, \mathbb{G}_0)$ , we have that

$$\lim_{n \to \infty} P_n \left( J_{\epsilon_n} \left( x_n, P_n \right) \le J_n \left( x_n, P_n \right) + \epsilon \right) = 1.$$
(39)

Note that Assumption A.2 holds for  $J_n(\cdot, P_n)$  and  $J_n(\cdot, \mathbb{G}_0)$  when  $J_n(\cdot, P_n) \rightsquigarrow \phi'_{\theta_0,\alpha_0}(\mathbb{G}_{0,\theta}, 0)$  by Fang and Santos (2014) Theorem 2.1, and when  $x_n$  belongs to the set of continuity points of the limiting  $\phi'_{\theta_0,\alpha_0}(\mathbb{G}_{0,\theta}, 0)$ .

When Assumption 3.9 holds instead,  $J_{\epsilon_n}(\cdot, \mathbb{G}_0)$  and  $J_n(\cdot, \mathbb{G}_0)$  are further approximated by

$$J_{\epsilon_n}'(\cdot, \mathbb{G}_0) = \phi_{\theta_0, \alpha_0}'\left(\frac{\theta_n - \theta_0}{\epsilon_n} + \mathbb{G}_{0, \theta}, \frac{\alpha_n - \alpha_0}{\epsilon_n}\right) - \phi_{\theta_0, \alpha_0}'\left(\frac{\theta_n - \theta_0}{\epsilon_n}, \frac{\alpha_n - \alpha_0}{\epsilon_n}\right)$$

and  $J'_{n}(\cdot, \mathbb{G}_{0}) = \phi'_{\theta_{0},\alpha_{0}}(r_{n}(\theta_{n} - \theta_{0}) + \mathbb{G}_{0,\theta}, r_{n}(\alpha_{n} - \alpha_{0})) - \phi'_{\theta_{0},\alpha_{0}}(r_{n}(\theta_{n} - \theta_{0}), r_{n}(\alpha_{n} - \alpha_{0})).$  So that  $\rho_{BL_{1}}\left(J_{\epsilon_{n}}(\cdot, \mathbb{G}_{0}), J'_{\epsilon_{n}}(\cdot, \mathbb{G}_{0})\right) = o_{P_{n}}(1)$ , and  $\rho_{BL_{1}}\left(J_{n}(\cdot, \mathbb{G}_{0}), J'_{n}(\cdot, \mathbb{G}_{0})\right) = o(1)$ . Then we use  $J'_{\epsilon_{n}}(x, \mathbb{G}_{0}) \leq J'_{n}(x, \mathbb{G}_{0})$  to conclude that (39) holds.

**Proof for Theorem 4.1** The first part of the theorem is exactly Theorem 2 in Römisch (2005). The second part will be argued using Lemma A.1. Define  $g_n(h) = \frac{1}{\epsilon_n^2} \left( \phi(\theta_0 + \epsilon_n h) - \phi(\theta_0) - \phi'_{\theta_0}(h) \right)$ . By definition of (20), for  $h_n \to h$ ,  $h \in \mathbb{D}_0$ ,  $\theta + \epsilon_n h_n \in \mathbb{D}$ ,  $g_n(h_n) \to g(h) = \frac{1}{2} \phi''_{\theta_0}(h)$ . Then write, noting that  $\phi'_{\theta_0}(h) \equiv 0$ ,

$$\frac{\phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*) - \phi(\hat{\theta}_n)}{\epsilon_n^2} = \frac{\phi\left(\theta_0 + \epsilon_n \left(\mathbb{Z}_n^* + \frac{\hat{\theta}_n - \theta_0}{\epsilon_n}\right)\right) - \phi(\theta_0)}{\epsilon_n^2} - \frac{\phi\left(\theta_0 + \epsilon_n \left(\epsilon_n^{-1} \left(\hat{\theta}_n - \theta_0\right)\right)\right) - \phi(\theta_0)}{\epsilon_n^2}}{\epsilon_n^2} = g_n \left(\mathbb{Z}_n^* + \frac{\hat{\theta}_n - \theta_0}{\epsilon_n}\right) - g_n \left(\epsilon_n^{-1} \left(\hat{\theta}_n - \theta_0\right)\right).$$

Since  $\mathbb{Z}_n^* \xrightarrow{\mathbb{P}} \mathbb{G}_0$ ,  $\epsilon_n^{-1} \left( \hat{\theta}_n - \theta_0 \right) = o_P(1)$ ,  $\mathbb{Z}_n^* + \epsilon_n^{-1} \left( \hat{\theta}_n - \theta_0 \right) \xrightarrow{\mathbb{P}} \mathbb{G}_0$  (also  $\rightsquigarrow \mathbb{G}_0$ ). Apply Lemma A.1 to the first term on the right side, the first term  $\xrightarrow{\mathbb{P}} \frac{1}{2} \phi_{\theta_0}''(\mathbb{G}_0)$  (and also  $\rightsquigarrow \frac{1}{2} \phi_{\theta_0}''(\mathbb{G}_0)$  by van der Vaart and Wellner (1996) Theorem 1.11.1). The second term is  $o_P(1)$  by van der Vaart and Wellner (1996) Theorem 1.11.1. Since  $X_n \xrightarrow{\mathbb{P}} X$  and  $Y_n = o_P(1)$  implies  $X_n + Y_n \xrightarrow{\mathbb{P}} X$ , summing the two terms on the right hand side leads to  $\frac{\phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*) - \phi(\hat{\theta}_n)}{\epsilon_n^2} \xrightarrow{\mathbb{P}} \frac{1}{2} \phi_{\theta_0}''(\mathbb{G}_0)$ .

**Proof for Theorem 4.2** Note  $\frac{1}{\epsilon_n} \left( \hat{\theta}_n - \theta_0 \right) \xrightarrow{p} 0$ ,  $\frac{1}{\epsilon_n} \left( \hat{\theta}_n - \theta_0 + \epsilon_n \mathbb{Z}_n^* \right) \xrightarrow{\mathbb{P}} \mathbb{G}_0$ . For  $g_n(h) = \frac{1}{\epsilon_n^2} \left( \phi(\theta_0 + \epsilon_n h) - \phi(\theta_0) - \epsilon_n \phi'_{\theta_0}(h) \right)$  and  $g(h) = \frac{1}{2} \phi''_{\theta_0}(h)$ ,  $g_n(h_n) \to g(h)$  when  $h_n \to h$ . Then by

van der Vaart and Wellner (1996) Theorem 1.11.1 and Lemma A.1, jointly,

$$\frac{1}{\epsilon_n^2} \left[ \phi(\hat{\theta}_n + 2\epsilon_n \mathbb{Z}_n^*) - \phi(\theta_0) - \phi_{\theta_0}'(\hat{\theta}_n - \theta_0 + 2\epsilon_n \mathbb{Z}_n^*) \right] = g_n \left( 2\mathbb{Z}_n^* + \frac{\hat{\theta}_n - \theta_0}{\epsilon_n} \right) \xrightarrow{\mathbb{P}} \frac{1}{2} \phi_{\theta_0}''(2\mathbb{G}_0)$$
$$\frac{1}{\epsilon_n^2} \left[ \phi(\hat{\theta}_n + \epsilon_n \mathbb{Z}_n^*) - \phi(\theta_0) - \phi_{\theta_0}'(\hat{\theta}_n - \theta_0 + \epsilon_n \mathbb{Z}_n^*) \right] = g_n \left( \mathbb{Z}_n^* + \frac{\hat{\theta}_n - \theta_0}{\epsilon_n} \right) \xrightarrow{\mathbb{P}} \frac{1}{2} \phi_{\theta_0}''(\mathbb{G}_0)$$

Furthermore,  $\frac{1}{\epsilon_n^2} \left[ \phi(\hat{\theta}_n) - \phi(\theta_0) - \phi'_{\theta_0}(\hat{\theta}_n - \theta_0) \right] = g_n \left( \frac{\hat{\theta}_n - \theta_0}{\epsilon_n} \right) \xrightarrow{p} g(0) = 0. \text{ By linearity of } \phi'_{\theta_0}(h),$  $R(\hat{\theta}_n, \theta_0, h) = \phi'_{\theta_0} \left( \frac{1}{\epsilon_n} \left( \hat{\theta}_n - \theta_0 + 2\epsilon_n h \right) \right) - 2\phi'_{\theta_0} \left( \frac{1}{\epsilon_n} \left( \hat{\theta}_n - \theta_0 + \epsilon_n h \right) \right) + \phi'_{\theta_0} \left( \frac{1}{\epsilon_n} \left( \hat{\theta}_n - \theta_0 \right) \right) = 0$ 

Therefore by the above joint convergence and continuous mapping,

$$\begin{split} \hat{\phi}_{n}^{\prime\prime}(\mathbb{Z}_{n}^{*}) &= \frac{1}{\epsilon_{n}^{2}} \left[ \phi(\hat{\theta}_{n} + 2\epsilon_{n}\mathbb{Z}_{n}^{*}) - 2\phi(\hat{\theta}_{n} + \epsilon_{n}\mathbb{Z}_{n}^{*}) + \phi(\hat{\theta}_{n}) \right] \\ &= g_{n} \left( 2\mathbb{Z}_{n}^{*} + \frac{\hat{\theta}_{n} - \theta_{0}}{\epsilon_{n}} \right) - g_{n} \left( \mathbb{Z}_{n}^{*} + \frac{\hat{\theta}_{n} - \theta_{0}}{\epsilon_{n}} \right) + g_{n} \left( \frac{\hat{\theta}_{n} - \theta_{0}}{\epsilon_{n}} \right) + \frac{1}{\epsilon_{n}} R(\hat{\theta}_{n}, \theta_{0}, \mathbb{Z}_{n}^{*}, \mathbb{Z}_{n}^{*}) \\ &\to \frac{1}{2} \phi_{\theta_{0}}^{\prime\prime}(2h) - 2\frac{1}{2} \phi_{\theta_{0}}^{\prime\prime}(h) + \frac{1}{2} \phi_{\theta_{0}}^{\prime\prime}(0) = \frac{1}{2} 4 \phi_{\theta_{0}}^{\prime\prime}(h) - 2\frac{1}{2} \phi_{\theta_{0}}^{\prime\prime}(h) = \phi_{\theta_{0}}^{\prime\prime}(h) \end{split}$$

**Proof for Theorem 4.3** Note that for  $g(t_n, h_n^1, h_n^2) = t_n^{-1} \left( \phi'_{\theta_0 + t_n h_n^1} \left( h_n^2 \right) - \phi'_{\theta_0} \left( h_n^2 \right) \right)$ ,

$$\bar{\phi}_n''(\mathbb{Z}_n^*,\mathbb{Z}_n^*) = g\left(\epsilon_n, \mathbb{Z}_n^* + \epsilon_n^{-1}\left(\hat{\theta}_n - \theta_0\right), \mathbb{Z}_n^*\right)$$

Since  $\left(\mathbb{Z}_{n}^{*} + \epsilon_{n}^{-1}\left(\hat{\theta}_{n} - \theta_{0}\right), \mathbb{Z}_{n}^{*}\right) \xrightarrow{\mathbb{P}} (\mathbb{G}_{0}, \mathbb{G}_{0})$  and  $g\left(t_{n}, h_{n}^{1}, h_{n}^{2}\right) \rightarrow g\left(h_{1}, h_{2}\right)$  when  $\left(h_{n}^{1}, h_{n}^{2}\right) \rightarrow (h_{1}, h_{2})$ , the result follows from Vaart and Wellner (1996) Theorem 1.11.1 and Lemma A.1.

**Proof for Theorem 4.4** Using the triangle inequality,

$$\begin{split} \left\| \bar{\phi}_{n}^{\prime\prime}(h_{1},h_{1}) - \bar{\phi}_{n}^{\prime\prime}(h_{2},h_{2}) \right\|_{\mathbb{E}} &= \left\| \frac{\phi_{\hat{\theta}_{n}+\epsilon_{n}h_{1}}^{\prime}(h_{1}) - \phi_{\hat{\theta}_{n}}^{\prime}(h_{1}) - \phi_{\hat{\theta}_{n}+\epsilon_{n}h_{2}}^{\prime}(h_{2}) + \phi_{\hat{\theta}_{n}}^{\prime}(h_{2})}{\epsilon_{n}} \right\|_{\mathbb{E}} \\ &= \left\| \frac{\phi_{\hat{\theta}_{n}+\epsilon_{n}h_{1}}^{\prime\prime}(h_{1}) - \phi_{\hat{\theta}_{n}+\epsilon_{n}h_{2}}^{\prime\prime}(h_{1}) + \phi_{\hat{\theta}_{n}}^{\prime}(h_{2}) - \phi_{\hat{\theta}_{n}}^{\prime}(h_{1}) + \phi_{\hat{\theta}_{n}+\epsilon_{n}h_{2}}^{\prime\prime}(h_{1}) - \phi_{\hat{\theta}_{n}+\epsilon_{n}h_{2}}^{\prime\prime}(h_{2})}{\epsilon_{n}} \right\|_{\mathbb{E}} \\ &\leq C_{\theta} \left\| h_{1} - h_{2} \right\|_{\mathbb{D}} + C_{h} \left\| h_{1} - h_{2} \right\|_{\mathbb{D}} + C_{h} \left\| h_{1} - h_{2} \right\|_{\mathbb{D}} \end{split}$$

where  $C_{\theta}$  is the Lipschitz constant on  $\theta$  and  $C_h$  is the Lipschitz constant on h for  $\phi'_{\theta}(h)$ .

**Proof for Theorem 6.1** The arguments are similar to those of Theorem 3.1. Let

$$\bar{L}_{\theta,p}^{\epsilon_n}(h_0, h_2, \dots, h_p) = \frac{1}{\epsilon_n} \left[ a_0 \phi \left( \theta + \epsilon_n h_0 \right) + \sum_{l=1}^p a_j \phi \left( \theta + \epsilon_n l h_l \right) \right]$$

For  $h_0 \to 0$ ,  $h_l \to h$  for l = 1, ..., p, and  $\epsilon_n \to 0$ , it follows from (26) and (27) that

$$\bar{L}^{\epsilon_n}_{\theta,p}(h_0,\dots,h_p) = \phi^{(1)}_{\theta}(h) + O\left(\hat{h} - h\right) + O\left(\epsilon^p_n\right)$$
(40)

Define  $g_n(h_0,\ldots,h_p) = \overline{L}_{\theta_0,p}^{\epsilon_n}(h_0,\ldots,h_p)$ , and  $g(h) = \phi_{\theta}^{(1)}(h)$ . Then by (40),  $g_n(h_0,\ldots,h_p) \rightarrow g(h)$ . Next write

$$\hat{\phi}_{n}'\left(\mathbb{Z}_{n}^{*};p\right) \equiv L_{\hat{\theta},p}^{\epsilon_{n}}\left(\mathbb{Z}_{n}^{*}\right) = \bar{L}_{\theta_{0},p}^{\epsilon_{n}}\left(\frac{\hat{\theta}_{n}-\theta_{0}}{\epsilon_{n}},\mathbb{Z}_{n}^{*}+\frac{\hat{\theta}_{n}-\theta_{0}}{l\epsilon_{n}},l=1,\ldots,p\right)$$

Since  $\left(\frac{\hat{\theta}_n - \theta_0}{\epsilon_n}, \mathbb{Z}_n^* + \frac{\hat{\theta}_n - \theta_0}{l\epsilon_n}, l = 1, \dots, p\right) \xrightarrow{\mathbb{P}} (0, \mathbb{G}_0, \dots, \mathbb{G}_0)$ , it follows from the bootstrap extended continuous mapping Theorem Lemma A.1 that  $\hat{\phi}'_n(\mathbb{Z}_n^*; p) \equiv L^{\epsilon_n}_{\hat{\theta}, p}(\mathbb{Z}_n^*) \xrightarrow{\mathbb{P}} \phi^{(1)}_{\theta_0}(\mathbb{Z}_n^*)$ .

**Lemma A.1 (Bootstrap extended continuous mapping theorem)** Under the conditions of Vaart and Wellner (1996) Theorem 1.11.1 and Kosorok (2007) Theorem 10.8,  $g_n\left(\hat{X}_n\right) \stackrel{\mathbb{P}}{\leadsto} g(X)$ .

**Proof:** We use the notation in Kosorok (2007) Theorem 10.8 which argues that  $\hat{X}_n \stackrel{\mathbb{P}}{\leadsto} X$  implies that unconditionally  $\hat{X}_n \rightsquigarrow X$ , so that  $g_n(\hat{X}_n) \rightsquigarrow g(X)$  by van der Vaart and Wellner (1996) Theorem 1.11.1. Let  $E_M$  denote the expectation conditional on the data. Write

$$\sup_{h\in BL_{1}}\left\|E_{M}h\left(g_{n}\left(\hat{X}_{n}\right)\right)-Eh\left(g\left(X\right)\right)\right\|\leq \sup_{h\in BL_{1}}\left\|E_{M}h\left(g_{n}\left(\hat{X}_{n}\right)\right)-E_{M}h\left(g\left(\hat{X}_{n}\right)\right)\right\|$$
$$+\sup_{h\in BL_{1}}\left\|E_{M}h\left(g\left(\hat{X}_{n}\right)\right)-Eh\left(g\left(X\right)\right)\right\|$$

Since  $h \in BL_1$ ,  $\left\| E_M h\left(g_n\left(\hat{X}_n\right)\right) - E_M h\left(g\left(\hat{X}_n\right)\right) \right\| \le E_M \left[ \|g_n\left(\hat{X}_n\right) - g\left(\hat{X}_n\right)\| \land 2 \right]$ . Next by van der Vaart and Wellner (1996) Theorem 1.11.1,  $\left(g_n\left(\hat{X}_n\right), g\left(\hat{X}_n\right)\right) \rightsquigarrow (g(X), g(X))$  and  $g_n\left(\hat{X}_n\right) - g\left(\hat{X}_n\right) \rightsquigarrow 0$ , so that argue as before equation 10.8 on page 190 of Kosorok (2007),

$$E^*E_M\left[\|g_n\left(\hat{X}_n\right) - g\left(\hat{X}_n\right)\| \wedge 2\right] \to 0 \implies E_M\left[\|g_n\left(\hat{X}_n\right) - g\left(\hat{X}_n\right)\| \wedge 2\right] = o_P(1).$$

By Kosorok (2007) Theorem 10.8,  $\sup_{h \in BL_1} \left\| E_M h\left(g\left(\hat{X}_n\right)\right) - Eh\left(g\left(X\right)\right) \right\| = o_P(1)$ . It then follows that  $\sup_{h \in BL_1} \left\| E_M h\left(g_n\left(\hat{X}_n\right)\right) - Eh\left(g\left(X\right)\right) \right\| = o_P(1)$ .

Monte Carlo Simulations for the Normal Mean Model  $\phi(\theta)=1.5\theta1(\theta>0)-0.5\theta1(\theta<0)$  Number of Monte Carlo Simulations is 2000

Ta	able 1: I	Empirical	Two-Side	d Symn	netric C	overage F	Frequenc	ies for $\epsilon_i$	$n = n^{-1}$	$^{/6}$ using	$\mathbb{Z}_n = N(0, \hat{\sigma})$
$ heta_n$	-2	$-n^{-1/6}$	$-n^{-1/3}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	$n^{-1/10}$	2
n = 2000	0.945	0.948	0.999	0.952	0.948	0.949	0.961	0.900	0.886	0.906	0.961
n = 4000	0.953	0.948	1.000	0.956	0.948	0.952	0.949	0.896	0.904	0.908	0.952
n = 6000	0.950	0.952	1.000	0.955	0.954	0.946	0.953	0.905	0.893	0.910	0.957
n = 8000	0.943	0.952	1.000	0.955	0.956	0.948	0.941	0.897	0.902	0.919	0.949
n = 10000	0.952	0.940	1.000	0.945	0.952	0.953	0.952	0.900	0.899	0.929	0.953
Ta	ble 2: E	mpirical '	Two-Sideo	l Equal-	tailed (	Coverage	Frequen	cies for $\epsilon$	$\epsilon_n = n^-$	1/6 using	$\mathbb{Z}_n = N(0, \hat{\sigma})$
$\theta_n$	-2	$-n^{-1/6}$	$-n^{-1/3}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	$n^{-1/10}$	2
n = 2000	0.945	0.801	0.583	0.952	0.917	0.612	0.500	0.566	0.780	0.901	0.961
n = 4000	0.952	0.815	0.587	0.953	0.924	0.630	0.479	0.559	0.808	0.915	0.951
n = 6000	0.951	0.800	0.566	0.958	0.935	0.644	0.492	0.565	0.794	0.916	0.957
n = 8000	0.943	0.809	0.553	0.958	0.943	0.672	0.497	0.550	0.798	0.923	0.949
n = 10000	0.953	0.812	0.548	0.951	0.930	0.673	0.478	0.551	0.801	0.929	0.953
	Table 3	8: Empirio	al upper	One-Sid	ed Cov	erage Fre	mencies	for $\epsilon_{\rm m} =$	$= n^{-1/6}$	using Z <sub>m</sub>	$= N(0, \hat{\sigma})$
$\theta_{r}$	-2	$-n^{-1/6}$	$-n^{-1/3}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	$n^{-1/10}$	2
n = 2000	0.948	0.792	0.566	0.946	0.916	0.623	0.511	0.582	0.802	0.911	0.961
n = 4000	0.950	0.808	0.571	0.952	0.922	0.632	0.503	0.584	0.822	0.929	0.949
n = 6000	0.956	0.800	0.551	0.952	0.934	0.655	0.504	0.578	0.818	0.932	0.949
n = 8000	0.944	0.801	0.540	0.957	0.945	0.680	0.512	0.571	0.817	0.944	0.943
n = 10000	0.953	0.810	0.529	0.950	0.936	0.680	0.491	0.566	0.822	0.948	0.953
	Table	1. Empiri	anl lower	Ono Sid	ad Cow	rogo Fro	mondia	for c -	$-n^{-1/6}$	using 7	$-N(0\hat{\sigma})$
θ		$-n^{-1/6}$	$n^{-1/3}$	0110-510	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	$n^{-1/10}$	= N(0, 0)
n = 2000	0.945	0.963	0.999	0.952	0.948	0.949	0.961	0.950	0.947	0.959	0.956
n = 4000	0.946	0.965	1.000	0.956	0.948	0.952	0.949	0.946	0.952	0.950	0.955
n = 6000	0.944	0.969	1.000	0.955	0.954	0.946	0.953	0.955	0.942	0.950	0.950
n = 8000	0.951	0.974	1.000	0.955	0.956	0.948	0.941	0.952	0.950	0.947	0.954
n = 10000	0.956	0.963	1.000	0.945	0.952	0.953	0.952	0.952	0.948	0.947	0.957
$n = 10000 \ 0.956 \ 0.963 \ 1.000 \ 0.945 \ 0.952 \ 0.953 \ 0.952 \ 0.952 \ 0.952 \ 0.948 \ 0.947 \ 0.957$											
T	blo 5. I	Empirical	Two Side	d Symn	otric C	ovorago F	Froquere	ios for c	$-n^{-1}$	/3 using	$\mathbb{Z} = N(0, \hat{\sigma})$
$\theta_n$ Ta	able 5: I $-2$	Empirical $-n^{-1/6}$	Two-Side $-n^{-1/3}$	d Symn 0	netric C $n^{-1}$	overage F $n^{-1/1.5}$	Frequenc $n^{-1/2}$	ies for $\epsilon_n^{-1/3}$	$n = n^{-1}$ $n^{-1/6}$	$n^{1/3}$ using $n^{-1/10}$	$\mathbb{Z}_n = N(0, \hat{\sigma})$
$\begin{aligned} & \theta_n \\ & n = 2000 \end{aligned}$	able 5: 1 —2 0.945	Empirical $-n^{-1/6}$ 0.953	Two-Side $-n^{-1/3}$ 0.956	d Symm 0 0.946	netric C $n^{-1}$ 0.942	$\begin{array}{c} \text{overage I} \\ n^{-1/1.5} \\ 0.949 \end{array}$	Frequenc $n^{-1/2}$ 0.961	ies for $\epsilon_n$ $n^{-1/3}$ 0.900	$n = n^{-1}$ $n^{-1/6}$ 0.938	$n^{1/3}$ using $n^{-1/10}$ 0.949	$\mathbb{Z}_n = N(0, \hat{\sigma})$ 2 0.961
$Ta \\ \theta_n \\ n = 2000 \\ n = 4000$	able 5: I —2 0.945 0.953	Empirical $-n^{-1/6}$ 0.953 0.952	Two-Side $-n^{-1/3}$ 0.956 0.955	d Symn 0 0.946 0.949	netric C $n^{-1}$ 0.942 0.946	overage F $n^{-1/1.5}$ 0.949 0.951	Frequenc $n^{-1/2}$ 0.961 0.949	ies for $\epsilon_{4}$ $n^{-1/3}$ 0.900 0.896	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956	$n^{1/3}$ using $n^{-1/10}$ 0.949 0.953	$\mathbb{Z}_n = N(0, \hat{\sigma})$ $\begin{array}{c} 2\\ 0.961\\ 0.952 \end{array}$
$T_{n} = 2000$ $n = 4000$ $n = 6000$	able 5: I -2 0.945 0.953 0.950	Empirical $-n^{-1/6}$ 0.953 0.952 0.953	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946	d Symn 0 0.946 0.949 0.950	netric C $n^{-1}$ 0.942 0.946 0.951	$\begin{array}{c} \text{overage I} \\ n^{-1/1.5} \\ 0.949 \\ 0.951 \\ 0.946 \end{array}$	Frequenc $n^{-1/2}$ 0.961 0.949 0.953	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946	$n^{-1/10}$ using $n^{-1/10}$ 0.949 0.953 0.943	$\mathbb{Z}_n = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000	able 5: $I$ -2 0.945 0.953 0.950 0.943	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960	d Symn 0 0.946 0.949 0.950 0.951	$\begin{array}{c} \text{netric C} \\ n^{-1} \\ 0.942 \\ 0.946 \\ 0.951 \\ 0.950 \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948	$\begin{array}{c} \text{Frequenc} \\ n^{-1/2} \\ 0.961 \\ 0.949 \\ 0.953 \\ 0.941 \end{array}$	ies for $\epsilon_n$ $n^{-1/3}$ 0.900 0.896 0.905 0.897	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.936 0.946 0.947	$n^{-1/10}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000	able 5: $I -2 0.945 0.953 0.950 0.943 0.952$	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.953 0.951	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958	d Symm 0 0.946 0.949 0.950 0.951 0.941	$\begin{array}{c} \text{netric C} \\ n^{-1} \\ 0.942 \\ 0.946 \\ 0.951 \\ 0.950 \\ 0.949 \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953	$\begin{array}{c} \text{Frequenc} \\ n^{-1/2} \\ 0.961 \\ 0.949 \\ 0.953 \\ 0.941 \\ 0.952 \end{array}$	ies for $\epsilon_{n}^{-1/3}$ 0.900 0.896 0.905 0.897 0.900	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955	$n^{-1/10}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947	$\mathbb{Z}_n = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Ta	able 5: $I -2$ 0.945 0.953 0.950 0.943 0.952 ble 6: F	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical '	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958	d Symn 0.946 0.949 0.950 0.951 0.941	netric C $n^{-1}$ 0.942 0.946 0.951 0.950 0.949	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequenc	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $e = n^{-1}$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$
$\begin{aligned} & \theta_n \\ n &= 2000 \\ n &= 4000 \\ n &= 6000 \\ n &= 8000 \\ n &= 10000 \end{aligned}$ $Ta \\ \theta_n \end{aligned}$	able 5: $I -2$ 0.945 0.953 0.950 0.943 0.952 ble 6: $E -2$	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sideo $-n^{-1/3}$	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0	$\begin{array}{c} \text{netric C} \\ n^{-1} \\ 0.942 \\ 0.946 \\ 0.951 \\ 0.950 \\ 0.949 \\ \text{tailed C} \\ n^{-1} \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage F $n^{-1/1.5}$	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequenc $n^{-1/2}$	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $\epsilon_n = n^{-1}$ $n^{-1/6}$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2
$\begin{aligned} \theta_n \\ n &= 2000 \\ n &= 4000 \\ n &= 6000 \\ n &= 8000 \\ n &= 10000 \end{aligned}$ $Ta \\ \theta_n \\ n &= 2000 \end{aligned}$	able 5: $I$ -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sideo $-n^{-1/3}$ 0.744	d Symn 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925 \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequenc $n^{-1/2}$ 0.553	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $\epsilon_n = n^{-1/6}$ 0.939	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961
$\begin{aligned} \theta_n \\ n &= 2000 \\ n &= 4000 \\ n &= 6000 \\ n &= 8000 \\ n &= 10000 \end{aligned}$ $\begin{aligned} \text{Ta} \\ \theta_n \\ n &= 2000 \\ n &= 4000 \end{aligned}$	able 5: $I$ -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951 0.951	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidee $-n^{-1/3}$ 0.744 0.760	d Symn 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.955	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage F $n^{-1/1.5}$ 0.652 0.679	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequenc $n^{-1/2}$ 0.553 0.541	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $\epsilon_n = n^{-1}$ $n^{-1/6}$ 0.939 0.956	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951
$\begin{aligned} \theta_n \\ n &= 2000 \\ n &= 4000 \\ n &= 6000 \\ n &= 8000 \\ n &= 10000 \end{aligned}$ $\begin{aligned} \text{Ta} \\ \theta_n \\ n &= 2000 \\ n &= 4000 \\ n &= 6000 \end{aligned}$	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.952 0.951	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951 0.951 0.953	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sideo $-n^{-1/3}$ 0.744 0.760 0.759	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.955 0.955 0.960	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686	$\begin{array}{c} \text{Frequenc} \\ n^{-1/2} \\ 0.961 \\ 0.949 \\ 0.953 \\ 0.941 \\ 0.952 \\ \text{Frequenc} \\ n^{-1/2} \\ 0.553 \\ 0.541 \\ 0.546 \end{array}$	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $\epsilon_n = n^{-1/6}$ 0.939 0.956 0.946	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957
$\begin{array}{c} & & \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \end{array}$ $\begin{array}{c} & \\ & \\ \Pi_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \end{array}$	able 5: $I$ -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.952 0.951 0.943	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951 0.951 0.953 0.952	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidee $-n^{-1/3}$ 0.744 0.760 0.759 0.754	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.955 0.955 0.960 0.961	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage F $n^{-1/1.5}$ 0.652 0.679 0.686 0.709	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequenc $n^{-1/2}$ 0.553 0.541 0.546 0.549	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.756	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $\epsilon_n = n^{-1}$ $n^{-1/6}$ 0.939 0.956 0.946 0.948	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 5 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000	able 5: $I$ -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.945 0.952 0.951 0.943 0.953	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical 0.951 0.951 0.951 0.953 0.952 0.951 0.953 0.952 0.953 0.953	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sideo $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.955 0.960 0.961 0.952	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \end{array}$	overage $F_{n^{-1/1.5}}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequenc $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.756 0.765	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $\epsilon_n = n^{-1/6}$ 0.939 0.956 0.946 0.948 0.955	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 10000 Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000	able 5: $I$ -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.951 0.943 0.953 Table 7	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical 7 $-n^{-1/6}$ 0.951 0.951 0.953 0.952 0.951 2.951 2.951 2.951 2.951 2.951 3.952 0.951 3.952 0.953 3.952 3.951 3.952 3.951 3.951 3.952 3.951 3.952 3.953 3.9555 3.9555 3.9555 3.9555 3.9555 3.9555 3.9555 3.9555 3.9555 3.9555 3.95	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidee $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.955 0.960 0.961 0.952 One-Sid	netric C $n^{-1}$ 0.942 0.946 0.951 0.950 0.949 tailed C $n^{-1}$ 0.925 0.931 0.942 0.944 0.936	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage F $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequenc $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 guencies	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.765 for $\epsilon_n^{-1/3}$	$n = n^{-1}$ $n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $\epsilon_n = n^{-1}$ $n^{-1/6}$ 0.939 0.956 0.946 0.948 0.955 $n^{-1/3}$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947 using $\mathbb{Z}$	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $- N(0, \hat{\sigma})$
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.951 0.953 Table 7 -2	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical 0.951 0.951 0.951 0.953 0.952 0.951 2.951 2.951 2.951 2.951 2.951 2.951 2.951 2.951 2.951 2.951 2.951 2.951 2.951 2.951 2.951 2.953 0.952 0.953 0.953 0.953 0.951 2.951 2.951 2.953 0.953 0.953 0.951 2.951 2.951 2.951 2.951 2.953 0.953 0.951 2.955 2.955 2.9551 2.95	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidec $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.955 0.960 0.961 0.952 One-Sid 0	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequenc $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.905 0.900 cies for $\epsilon_n^{-1/3}$ 0.749 0.765 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{n^{-1/3}}$	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1}$ $n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.957 0.949 0.953
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Ta $\theta_n$ n = 2000 n = 6000 n = 8000 n = 10000 $\theta_n$ n = 2000	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.952 0.951 0.943 0.953 Table 7 -2 0.948	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951 0.951 0.953 0.952 0.951 ': Empirical ' $-n^{-1/6}$ 0.952	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidec $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0 0.953	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ 0.929\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$ 0.571	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.749 0.765 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{0.753}$	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1}$ $n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.940$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.957 0.949 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Ta $\theta_n$ n = 2000 n = 6000 n = 8000 n = 10000 $\theta_n$ n = 2000 n = 2000 n = 4000	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.951 0.943 0.953 Table 7 -2 0.948 0.950	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951 0.951 0.951 0.953 0.952 0.951 ': Empirical $-n^{-1/6}$ 0.952 0.949	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sideo $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0 0.953 0.959	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{ded Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685	Frequence $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$ 0.571 0.569	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.756 0.765 for $\epsilon_n = \frac{n^{-1/3}}{n^{-1/3}}$ 0.753 0.769	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\pi^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.940$ $0.953$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $= N(0, \hat{\sigma})$ 2 0.961 0.949
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Ta $\theta_n$ n = 2000 n = 4000 n = 10000 $\theta_n$ n = 2000 n = 2000 n = 4000 n = 4000 n = 6000	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.951 0.943 0.953 Table 7 -2 0.948 0.950 0.950 0.950 0.953	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951 0.951 0.951 0.952 0.951 ': Empirical ' $-n^{-1/6}$ 0.952 0.951 ': Empirical ' 0.952 0.951	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidec $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765 0.774	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0 0.953 0.959 0.960	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ 0.944\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685 0.694	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$ 0.571 0.569 0.557	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{n^{-1/3}}$ 0.753 0.769 0.780	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $e_n = n^{-1}$ $n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.940$ $0.953$ $0.952$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945 0.945	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961 0.949 0.949
Ta $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Ta $\theta_n$ n = 2000 n = 4000 n = 8000 n = 10000 $\theta_n$ n = 2000 n = 4000 n = 4000 n = 4000 n = 4000 n = 8000 n = 8000	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.951 0.943 0.953 Table 7 -2 0.948 0.950 0.956 0.944	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951 0.951 0.951 0.952 0.951 ': Empirical ' $-n^{-1/6}$ 0.952 0.951 ': Empirical ' 0.952 0.951 ': Empirical ' 0.952 0.951 ': Empirical ' 0.952 0.951 ': Empirical ' 0.952 0.953 0.952 0.953 0.953 0.953 0.951 0.953 0.953 0.951 0.953 0.953 0.951 0.953 0.953 0.953 0.953 0.951 0.953 0.953 0.953 0.953 0.953 0.953 0.951 0.953 0.953 0.953 0.953 0.955 0.953 0.955 0.953 0.955 0.95 0.9	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidee $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765 0.774 0.755	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0 0.953 0.959 0.960 0.963	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ 0.944\\ 0.950\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage F $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685 0.694 0.722	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$ 0.571 0.569 0.557 0.566	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{n^{-1/3}}$ 0.753 0.769 0.780 0.777	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1}$ $n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.940$ $0.953$ $0.952$ $0.952$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945 0.945 0.952	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $= N(0, \hat{\sigma})$ 2 0.961 0.949 0.949 0.943
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \text{Ta} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 4000 \\ n = 6000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \end{array}$	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.951 0.943 0.953 Table 7 -2 0.948 0.950 0.950 0.948 0.950 0.956 0.944 0.953	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical 0.951 0.951 0.951 0.951 0.952 0.951 2: Empirical $-n^{-1/6}$ 0.952 0.951 2: Empirical 0.952 0.955 0.955 0.950 0.956	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidee $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765 0.774 0.755 0.756	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0 0.953 0.959 0.960 0.963 0.958	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ 0.944\\ 0.950\\ 0.942\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685 0.694 0.722 0.720	$\begin{array}{c} & requenc \\ n^{-1/2} \\ 0.961 \\ 0.949 \\ 0.953 \\ 0.941 \\ 0.952 \\ \end{array}$ $\begin{array}{c} & requenc \\ n^{-1/2} \\ 0.553 \\ 0.541 \\ 0.546 \\ 0.549 \\ 0.527 \\ \end{array}$ $\begin{array}{c} & quencies \\ n^{-1/2} \\ 0.571 \\ 0.569 \\ 0.557 \\ 0.566 \\ 0.544 \end{array}$	ies for $\epsilon_n$ $n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{0.753}$ 0.769 0.780 0.777 0.781	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\pi^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.940$ $0.953$ $0.952$ $0.952$ $0.952$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.945 0.945 0.945 0.952 0.953	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961 0.949 0.949 0.949 0.943 0.953
$\begin{aligned} & \theta_n \\ n &= 2000 \\ n &= 4000 \\ n &= 6000 \\ n &= 8000 \\ n &= 10000 \end{aligned}$ $\begin{aligned} & \text{Ta} \\ & \theta_n \\ n &= 2000 \\ n &= 4000 \\ n &= 8000 \\ n &= 10000 \end{aligned}$ $\begin{aligned} & \theta_n \\ n &= 2000 \\ n &= 4000 \\ n &= 6000 \\ n &= 8000 \\ n &= 10000 \end{aligned}$	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.945 0.953 Table 7 -2 0.948 0.950 0.956 0.944 0.953 Table 5	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951 0.951 0.951 0.952 0.951 ': Empirical ' $-n^{-1/6}$ 0.952 0.951 ': Empirical ' 0.955 0.952 0.955 0.956 0.955 0.956 0.95	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidec $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765 0.774 0.755 0.756 cal lower 4	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0 0.953 0.959 0.960 0.953 0.958 0.958	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ 0.929\\ 0.934\\ 0.942\\ 0.950\\ 0.942\\ \text{ed Cove}\\ n^{-1}\\ 0.929\\ 0.942\\ \text{ed Cove}\\ n^{-1}\\ 0.950\\ 0$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685 0.694 0.722 0.720	Frequence $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$ 0.571 0.569 0.557 0.566 0.544 quencies	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{n^{-1/3}}$ 0.753 0.769 0.780 0.777 0.781 for $\epsilon = \frac{n^{-1}}{n^{-1}}$	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $E_n = n^{-1}$ $n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.940$ $0.953$ $0.952$ $0.952$ $0.952$ $n^{-1/3}$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945 0.945 0.952 0.953 using $\mathbb{Z}$	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961 0.949 0.949 0.943 0.943 0.953 $- N(0, \hat{\sigma})$
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \text{Ta} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.951 0.943 0.953 Table 7 -2 0.948 0.950 0.948 0.950 0.956 0.944 0.953 Table 8 -2	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.953 0.951 Empirical 7 $-n^{-1/6}$ 0.951 0.951 0.951 0.952 0.951 7: Empirical 7 $-n^{-1/6}$ 0.953 0.952 0.951 7: Empirical 7 $-n^{-1/6}$ 0.953 0.955 0.951 8: Empirical 7 $-n^{-1/6}$	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidee $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765 0.774 0.755 0.756 cal lower 9 $-n^{-1/3}$	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0 0.953 0.959 0.960 0.963 0.958 One-Sid 0	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{ctailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ 0.944\\ 0.950\\ 0.942\\ \text{ed Cove}\\ n^{-1}\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685 0.694 0.722 0.720 erage Free $n^{-1/1.5}$	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$ 0.571 0.569 0.557 0.566 0.544 quencies $n^{-1/2}$	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{0.753}$ 0.769 0.780 0.777 0.781 for $\epsilon_n = \frac{n^{-1/3}}{0.781}$	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1}$ $n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.940$ $0.953$ $0.952$ $0.952$ $0.952$ $n^{-1/3}$ $n^{-1/6}$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945 0.945 0.952 0.953 using $\mathbb{Z}_n$ $n^{-1/10}$	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961 0.949 0.949 0.943 0.943 0.953 $= N(0, \hat{\sigma})$ 2 0.961 0.949 0.943 0.953 $= N(0, \hat{\sigma})$ 2 0.961 0.943 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961 0.943 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.954 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.955 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \text{Ta} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 4000 \\ n = 6000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.945 0.953 Table 7 -2 0.948 0.950 0.956 0.944 0.953 Table 8 -2 0.945	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical 0.951 0.951 0.951 0.951 0.952 0.951 2: Empirical $-n^{-1/6}$ 0.952 0.951 2: Empirical $-n^{-1/6}$ 0.952 0.955 0.955 0.950 0.955 0.956 8: Empirical $-n^{-1/6}$ 0.949	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidec $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765 0.774 0.755 0.756 cal lower $-n^{-1/3}$ 0.956	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0 0.953 0.959 0.960 0.963 0.958 One-Sid 0 0.946	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ 0.944\\ 0.950\\ 0.942\\ \text{ed Cove}\\ n^{-1}\\ 0.942\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685 0.694 0.722 0.720 erage Free $n^{-1/1.5}$ 0.694	Frequenc $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$ 0.571 0.569 0.557 0.566 0.544 quencies $n^{-1/2}$ 0.941 0.952	ies for $\epsilon_n$ $n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{0.753}$ 0.769 0.780 0.777 0.781 for $\epsilon_n = \frac{n^{-1/3}}{0.781}$	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1}$ $n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.952$ $0.952$ $0.952$ $0.952$ $n^{-1/3}$ $n^{-1/6}$ $0.947$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.943 0.943 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945 0.945 0.945 0.952 0.953 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.959	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.957 0.949 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961 0.949 0.949 0.949 0.949 0.949 0.943 0.953 $= N(0, \hat{\sigma})$ 2 0.956
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 8000 \\ n = 10000 \\ \end{array}$	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.945 0.953 Table 7 -2 0.948 0.950 0.956 0.944 0.953 Table 8 -2 0.945 0.945 0.944 0.953 Table 8 -2 0.945 0.945 0.945 0.944 0.953 0.956 0.944 0.953 0.956 0.944 0.953 0.944 0.953 0.956 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.944 0.953 0.945 0.94	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical 0.951 0.951 0.951 0.951 0.951 0.952 0.951 2. Empirical $-n^{-1/6}$ 0.952 0.951 2. Empirical $-n^{-1/6}$ 0.952 0.955 0.950 0.955 0.950 0.955 0.950 0.955 0.950 0.952 0.955 0.950 0.955 0.952 0.952 0.953 0.952 0.951 2. Empirical $-n^{-1/6}$ 0.952 0.955 0.95	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sided $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765 0.774 0.755 0.756 cal lower 1 $-n^{-1/3}$ 0.956 0.955	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0.953 0.960 0.963 0.958 One-Sid 0 0.946 0.949	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{ded Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ 0.944\\ 0.950\\ 0.942\\ \text{ed Cove}\\ n^{-1}\\ 0.942\\ 0.946\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685 0.694 0.722 0.720 erage Free $n^{-1/1.5}$ 0.694 0.722 0.720	Frequence $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 cuencies $n^{-1/2}$ 0.571 0.569 0.557 0.566 0.544 quencies $n^{-1/2}$ 0.544 quencies $n^{-1/2}$	ies for $\epsilon_n$ $n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n$ $n^{-1/3}$ 0.731 0.749 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{0.753}$ 0.769 0.780 0.777 0.781 for $\epsilon_n = \frac{n^{-1/3}}{0.781}$ for $\epsilon_n = \frac{n^{-1/3}}{0.781}$ for $\epsilon_n = \frac{n^{-1/3}}{0.950}$ 0.946	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1}$ $n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.952$ $0.952$ $0.952$ $n^{-1/3}$ $n^{-1/6}$ $0.947$ $0.952$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945 0.945 0.945 0.952 0.953 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.959 0.950	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961 0.949 0.943 0.943 0.943 0.953 $= N(0, \hat{\sigma})$ 2 0.956 0.955
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.945 0.953 Table 7 -2 0.948 0.950 0.948 0.950 0.944 0.953 Table 8 -2 0.945 0.945 0.944 0.953 Table 8 -2 0.945 0.945 0.944 0.953 Table 8 -2 0.945 0.945 0.944 0.953 Table 8 -2 0.945 0.944 0.953 Table 8 -2 0.945 0.944 0.954 0.954 0.956 0.944 0.953 Table 8 -2 0.945 0.945 0.945 0.945 0.944 0.954 0.945 0.945 0.945 0.945 0.944 0.954 0.945 0.945 0.944 0.945 0.946 0.944	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.953 0.951 Empirical ' $-n^{-1/6}$ 0.951 0.951 0.951 0.952 0.951 ': Empirical ' $-n^{-1/6}$ 0.952 0.951 ': Empirical ' $-n^{-1/6}$ 0.952 0.955 0.956 S: Empirical ' $-n^{-1/6}$ 0.955 0.956 S: Empirical ' $-n^{-1/6}$ 0.957 0.949 0.957 0.957 0.949 0.957 0.949 0.957 0.949 0.957 0.949 0.957 0.949 0.957 0.947	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidec $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765 0.774 0.755 0.756 cal lower 0 $-n^{-1/3}$ 0.956 0.955 0.946	d Symn 0 0.946 0.949 0.950 0.951 0.941 d Equal- 0 0.955 0.960 0.961 0.952 One-Sid 0 0.963 0.958 One-Sid 0 0.946 0.949 0.950	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ 0.944\\ 0.950\\ 0.942\\ \text{ed Cove}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685 0.694 0.722 0.720 erage Free $n^{-1/1.5}$ 0.949 0.951 0.946	Frequence $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$ 0.571 0.569 0.557 0.566 0.544 quencies $n^{-1/2}$ 0.961 0.949 0.953	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.897 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{0.753}$ 0.769 0.776 0.777 0.781 for $\epsilon_n = \frac{n^{-1/3}}{0.781}$ for $\epsilon_n = \frac{n^{-1/3}}{0.781}$	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $(n = n^{-1/6})$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.940$ $0.952$ $0.952$ $0.952$ $0.952$ $n^{-1/3}$ $n^{-1/6}$ $0.947$ $0.942$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945 0.945 0.945 0.952 0.953 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.959 0.950	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961 0.949 0.949 0.943 0.953 $= N(0, \hat{\sigma})$ 2 0.956 0.955 0.950
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 8000 \\ n = 10000 \\ \end{array}$ $\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 8000 \\ n = 8000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ \end{array}$	able 5: I -2 0.945 0.953 0.950 0.943 0.952 ble 6: E -2 0.945 0.952 0.951 0.943 0.953 Table 7 -2 0.948 0.950 0.948 0.950 0.948 0.950 0.944 0.953 Table 8 -2 0.945 0.944 0.953 Table 8 -2 0.945 0.945 0.944 0.953 Table 8 -2 0.945 0.945 0.952 0.948 0.953 0.956 0.944 0.953 0.953 0.956 0.944 0.953 0.953 0.956 0.944 0.953 0.945 0.944 0.953 0.945 0.946 0.951 0.951	Empirical $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirical 7 $-n^{-1/6}$ 0.951 0.951 0.951 0.951 0.952 0.951 7: Empirical 7 $-n^{-1/6}$ 0.952 0.951 7: Empirical 7 $-n^{-1/6}$ 0.952 0.955 0.950 0.955 0.956 8: Empirical 7 0.952 0.949 0.955 0.956 8: Empirical 7 0.952 0.949 0.955 0.956 1: Empirical 7 0.956 1: Empirical 7 0.95	Two-Side $-n^{-1/3}$ 0.956 0.955 0.946 0.960 0.958 Two-Sidea $-n^{-1/3}$ 0.744 0.760 0.759 0.754 0.751 cal upper $-n^{-1/3}$ 0.746 0.765 0.774 0.755 0.756 cal lower $-n^{-1/3}$ 0.956 0.955 0.946 0.960	d Symn 0 0.946 0.949 0.950 0.951 0.951 0.955 0.960 0.961 0.952 One-Sid 0 0.953 0.959 0.960 0.953 0.959 0.960 0.953 0.958 One-Sid 0 0.946 0.949 0.950 0.951	$\begin{array}{c} \text{netric C}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ 0.949\\ \text{tailed C}\\ n^{-1}\\ 0.925\\ 0.931\\ 0.942\\ 0.944\\ 0.936\\ \text{led Cove}\\ n^{-1}\\ 0.929\\ 0.934\\ 0.944\\ 0.950\\ 0.942\\ \text{ed Cove}\\ n^{-1}\\ 0.942\\ 0.946\\ 0.951\\ 0.950\\ \end{array}$	overage F $n^{-1/1.5}$ 0.949 0.951 0.946 0.948 0.953 Coverage $n^{-1/1.5}$ 0.652 0.679 0.686 0.709 0.712 erage Free $n^{-1/1.5}$ 0.671 0.685 0.694 0.722 0.720 erage Free $n^{-1/1.5}$ 0.949 0.951 0.946 0.948	Frequence $n^{-1/2}$ 0.961 0.949 0.953 0.941 0.952 Frequence $n^{-1/2}$ 0.553 0.541 0.546 0.549 0.527 quencies $n^{-1/2}$ 0.571 0.569 0.557 0.566 0.544 quencies $n^{-1/2}$ 0.961 0.949 0.953 0.941	ies for $\epsilon_n^{-1/3}$ 0.900 0.896 0.905 0.905 0.905 0.900 cies for $\epsilon_n^{-1/3}$ 0.731 0.749 0.765 0.765 for $\epsilon_n = \frac{n^{-1/3}}{0.753}$ 0.769 0.777 0.781 for $\epsilon_n = \frac{n^{-1/3}}{0.781}$ for $\epsilon_n = \frac{n^{-1/3}}{0.950}$ 0.946 0.955 0.952	$n = n^{-1}$ $n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1}$ $n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1/3}$ $n^{-1/6}$ $0.940$ $0.952$ $0.$	$^{/3}$ using $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{1/3}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945 0.952 0.953 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.959 0.950 0.950 0.947	$\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $\mathbb{Z} = N(0, \hat{\sigma})$ 2 0.961 0.949 0.943 0.943 0.953 $= N(0, \hat{\sigma})$ 2 0.956 0.955 0.950 0.954

Т	able 9:	Empirical $-1/6$	Two-Side $-1/3$	d Symn	$_{-1}^{\text{netric C}}$	overage F -1/1.5	requence $-1/2$	ies for $\epsilon_{-1/3}$	$n = n^{-1}$	$\frac{1}{2}$ using $\frac{1}{2}$	$\mathbb{Z}_n = N(0, \hat{\sigma})$
$\theta_n$	-2	$-n^{-1/2}$	$-n^{-1/6}$	0	$n^{-1}$	$n^{-1,10}$	$n^{-r/-}$	$n^{-1/6}$	$n^{-1/2}$	$n^{-1/10}$	2
n = 2000	0.945	0.955	0.954	0.900	0.924	0.972	0.900	0.955	0.938	0.949	0.901
n = 4000	0.955	0.952	0.945 0.041	0.908	0.920	0.975 0.070	0.940	0.945 0.052	0.950	0.900	0.952
n = 0000 n = 8000	0.950	0.955 0.053	0.941 0.048	0.900	0.910	0.970	0.904 0.058	0.955 0.047	0.940 0.047	0.943 0.044	0.957
n = 8000 n = 10000	0.945 0.052	0.955 0.051	0.948 0.048	0.912 0.005	0.925 0.016	0.907 0.073	0.958	0.947 0.051	0.947 0.055	0.944 0.047	0.949
n = 10000	0.952	0.951	0.940	0.905	0.910	0.975	0.959	0.951	0.955	0.947	0.955
Ta	ble 10:	Empirical	Two-Side	d Equal	-tailed	Coverage	Frequen	cies for	$\epsilon_n = n^-$	$^{-1/2}$ using	$\mathbb{Z}_n = N(0, \hat{\sigma})$
$\theta_n$	-2	$-n^{-1/6}$	$-n^{-1/3}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	$n^{-1/10}$	$\frac{2}{2}$
n = 2000	0.945	0.952	0.921	0.938	0.925	0.761	0.662	0.930	0.939	0.948	0.961
n = 4000	0.952	0.951	0.941	0.941	0.927	0.770	0.665	0.941	0.956	0.951	0.951
n = 6000	0.951	0.953	0.941	0.941	0.932	0.784	0.673	0.953	0.946	0.943	0.957
n = 8000	0.943	0.952	0.947	0.943	0.937	0.808	0.676	0.947	0.948	0.943	0.949
n = 10000	0.953	0.951	0.947	0.940	0.929	0.804	0.661	0.951	0.955	0.947	0.953
	Table 1	1: Empiri	cal upper	One-Sie	ded Cov	verage Fre	quencies	s for $\epsilon_n$ :	$= n^{-1/2}$	using $\mathbb{Z}_n$	$\hat{\sigma} = N(0, \hat{\sigma})$
$ heta_n$	-2	$-n^{-1/6}$	$-n^{-1/3}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	$n^{-1/10}$	2
n = 2000	0.948	0.953	0.922	0.975	0.957	0.774	0.677	0.948	0.940	0.937	0.961
n = 4000	0.950	0.949	0.950	0.973	0.957	0.784	0.697	0.951	0.953	0.945	0.949
n = 6000	0.956	0.955	0.952	0.972	0.966	0.799	0.688	0.951	0.952	0.945	0.949
n = 8000	0.944	0.950	0.950	0.978	0.967	0.824	0.696	0.945	0.952	0.952	0.943
n = 10000	0.953	0.956	0.945	0.976	0.963	0.820	0.679	0.948	0.952	0.953	0.953
									1 /0		
	Table 1	12: Empiri	ical lower	One-Sic	led Cov	erage Fre	quencies	for $\epsilon_n =$	$= n^{-1/2}$	using $\mathbb{Z}_n$	$= N(0, \hat{\sigma})$
$ heta_n$	-2	$-n^{-1/6}$	$-n^{-1/3}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	$n^{-1/10}$	2
n = 2000	0.945	0.949	0.952	0.906	0.911	0.946	0.961	0.950	0.947	0.959	0.956
n = 4000	0.946	0.952	0.952	0.908	0.909	0.946	0.949	0.946	0.952	0.950	0.955
n = 6000	0.944	0.947	0.939	0.906	0.910	0.940	0.953	0.955	0.942	0.950	0.950
n = 8000	0.051	0.953	0.952	0.912	0.912	0.940	0.941	0.952	0.950	0.947	0.954
10 0000	0.301	0.000	0.00-								
n = 10000	0.951 0.956	0.946	0.951	0.905	0.907	0.948	0.952	0.952	0.948	0.947	0.957
n = 10000	0.951 0.956 Table 13	0.946 • Empirice	0.951	0.905 led Sym	0.907	0.948 Coverage	0.952 Frequen	0.952	0.948	0.947	0.957 Z = N(0 $\hat{\sigma}$ )
n = 10000 T	0.951 0.956 Table 13 -2	0.946 : Empirica $-n^{-1/6}$	$0.951$ al Two-Sid $-n^{-1/3}$	0.905 led Sym	$\begin{array}{c} 0.907\\ \text{metric}\\ n^{-1} \end{array}$	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5} \end{array}$	$0.952$ Frequen $n^{-1/2}$	$0.952$ acies for $n^{-1/3}$	$0.948$ $\epsilon_n = n^{-1/6}$	0.947 $^{-1}$ using 2 $n^{-1/10}$	$0.957$ $\mathbb{Z}_n = N(0, \hat{\sigma})$
$n = 10000$ $n = 10000$ $T$ $\theta_n$ $n = 2000$	0.951 0.956 Table 13 -2 0.945	0.946 : Empirica $-n^{-1/6}$ 0.953	0.951 al Two-Sid $-n^{-1/3}$ 0.952	0.905 led Sym 0 0.950	0.907 metric $n^{-1}$ 0.955	0.948 Coverage $n^{-1/1.5}$ 0.972	0.952 Frequen $n^{-1/2}$ 0.831	0.952 ticles for $n^{-1/3}$ 0.955	0.948 $\epsilon_n = n^{-1/6}$ 0.938	$0.947 \\ ^{-1} \text{ using } 2 \\ n^{-1/10} \\ 0.949$	$0.957$ $\mathbb{Z}_n = N(0, \hat{\sigma})$ $2$ $0.961$
$n = 10000$ $n = 10000$ $T$ $\theta_n$ $n = 2000$ $n = 4000$	0.951 0.956 Table 13 -2 0.945 0.953	0.946 : Empirica $-n^{-1/6}$ 0.953 0.952	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952	0.905 led Sym 0 0.950 0.955	0.907 metric $n^{-1}$ 0.955 0.956	$\begin{array}{c} 0.948 \\ \text{Coverage} \\ n^{-1/1.5} \\ 0.972 \\ 0.976 \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827	0.952 acies for $n^{-1/3}$ 0.955 0.946	0.948 $\epsilon_n = n^{-1/6}$ 0.938 0.956	$\begin{array}{c} 0.947\\ ^{-1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953 \end{array}$	0.957 $\mathbb{Z}_n = N(0, \hat{\sigma})$ 0.961 0.952
n = 10000 n = 10000 n = 2000 n = 4000 n = 6000	$\begin{array}{c} 0.951 \\ 0.956 \\ \hline \text{able 13} \\ -2 \\ 0.945 \\ 0.953 \\ 0.950 \end{array}$	0.946 : Empirica $-n^{-1/6}$ 0.953 0.952 0.953	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943	0.905 led Sym 0.950 0.955 0.954	0.907 metric $n^{-1}$ 0.955 0.956 0.954	$\begin{array}{c} 0.948 \\ \text{Coverage} \\ n^{-1/1.5} \\ 0.972 \\ 0.976 \\ 0.972 \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846	0.952 acies for $n^{-1/3}$ 0.955 0.946 0.953	$ \begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946 \end{array} $	$\begin{array}{c} 0.947\\ ^{-1} \text{ using 2}\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\end{array}$	0.957 $\mathbb{Z}_n = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957
n = 10000 n = 10000 n = 2000 n = 4000 n = 6000 n = 8000	$\begin{array}{c} 0.351\\ 0.956\\ \hline \text{Table 13}\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ \end{array}$	0.946 : Empirica $-n^{-1/6}$ 0.953 0.952 0.953 0.953	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950	0.905 led Sym 0 0.950 0.955 0.954 0.956	$\begin{array}{c} 0.907 \\ \text{imetric} \\ n^{-1} \\ 0.955 \\ 0.956 \\ 0.954 \\ 0.956 \end{array}$	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833	$\begin{array}{c} 0.952 \\ \text{acies for} \\ n^{-1/3} \\ 0.955 \\ 0.946 \\ 0.953 \\ 0.947 \end{array}$	$0.948$ $\epsilon_n = n^{-1/6}$ 0.938 0.956 0.946 0.947	$\begin{array}{c} 0.947\\ ^{-1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944 \end{array}$	$0.957$ $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ $2$ $0.961$ $0.952$ $0.957$ $0.949$
n = 10000 $n = 10000$ $T$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$	$\begin{array}{c} 0.951 \\ 0.956 \\ \hline \\ \text{Table 13} \\ -2 \\ 0.945 \\ 0.953 \\ 0.950 \\ 0.943 \\ 0.952 \end{array}$	0.946 : Empirica $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951	0.951 al Two-Sic $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949	0.905 led Sym 0.950 0.955 0.954 0.956 0.949	$\begin{array}{c} 0.907 \\ \text{imetric} \\ n^{-1} \\ 0.955 \\ 0.956 \\ 0.954 \\ 0.956 \\ 0.951 \end{array}$	$\begin{array}{c} 0.948 \\ \text{Coverage} \\ n^{-1/1.5} \\ 0.972 \\ 0.976 \\ 0.972 \\ 0.971 \\ 0.973 \end{array}$	$\begin{array}{c} 0.952 \\ \text{Frequen} \\ n^{-1/2} \\ 0.831 \\ 0.827 \\ 0.846 \\ 0.833 \\ 0.822 \end{array}$	$\begin{array}{c} 0.952 \\ \text{acies for} \\ n^{-1/3} \\ 0.955 \\ 0.946 \\ 0.953 \\ 0.947 \\ 0.951 \end{array}$	$\begin{array}{c} 0.948 \\ \epsilon_n = n^{-1/6} \\ 0.938 \\ 0.956 \\ 0.946 \\ 0.947 \\ 0.955 \end{array}$	$\begin{array}{c} 0.947\\ ^{-1}\text{ using 2}\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944\\ 0.947\\ \end{array}$	$0.957$ $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ $2$ $0.961$ $0.952$ $0.957$ $0.949$ $0.953$
n = 10000 $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline \text{Table 13}\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \end{array}$	$\begin{array}{c} 0.935\\ 0.946\\ \vdots \ \text{Empirica}\\ -n^{-1/6}\\ 0.953\\ 0.952\\ 0.953\\ 0.953\\ 0.951\\ \end{array}$	0.951 al Two-Sic $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949	0.905 led Sym 0.950 0.955 0.954 0.956 0.949	$\begin{array}{c} 0.907 \\ \text{metric} \\ n^{-1} \\ 0.955 \\ 0.956 \\ 0.954 \\ 0.956 \\ 0.951 \end{array}$	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822	$\begin{array}{c} 0.952 \\ \text{acies for} \\ n^{-1/3} \\ 0.955 \\ 0.946 \\ 0.953 \\ 0.947 \\ 0.951 \end{array}$	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955 \end{array}$	$\begin{array}{c} 0.947\\ ^{-1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944\\ 0.947\\ \end{array}$	$\begin{array}{c} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953 \end{array}$
n = 10000 $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ Ta	$\begin{array}{c} 0.951\\ 0.956\\ \hline \text{Cable 13}\\ -2\\ 0.945\\ 0.953\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline \text{able 14:} \end{array}$	0.946 : Empirica $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirica	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Side	0.905 led Sym 0.950 0.955 0.954 0.956 0.949 ed Equa	0.907 metric $n^{-1}$ 0.955 0.956 0.954 0.956 0.951 d-tailed	0.948 Coverage $n^{-1/1.5}$ 0.972 0.976 0.972 0.971 0.973 Coverage	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Freques	0.952 acies for $n^{-1/3}$ 0.955 0.946 0.953 0.947 0.951 acies for	$0.948 \\ \epsilon_n = n^{-1/6} \\ 0.938 \\ 0.956 \\ 0.946 \\ 0.947 \\ 0.955 \\ \epsilon_n = n$	$\begin{array}{c} 0.947\\ ^{-1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944\\ 0.947\\ ^{-1} \text{ using } \end{array}$	$\begin{array}{c} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma}) \end{array}$
n = 10000 $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ Table and the second secon	$\begin{array}{c} 0.951\\ 0.956\\ \hline \text{Cable 13}\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline \text{Cable 14:}\\ -2\\ \end{array}$	0.946 : Empirica $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirica $-n^{-1/6}$	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$	0.905 led Sym 0.950 0.955 0.954 0.956 0.949 ed Equa 0	0.907 metric $n^{-1}$ 0.955 0.956 0.954 0.956 0.951 ll-tailed $n^{-1}$	$\begin{array}{c} 0.948 \\ \text{Coverage} \\ n^{-1/1.5} \\ 0.972 \\ 0.976 \\ 0.972 \\ 0.971 \\ 0.973 \\ \text{Coverage} \\ n^{-1/1.5} \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Freques $n^{-1/2}$	$\begin{array}{c} 0.952 \\ \text{acies for} \\ n^{-1/3} \\ 0.955 \\ 0.946 \\ 0.953 \\ 0.947 \\ 0.951 \\ \text{acies for} \\ n^{-1/3} \end{array}$	$0.948 \\ \epsilon_n = n^{-1/6} \\ 0.938 \\ 0.956 \\ 0.946 \\ 0.947 \\ 0.955 \\ \epsilon_n = n^{-1/6} $	$\begin{array}{c} 0.947\\ ^{-1}\text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944\\ 0.947\\ ^{-1}\text{ using }\\ n^{-1/10}\end{array}$	$\begin{array}{c} 0.957 \\ \mathbb{Z}_n = N(0, \hat{\sigma}) \\ 2 \\ 0.961 \\ 0.952 \\ 0.957 \\ 0.949 \\ 0.953 \\ \mathbb{Z}_n = N(0, \hat{\sigma}) \\ 2 \end{array}$
n = 10000 $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ Tai $\theta_n$ $n = 2000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline \\ \text{Table 13}\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline \\ \text{able 14:}\\ -2\\ 0.945\\ \end{array}$	0.946 : Empirica $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953	$\begin{array}{c} 0.905 \\ \text{led Sym} \\ 0 \\ 0.950 \\ 0.955 \\ 0.954 \\ 0.956 \\ 0.949 \\ \text{ed Equa} \\ 0 \\ 0.949 \end{array}$	0.907 metric $n^{-1}$ 0.955 0.956 0.954 0.956 0.951 d-tailed $n^{-1}$ 0.955	$\begin{array}{c} 0.948 \\ \text{Coverage} \\ n^{-1/1.5} \\ 0.972 \\ 0.976 \\ 0.972 \\ 0.971 \\ 0.973 \\ \text{Coverage} \\ n^{-1/1.5} \\ 0.961 \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Freques $n^{-1/2}$ 0.823	0.952 0.952 0.955 0.946 0.953 0.947 0.951 ncies for $n^{-1/3}$ 0.954	$0.948$ $\epsilon_n = n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $\epsilon_n = n^{-1/6}$ 0.939	$\begin{array}{c} 0.947\\ ^{-1}\text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944\\ 0.947\\ ^{-1}\text{ using }\\ n^{-1/10}\\ 0.948\end{array}$	$\begin{array}{c} 0.957 \\ \mathbb{Z}_n = N(0, \hat{\sigma}) \\ 2 \\ 0.961 \\ 0.952 \\ 0.957 \\ 0.949 \\ 0.953 \\ \mathbb{Z}_n = N(0, \hat{\sigma}) \\ 2 \\ 0.961 \end{array}$
$n = 10000$ $n = 10000$ $T$ $n = 2000$ $n = 4000$ $n = 8000$ $n = 10000$ $T$ $\theta_n$ $n = 2000$ $n = 4000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline 0.952\\ \hline 0.945\\ 0.952\\ \end{array}$	0.946 0.946 Empirica $-n^{-1/6}$ 0.953 0.953 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952 0.952 0.951	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Side $-n^{-1/3}$ 0.953 0.951	$\begin{array}{c} 0.905\\ \text{led Sym}\\ 0\\ 0.950\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ \text{ed Equa}\\ 0\\ 0.949\\ 0.953\\ \end{array}$	0.907 metric $n^{-1}$ 0.955 0.956 0.954 0.956 0.951 d-tailed $n^{-1}$ 0.955 0.954	$\begin{array}{c} 0.948 \\ \text{Coverage} \\ n^{-1/1.5} \\ 0.972 \\ 0.976 \\ 0.972 \\ 0.971 \\ 0.973 \\ \text{Coverage} \\ n^{-1/1.5} \\ 0.961 \\ 0.971 \\ \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Freque: $n^{-1/2}$ 0.823 0.819	0.952 0.952 0.955 0.946 0.953 0.947 0.951 ncies for $n^{-1/3}$ 0.954 0.946	$0.948$ $\epsilon_n = n^{-1/6}$ 0.938 0.956 0.946 0.947 0.955 $\epsilon_n = n^{-1/6}$ 0.939 0.956	$\begin{array}{c} 0.947\\ ^{-1}\text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944\\ 0.947\\ ^{-1}\text{ using }\\ n^{-1/10}\\ 0.948\\ 0.951\end{array}$	$\begin{array}{c} 0.957 \\ \mathbb{Z}_n = N(0, \hat{\sigma}) \\ 2 \\ 0.961 \\ 0.952 \\ 0.957 \\ 0.949 \\ 0.953 \\ \mathbb{Z}_n = N(0, \hat{\sigma}) \\ 2 \\ 0.961 \\ 0.951 \end{array}$
$n = 10000$ $n = 10000$ $T$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 10000$ $T$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline 0.948\\ 0.953\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline able 14:\\ -2\\ 0.945\\ 0.952\\ 0.951\\ \hline \end{array}$	0.946 0.946 Empirica $-n^{-1/6}$ 0.953 0.953 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952 0.951 0.953	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943	$\begin{array}{c} 0.905\\ \hline 0.905\\ 0.950\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ ed \ Equa\\ 0\\ 0.949\\ 0.953\\ 0.954\\ \end{array}$	$\begin{array}{c} 0.907\\ \text{imetric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{il-tailed}\\ n^{-1}\\ 0.955\\ 0.954\\ 0.954\\ 0.954\\ \end{array}$	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ \end{array}$	$\begin{array}{c} 0.952 \\ \text{Frequen} \\ n^{-1/2} \\ 0.831 \\ 0.827 \\ 0.846 \\ 0.833 \\ 0.822 \\ \text{e} \\ \text{Frequee} \\ n^{-1/2} \\ 0.823 \\ 0.819 \\ 0.839 \\ \end{array}$	$\begin{array}{c} 0.952 \\ \text{acies for} \\ n^{-1/3} \\ 0.955 \\ 0.946 \\ 0.953 \\ 0.947 \\ 0.951 \\ \text{acies for} \\ n^{-1/3} \\ 0.954 \\ 0.946 \\ 0.953 \end{array}$	$0.948$ $\epsilon_n = n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1/6}$ $0.939$ $0.956$ $0.946$	$\begin{array}{c} 0.947\\ ^{-1}\text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ ^{-1}\text{ using }\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ \end{array}$	$\begin{array}{c} 0.957 \\ \mathbb{Z}_n = N(0, \hat{\sigma}) \\ 2 \\ 0.961 \\ 0.952 \\ 0.957 \\ 0.949 \\ 0.953 \\ \mathbb{Z}_n = N(0, \hat{\sigma}) \\ 2 \\ 0.961 \\ 0.951 \\ 0.957 \end{array}$
$n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $Ta$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline 0.956\\ 0.943\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline able 14:\\ -2\\ 0.945\\ 0.952\\ 0.951\\ 0.943\\ \end{array}$	0.946 0.946 Empirica $-n^{-1/6}$ 0.953 0.953 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952 0.951 0.953 0.953 0.953 0.953	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.943 0.949	$\begin{array}{c} 0.905\\ \text{led Sym}\\ 0\\ 0.950\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ \text{ed Equa}\\ 0\\ 0.949\\ 0.953\\ 0.954\\ 0.956\\ \end{array}$	$\begin{array}{c} 0.907\\ \text{imetric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{il-tailed}\\ n^{-1}\\ 0.955\\ 0.954\\ 0.954\\ 0.955\\ \end{array}$	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Freques $n^{-1/2}$ 0.823 0.819 0.839 0.831	$\begin{array}{c} 0.952 \\ \text{acies for} \\ n^{-1/3} \\ 0.955 \\ 0.946 \\ 0.953 \\ 0.947 \\ 0.951 \\ \text{acies for} \\ n^{-1/3} \\ 0.954 \\ 0.946 \\ 0.953 \\ 0.947 \end{array}$	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ \end{array}$	$\begin{array}{c} 0.947\\ ^{-1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.947\\ ^{-1} \text{ using }\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.943\\ \end{array}$	$\begin{array}{c} 0.957\\ \\ \mathbb{Z}_n = N(0, \hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \\ \mathbb{Z}_n = N(0, \hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949 \end{array}$
n = 10000 $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline able 13\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline able 14:\\ -2\\ 0.945\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline \end{array}$	$\begin{array}{c} 0.935\\ 0.946\\ \vdots \ \mathrm{Empirica}\\ -n^{-1/6}\\ 0.953\\ 0.952\\ 0.953\\ 0.953\\ 0.951\\ \end{array}$ $\begin{array}{c} \mathrm{Empirica}\\ -n^{-1/6}\\ 0.952\\ 0.951\\ 0.953\\ 0.952\\ 0.951\\ \end{array}$	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.943 0.949 0.948	$\begin{array}{c} 0.905\\ \hline 0.905\\ 0.950\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ \hline 0.949\\ 0.953\\ 0.953\\ 0.954\\ 0.956\\ 0.949\\ \end{array}$	$\begin{array}{c} 0.907\\ \text{metric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.956\\ 0.951\\ \text{oleft}\\ 0.951\\ \text{oleft}\\ 1-\text{tailed}\\ n^{-1}\\ 0.955\\ 0.954\\ 0.955\\ 0.951\\ \end{array}$	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \end{array}$	$\begin{array}{c} 0.952 \\ \text{Frequen} \\ n^{-1/2} \\ 0.831 \\ 0.827 \\ 0.846 \\ 0.833 \\ 0.822 \\ \text{e} \text{Freques} \\ n^{-1/2} \\ 0.823 \\ 0.819 \\ 0.831 \\ 0.820 \\ \end{array}$	$\begin{array}{c} 0.952 \\ \text{acies for} \\ n^{-1/3} \\ 0.955 \\ 0.946 \\ 0.953 \\ 0.947 \\ 0.951 \\ \text{acies for} \\ n^{-1/3} \\ 0.954 \\ 0.954 \\ 0.953 \\ 0.947 \\ 0.951 \\ \end{array}$	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ \end{array}$	$\begin{array}{c} 0.947\\ -^1  \text{using 2}\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ \end{array}$	$\begin{array}{c} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.951\\ 0.957\\ 0.949\\ 0.953 \end{array}$
$n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 10000$ $r_{a}$ $\theta_{n}$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline able 13\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline able 14:\\ -2\\ 0.945\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline able 33\\ \hline$	0.946 0.946 Empirica $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952 0.951 0.953 0.952 0.951 0.953 0.952 0.951 15. Empirica	0.951 0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.943 0.949 0.948 biological antipole	$\begin{array}{c} 0.905\\ \text{led Sym}\\ 0\\ 0.950\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ \text{ed Equa}\\ 0\\ 0.949\\ 0.953\\ 0.954\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ \end{array}$	0.907 metric $n^{-1}$ 0.955 0.956 0.954 0.956 0.951 d-tailed $n^{-1}$ 0.955 0.954 0.955 0.954 0.955 0.951 d-955 0.951	0.948 Coverage $n^{-1/1.5}$ 0.972 0.976 0.972 0.971 0.973 Coverage $n^{-1/1.5}$ 0.961 0.965 0.967 0.971	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Freques $n^{-1/2}$ 0.823 0.819 0.839 0.831 0.820	0.952 0.952 0.955 0.946 0.953 0.947 0.951 ncies for $n^{-1/3}$ 0.954 0.954 0.953 0.947 0.951 0.951 0.951 0.951 0.951 0.951 0.955 0.946 0.953 0.955 0.955 0.955 0.946 0.955 0.946 0.955 0.947 0.955 0.955 0.947 0.955 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.955 0.947 0.955 0.955 0.947 0.955 0.955 0.947 0.955 0.955 0.947 0.955 0.955 0.955 0.947 0.955 0.955 0.955 0.947 0.955	$0.948$ $\epsilon_n = n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $m^{-1}$	$\begin{array}{c} 0.947\\ -^{1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ -^{1} \text{ using }\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.947\\ \end{array}$	$\begin{array}{c} 0.957\\ 0.957\\ \mathbb{Z}_n = N(0, \hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0, \hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ N(0, \hat{\sigma})\\ \end{array}$
n = 10000 $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline able 13\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline able 14:\\ -2\\ 0.945\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline able\\ 2\end{array}$	0.946 0.946 Empirica $-n^{-1/6}$ 0.953 0.953 0.953 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952 0.951 0.953 0.952 0.951 15: Empirica $n^{-1/6}$	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.943 0.949 0.948 cical upper	$\begin{array}{c} 0.905\\ 0.905\\ 0.950\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ 0.949\\ 0.953\\ 0.954\\ 0.953\\ 0.954\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ \end{array}$	0.907 metric $n^{-1}$ 0.955 0.956 0.954 0.956 0.951 d-tailed $n^{-1}$ 0.955 0.954 0.954 0.955 0.954 0.955 0.951 ided Cc	0.948 Coverage $n^{-1/1.5}$ 0.972 0.976 0.972 0.971 0.973 Coverage $n^{-1/1.5}$ 0.961 0.971 0.965 0.967 0.971 verage Fr $n^{-1/1.5}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Freques $n^{-1/2}$ 0.823 0.819 0.839 0.831 0.820 equencies $n^{-1/2}$	0.952 0.952 0.955 0.946 0.953 0.947 0.951 ncies for $n^{-1/3}$ 0.954 0.954 0.946 0.953 0.947 0.951 es for $\epsilon_n$ $n^{-1/3}$	$0.948$ $\epsilon_n = n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1}$ $n^{-1/6}$	$\begin{array}{c} 0.947\\ -^{1} using \\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944\\ 0.947\\ -^{1} using \\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.947\\ using \\ \mathbb{Z}_{n}\\ n^{-1/10}\\ \end{array}$	$\begin{array}{c} 0.957\\ 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\end{array}$
$n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $Ta$ $\theta_{n}$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $\theta_{n}$ $m = 2000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline able 13\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline able 14:\\ -2\\ 0.945\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\end{array}$	0.946 0.946 Empirica $-n^{-1/6}$ 0.953 0.953 0.953 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952 0.951 0.953 0.952 0.951 15: Empir $-n^{-1/6}$ 0.952	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.942	$\begin{array}{c} 0.905\\ \hline 0.905\\ 0.950\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ \hline 0.949\\ 0.953\\ 0.953\\ 0.954\\ 0.956\\ 0.949\\ \hline 0.956\\ 0.949\\ \hline r \ One-S\\ 0\\ 0.900\\ \hline \end{array}$	0.907 metric $n^{-1}$ 0.955 0.956 0.954 0.956 0.951 d-tailed $n^{-1}$ 0.955 0.954 0.954 0.955 0.951 ided Ccc $n^{-1}$ 0.900	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \text{werage Fr}\\ n^{-1/1.5}\\ 0.982\\ \end{array}$	0.952 Frequence $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Frequest $n^{-1/2}$ 0.823 0.819 0.839 0.831 0.820 equencie $n^{-1/2}$ 0.823	0.952 0.952 0.955 0.946 0.953 0.947 0.951 ncies for $n^{-1/3}$ 0.954 0.946 0.953 0.947 0.951 0.951 0.951 0.951 0.951	$0.948$ $\epsilon_n = n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1}$ $n^{-1/6}$ $0.940$	$\begin{array}{c} 0.947\\ -^{1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ -^{1} \text{ using }\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.947\\ \\ \text{using } \mathbb{Z}_{n}\\ n^{-1/10}\\ 0.927\end{array}$	$0.957$ $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $= N(0, \hat{\sigma})$ 2 0.961
$n = 10000$ $n = 10000$ $n = 10000$ $n = 2000$ $n = 6000$ $n = 10000$ $n = 4000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $m = 2000$ $m = 4000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline able 13\\ -2\\ 0.945\\ 0.953\\ 0.953\\ 0.952\\ \hline able 14:\\ -2\\ 0.945\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\\ 0.950\\ \end{array}$	0.946 0.946 Empirica $-n^{-1/6}$ 0.953 0.953 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952 0.951 0.953 0.952 0.951 15: Empiri $-n^{-1/6}$ 0.953 0.953 0.953	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.942 0.955	$\begin{array}{c} 0.905\\ 0.905\\ 0.955\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ 0.949\\ 0.953\\ 0.954\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ r \text{ One-S}\\ 0\\ 0.999\\ 0.900\\ 0.900\\ \end{array}$	0.907 metric $n^{-1}$ 0.955 0.956 0.954 0.956 0.951 dl-tailed $n^{-1}$ 0.955 0.954 0.955 0.954 0.955 0.951 ided Co $n^{-1}$ 0.999 0.908	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \text{verage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.906\\ \end{array}$	$\begin{array}{c} 0.952 \\ \text{Frequen} \\ n^{-1/2} \\ 0.831 \\ 0.827 \\ 0.846 \\ 0.833 \\ 0.822 \\ \text{e} \\ \text{Freque:} \\ n^{-1/2} \\ 0.823 \\ 0.819 \\ 0.839 \\ 0.831 \\ 0.820 \\ \text{equencie} \\ n^{-1/2} \\ 0.832 \\ 0.841 \\ \end{array}$	0.952 0.952 0.955 0.946 0.953 0.947 0.951 ncies for $n^{-1/3}$ 0.954 0.953 0.947 0.954 0.953 0.947 0.951 es for $\epsilon_n$ $n^{-1/3}$ 0.951 0.951	$0.948$ $\epsilon_n = n^{-1/6}$ $0.938$ $0.956$ $0.946$ $0.947$ $0.955$ $\epsilon_n = n^{-1/6}$ $0.939$ $0.956$ $0.946$ $0.948$ $0.955$ $= n^{-1}$ $n^{-1/6}$ $0.940$ $0.952$	$\begin{array}{c} 0.947\\ -^{1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ \end{array}$ $\begin{array}{c} -^{1} \text{ using }\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.943\\ 0.947\\ \end{array}$ $\begin{array}{c} \text{using } \mathbb{Z}_{n}\\ n^{-1/10}\\ 0.937\\ 0.945\\ \end{array}$	$\begin{array}{c} 0.957\\ \hline \\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \hline \\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ \hline \\ = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.940\\ \end{array}$
$n = 10000$ $n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 10000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 4000$ $n = 6000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline 0.956\\ \hline 0.956\\ 0.943\\ 0.950\\ 0.943\\ 0.952\\ \hline 0.952\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\\ 0.950\\ 0.956\\ \hline \end{array}$	0.946 0.946 Empirica $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952 0.951 0.953 0.952 0.951 15: Empir $-n^{-1/6}$ 0.953 0.953 0.953 0.953 0.955	0.951 0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Side $-n^{-1/3}$ 0.953 0.951 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.942 0.955 0.954	$\begin{array}{c} 0.905\\ \hline 0.905\\ 0.955\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ \hline 0.949\\ 0.953\\ 0.954\\ 0.953\\ 0.954\\ 0.956\\ 0.949\\ \hline 0.956\\ 0.949\\ \hline 0.999\\ 0.999\\ 0.999\\ 1.000\\ \end{array}$	$\begin{array}{c} 0.907\\ \text{metric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{dl-tailed}\\ n^{-1}\\ 0.955\\ 0.954\\ 0.955\\ 0.951\\ \text{ided Cc}\\ n^{-1}\\ 0.999\\ 0.998\\ 0.908\\ \end{array}$	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \text{overage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.002\\ \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Freques $n^{-1/2}$ 0.823 0.819 0.839 0.831 0.820 equencies $n^{-1/2}$ 0.832 0.841 0.844	0.952 0.952 0.955 0.946 0.953 0.947 0.951 ncies for $n^{-1/3}$ 0.954 0.953 0.947 0.954 0.953 0.947 0.951 0.951 0.951 0.951	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \hline\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ \end{array}$	$\begin{array}{c} 0.947\\ -^{1} \text{ using } \\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ \end{array}$ $\begin{array}{c} -^{1} \text{ using } \\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.947\\ \end{array}$ $\begin{array}{c} \text{using } \\ \mathbb{Z}_{n}\\ n^{-1/10}\\ 0.937\\ 0.945\\ 0.945\\ 0.945\\ \end{array}$	$0.957$ $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.952 0.957 0.949 0.953 $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ 2 0.961 0.951 0.957 0.949 0.953 $= N(0, \hat{\sigma})$ 2 0.961 0.949 0.949
$n = 10000$ $n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 4000$ $n = 6000$ $n = 6000$ $n = 8000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline 0.956\\ \hline 0.943\\ 0.950\\ 0.943\\ 0.952\\ \hline 0.952\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\\ 0.950\\ 0.956\\ 0.944\\ \end{array}$	$\begin{array}{c} 0.946\\ 0.946\\ \vdots \ \mathrm{Empirica}\\ -n^{-1/6}\\ 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.951\\ \end{array}$ $\begin{array}{c} \mathrm{Empirica}\\ -n^{-1/6}\\ 0.952\\ 0.951\\ 0.953\\ 0.952\\ 0.951\\ \end{array}$ $\begin{array}{c} 15: \ \mathrm{Empiri}\\ -n^{-1/6}\\ 0.953\\ 0.949\\ 0.955\\ 0.950\\ \end{array}$	0.951 0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.942 0.955 0.954 0.950	$\begin{array}{c} 0.905\\ 0.905\\ 0.955\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ 0.953\\ 0.953\\ 0.954\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ r \text{ One-S}\\ 0\\ 0.999\\ 0.999\\ 1.000\\ 1.000\\ 1.000\\ 0.909\end{array}$	$\begin{array}{c} 0.907\\ \text{imetric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{il-tailed}\\ n^{-1}\\ 0.955\\ 0.954\\ 0.955\\ 0.951\\ \text{ided Cc}\\ n^{-1}\\ 0.999\\ 0.998\\ 0.998\\ 1.000\\ \end{array}$	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \text{verage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.993\\ 0.007\\ \end{array}$	0.952 Frequent $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Frequest $n^{-1/2}$ 0.823 0.819 0.839 0.831 0.820 equencies $n^{-1/2}$ 0.832 0.841 0.844 0.844 0.846	0.952 0.952 0.955 0.946 0.953 0.947 0.951 0.954 0.954 0.953 0.947 0.954 0.953 0.947 0.951 0.951 0.951 0.951 0.945	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ 0.952\\ 0.952\end{array}$	$\begin{array}{c} 0.947\\ -^{1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ -^{1} \text{ using }\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.943\\ 0.947\\ \text{using } \mathbb{Z}_{n}\\ n^{-1/10}\\ 0.937\\ 0.945\\ 0.945\\ 0.952\\ \end{array}$	$0.957$ $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ $2$ $0.961$ $0.952$ $0.957$ $0.949$ $0.953$ $\mathbb{Z}_{n} = N(0, \hat{\sigma})$ $2$ $0.961$ $0.953$ $= N(0, \hat{\sigma})$ $2$ $0.961$ $0.949$ $0.949$ $0.949$ $0.949$
$n = 10000$ $n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 8000$ $n = 10000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 8000$ $n = 10000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline able 13\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline able 14:\\ -2\\ 0.945\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\\ 0.950\\ 0.956\\ 0.944\\ 0.953\\ \end{array}$	$\begin{array}{c} 0.946\\ 0.946\\ \vdots \ \mathrm{Empirica}\\ -n^{-1/6}\\ 0.953\\ 0.952\\ 0.953\\ 0.953\\ 0.951\\ \end{array}$ $\begin{array}{c} \mathrm{Empirica}\\ -n^{-1/6}\\ 0.952\\ 0.951\\ 0.953\\ 0.952\\ 0.951\\ \end{array}$ $\begin{array}{c} 15: \ \mathrm{Empiri}\\ -n^{-1/6}\\ 0.953\\ 0.953\\ 0.949\\ 0.955\\ 0.950\\ 0.956\\ \end{array}$	0.951 0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.943 0.948 cical upper $-n^{-1/3}$ 0.942 0.955 0.954 0.950 0.945	$\begin{array}{c} 0.905\\ 0.905\\ 0.955\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ 0.953\\ 0.953\\ 0.954\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ 0.999\\ 1.000\\ 1.000\\ 1.000\\ 1.000\\ 0.909\end{array}$	$\begin{array}{c} 0.907\\ \text{imetric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{il-tailed}\\ n^{-1}\\ 0.955\\ 0.954\\ 0.955\\ 0.954\\ 0.955\\ 0.951\\ \text{ided Cc}\\ n^{-1}\\ 0.999\\ 0.998\\ 1.000\\ 1.000\\ 1.000\\ \end{array}$	$\begin{array}{c} 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \text{everage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.993\\ 0.997\\ 0.908\\ \end{array}$	0.952 Frequen $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Freques $n^{-1/2}$ 0.823 0.819 0.839 0.831 0.820 equencia $n^{-1/2}$ 0.832 0.841 0.844 0.846 0.835	0.952 0.952 0.955 0.946 0.953 0.947 0.951 0.954 0.954 0.953 0.947 0.954 0.953 0.947 0.951 0.951 0.951 0.951 0.945 0.948	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ 0.9$	$\begin{array}{c} 0.947\\ -^{1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944\\ 0.947\\ -^{1} \text{ using }\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.943\\ 0.947\\ \end{array}$ $\begin{array}{c} \text{using } \mathbb{Z}_{n}\\ n^{-1/10}\\ 0.937\\ 0.945\\ 0.945\\ 0.952\\ 0.953\\ \end{array}$	$\begin{array}{c} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.949\\ 0.949\\ 0.943\\ 0.953\end{array}$
$n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 4000$ $n = 4000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline able 13\\ -2\\ 0.945\\ 0.953\\ 0.950\\ 0.943\\ 0.952\\ \hline 0.943\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\\ 0.950\\ 0.956\\ 0.944\\ 0.953\\ \end{array}$	$\begin{array}{c} 0.946\\ 0.946\\ \vdots \ \mathrm{Empirica}\\ -n^{-1/6}\\ 0.953\\ 0.952\\ 0.953\\ 0.953\\ 0.951\\ \end{array}$ $\begin{array}{c} \mathrm{Empirica}\\ -n^{-1/6}\\ 0.952\\ 0.951\\ 0.953\\ 0.952\\ 0.951\\ \end{array}$ $\begin{array}{c} 15: \ \mathrm{Empir}\\ -n^{-1/6}\\ 0.953\\ 0.953\\ 0.949\\ 0.955\\ 0.950\\ 0.956\\ \end{array}$	0.951 0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.942 0.955 0.954 0.950 0.945	$\begin{array}{c} 0.905\\ 0.905\\ 0.955\\ 0.955\\ 0.954\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ 0.953\\ 0.954\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ 0.956\\ 0.949\\ 0.999\\ 0.999\\ 1.000\\ 1.000\\ 1.000\\ 0.$	$\begin{array}{c} 0.907\\ \text{metric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{d} \\ \text{d} \\ n^{-1}\\ 0.955\\ 0.954\\ 0.955\\ 0.951\\ \text{ided Cc}\\ n^{-1}\\ 0.999\\ 0.998\\ 0.998\\ 1.000\\ 1.000\\ \end{array}$	$\begin{array}{c} 0.948\\ 0.948\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \text{verage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.993\\ 0.997\\ 0.998\\ \end{array}$	$\begin{array}{c} 0.952 \\ \text{Frequen} \\ n^{-1/2} \\ 0.831 \\ 0.827 \\ 0.846 \\ 0.833 \\ 0.822 \\ \text{e} \text{Frequee} \\ n^{-1/2} \\ 0.823 \\ 0.819 \\ 0.839 \\ 0.831 \\ 0.820 \\ \text{equencie} \\ n^{-1/2} \\ 0.832 \\ 0.831 \\ 0.820 \\ \text{equencie} \\ n^{-1/2} \\ 0.832 \\ 0.841 \\ 0.844 \\ 0.846 \\ 0.835 \\ \end{array}$	0.952 0.952 0.955 0.946 0.953 0.947 0.951 0.951 0.954 0.953 0.947 0.954 0.953 0.947 0.951 0.951 0.951 0.951 0.951 0.951 0.948	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ 0.952\\ 0.952\\ 0.952\\ \end{array}$	$\begin{array}{c} 0.947\\ 0.947\\ ^{-1} using 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.944\\ 0.947\\ ^{-1} using\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.943\\ 0.947\\ using \mathbb{Z}_n\\ n^{-1/10}\\ 0.937\\ 0.945\\ 0.945\\ 0.952\\ 0.953\\ \end{array}$	$\begin{array}{l} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.949\\ 0.943\\ 0.953\\ \end{array}$
$n = 10000$ $n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 10000$ $n = 4000$ $n = 4000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$	0.956 0.956 -2 0.945 0.953 0.950 0.943 0.952 0.952 0.952 0.952 0.952 0.952 0.953 0.953 Table -2 0.943 0.953 Table -2 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.948 0.950 0.951 0.952 0.943 0.953 Table	0.946 0.946 : Empirica $-n^{-1/6}$ 0.953 0.952 0.953 0.953 0.951 Empirica $-n^{-1/6}$ 0.952 0.951 0.952 0.951 15: Empirica $-n^{-1/6}$ 0.952 0.951 15: Empirica $-n^{-1/6}$ 0.953 0.952 0.951 15: Empirica 0.953 0.955 0.956 16: Empiri	0.951 0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.942 0.955 0.954 0.955 0.954 0.955 1.955 0.945 rical lower	0.905 led Sym 0 0.950 0.955 0.954 0.956 0.949 ed Equa 0 0.949 0.953 0.954 0.956 0.949 0.956 0.949 r One-S 0 0.999 1.000 1.000 1.000	0.907 metric $n^{-1}$ 0.955 0.956 0.954 0.956 0.951 d-tailed $n^{-1}$ 0.955 0.954 0.954 0.955 0.954 0.955 0.951 ided Co $n^{-1}$ 0.999 0.998 0.998 1.000 1.000 ded Co	$\begin{array}{c} 0.948\\ 0.948\\ \hline \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \hline \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \hline \text{verage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.993\\ 0.997\\ 0.998\\ \hline \text{verage Free}\\ p^{-1}(1.5)\\ $	0.952 Frequence $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Frequese $n^{-1/2}$ 0.823 0.819 0.839 0.831 0.820 equencie $n^{-1/2}$ 0.832 0.841 0.844 0.846 0.835 equencie equencie eque	0.952 0.952 0.955 0.946 0.953 0.947 0.951 0.954 0.954 0.953 0.947 0.954 0.953 0.947 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.955 0.946 0.953 0.947 0.955 0.946 0.953 0.947 0.955 0.946 0.953 0.947 0.955 0.946 0.953 0.947 0.955 0.946 0.953 0.947 0.955 0.946 0.953 0.947 0.955 0.946 0.953 0.955 0.946 0.953 0.955 0.946 0.953 0.955 0.947 0.955 0.946 0.953 0.955 0.947 0.955 0.946 0.953 0.947 0.955 0.947 0.955 0.946 0.953 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.947 0.955 0.945 0.945 0.955 0.945 0.945 0.955 0.945 0.945 0.955 0.945 0.945 0.955 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.9488 0.9488 0.9488 0.9488 0.9488 0.9488 0.9488 0.9488 0.9488	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ 0.952\\ 0.952\\ 0.952\\ = n^{-1}\\ n^{-1} \end{array}$	0.947 $^{-1}$ using 2 $n^{-1/10}$ 0.949 0.953 0.943 0.944 0.947 $^{-1}$ using $n^{-1/10}$ 0.948 0.951 0.943 0.943 0.943 0.947 using $\mathbb{Z}_n$ $n^{-1/10}$ 0.937 0.945 0.945 0.952 0.953 using $\mathbb{Z}_n$	$\begin{array}{l} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.949\\ 0.949\\ 0.949\\ 0.943\\ 0.953\\ = N(0,\hat{\sigma})\end{array}$
$n = 10000$ $n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 10000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $\theta_n$	0.956 0.956 -2 0.945 0.953 0.950 0.943 0.952 0.952 0.952 0.952 0.952 0.953 0.953 Table -2 0.943 0.953 Table -2 0.948 0.950 0.944 0.953 Table -2 0.944 0.953 Table -2 0.943 0.953 Table -2 0.943 0.953 Table -2 0.943 0.953 Table -2 0.943 0.953 Table -2 0.943 0.953 Table -2 0.943 0.953 Table -2 0.943 0.953 Table -2 0.943 0.953 Table -2 0.943 0.955 0.955 0.956 0.944 0.953 Table -2 0.943 0.956 0.944 0.953 Table -2 0.943 0.953 Table -2 0.944 0.953 Table -2 0.944 0.953 Table -2	$\begin{array}{l} 0.946\\ \hline 0.946\\ \hline 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.951\\ \hline \\ \text{Empirical}\\ -n^{-1/6}\\ 0.952\\ 0.951\\ 0.952\\ 0.951\\ 15: \text{ Empir}\\ -n^{-1/6}\\ 0.953\\ 0.949\\ 0.955\\ 0.950\\ 0.956\\ 16: \text{ Empir}\\ -n^{-1/6}\\ \end{array}$	0.951 0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Side $-n^{-1/3}$ 0.953 0.951 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.942 0.955 0.954 0.955 0.945 rical lower $-n^{-1/3}$	0.905 led Sym 0.950 0.955 0.954 0.956 0.949 ed Equa 0 0.949 0.953 0.953 0.954 0.956 0.949 r One-S 0 0.999 0.999 1.000 1.000 1.000	$\begin{array}{c} 0.907\\ \text{imetric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{il-tailed}\\ n^{-1}\\ 0.955\\ 0.954\\ 0.955\\ 0.951\\ \text{ided Cc}\\ n^{-1}\\ 0.999\\ 0.998\\ 1.000\\ 1.000\\ \text{ided Co}\\ n^{-1}\\ \end{array}$	$\begin{array}{c} 0.948\\ 0.948\\ \hline\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \hline\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \hline\\ \text{overage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.993\\ 0.997\\ 0.998\\ \hline\\ \text{verage Fre}\\ n^{-1/1.5}\\ \hline\end{array}$	0.952 Frequence $n^{-1/2}$ 0.831 0.827 0.846 0.833 0.822 e Frequesite $n^{-1/2}$ 0.823 0.819 0.839 0.831 0.820 equencie $n^{-1/2}$ 0.844 0.846 0.835 equencie $n^{-1/2}$	0.952 0.952 0.955 0.946 0.953 0.947 0.951 0.954 0.954 0.953 0.947 0.954 0.954 0.953 0.947 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.955 0.946 0.953 0.947 0.953 0.947 0.954 0.953 0.947 0.954 0.953 0.947 0.955 0.946 0.953 0.947 0.954 0.953 0.947 0.955 0.946 0.953 0.947 0.954 0.953 0.947 0.955 0.946 0.953 0.954 0.955 0.946 0.953 0.955 0.946 0.953 0.955 0.946 0.953 0.954 0.955 0.947 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.955 0.945 0.945 0.951 0.951 0.955 0.945 0.945 0.951 0.951 0.955 0.945 0.945 0.945 0.945 0.945 0.945 0.951 0.951 0.951 0.945 0	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.948\\ 0.955\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ 0.952\\ 0.952\\ 0.952\\ = n^{-1}\\ n^{-1/6}\\ \end{array}$	$\begin{array}{c} 0.947\\ -^{1} \text{ using } \\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ -^{1} \text{ using } \\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.943\\ 0.943\\ 0.947\\ 0.943\\ 0.945\\ 0.945\\ 0.945\\ 0.945\\ 0.952\\ 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.952\\ 0.953$	$\begin{array}{l} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.949\\ 0.943\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.9(1,\hat{\sigma})\\ 0.9(1,\hat{\sigma})\\$
$n = 10000$ $n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 10000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline 0.956\\ \hline 0.956\\ 0.943\\ 0.950\\ 0.943\\ 0.952\\ \hline 0.952\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\\ 0.950\\ 0.956\\ 0.944\\ 0.953\\ \hline Table\\ -2\\ 0.945\\ \hline \end{array}$	$\begin{array}{l} 0.946\\ \hline 0.946\\ \hline 0.946\\ \hline 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.951\\ \hline \\ \text{Empirical}\\ -n^{-1/6}\\ 0.952\\ 0.951\\ 0.953\\ 0.952\\ 0.951\\ \hline \\ 15: \text{ Empirical}\\ -n^{-1/6}\\ 0.953\\ 0.949\\ 0.955\\ 0.956\\ \hline \\ 16: \text{ Empirical}\\ -n^{-1/6}\\ 0.949\\ \hline \end{array}$	0.951 0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.955 0.954 0.955 0.954 0.955 0.954 0.955 0.955 0.954 0.955 0.952	0.905 led Sym 0 0.950 0.955 0.954 0.956 0.949 ed Equa 0 0.949 0.953 0.954 0.956 0.949 r One-S 0 0.999 1.000 1.000 1.000 0.903	$\begin{array}{c} 0.907\\ \text{metric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{dl-tailed}\\ n^{-1}\\ 0.955\\ 0.954\\ 0.955\\ 0.951\\ \text{ided Co}\\ n^{-1}\\ 0.999\\ 0.998\\ 1.000\\ 1.000\\ \text{ded Co}\\ n^{-1}\\ 0.909 \end{array}$	$\begin{array}{c} 0.948\\ 0.948\\ \hline\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \hline\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \hline\\ \text{werage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.993\\ 0.997\\ 0.998\\ \hline\\ \text{werage Fre}\\ n^{-1/1.5}\\ 0.946\\ \hline\end{array}$	$\begin{array}{c} 0.952 \\ \text{Frequen} \\ n^{-1/2} \\ 0.831 \\ 0.827 \\ 0.846 \\ 0.833 \\ 0.822 \\ \text{e} \text{Frequee} \\ n^{-1/2} \\ 0.823 \\ 0.819 \\ 0.839 \\ 0.831 \\ 0.820 \\ \text{equencie} \\ n^{-1/2} \\ 0.832 \\ 0.841 \\ 0.846 \\ 0.835 \\ \text{equencie} \\ n^{-1/2} \\ 0.961 \\ \end{array}$	$\begin{array}{c} 0.952 \\ \text{acies for } n^{-1/3} \\ 0.955 \\ 0.946 \\ 0.953 \\ 0.947 \\ 0.951 \\ \text{acies for } n^{-1/3} \\ 0.954 \\ 0.953 \\ 0.947 \\ 0.951 \\ 0.951 \\ 0.951 \\ 0.951 \\ 0.951 \\ 0.948 \\ \text{s for } \epsilon_n \\ n^{-1/3} \\ 0.950 \\ \end{array}$	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ 0.952\\ 0.952\\ 0.952\\ = n^{-1}\\ n^{-1/6}\\ 0.947\\ \end{array}$	$\begin{array}{c} 0.947\\ -^{1} \text{ using } 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ -^{1} \text{ using }\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.943\\ 0.943\\ 0.943\\ 0.945\\ 0.945\\ 0.945\\ 0.945\\ 0.952\\ 0.953\\ \text{using } \mathbb{Z}_{n}\\ n^{-1/10}\\ 0.959\\ \end{array}$	$\begin{array}{c} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.949\\ 0.943\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.956\end{array}$
$n = 10000$ $n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 8000$ $n = 10000$ $n = 4000$ $n = 4000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline 0.956\\ \hline 0.956\\ 0.943\\ 0.950\\ 0.943\\ 0.952\\ \hline 0.943\\ 0.952\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\\ 0.950\\ 0.956\\ 0.944\\ 0.953\\ \hline Table\\ -2\\ 0.945\\ 0.945\\ 0.946\\ \end{array}$	$\begin{array}{c} 0.946\\ 0.946\\ \vdots \ \mathrm{Empirica}\\ -n^{-1/6}\\ 0.953\\ 0.952\\ 0.953\\ 0.953\\ 0.951\\ \end{array}$ $\begin{array}{c} \mathrm{Empirica}\\ -n^{-1/6}\\ 0.952\\ 0.951\\ 0.953\\ 0.952\\ 0.951\\ \end{array}$ $\begin{array}{c} 15: \ \mathrm{Empir}\\ -n^{-1/6}\\ 0.953\\ 0.949\\ 0.955\\ 0.956\\ \end{array}$ $\begin{array}{c} 16: \ \mathrm{Empir}\\ -n^{-1/6}\\ 0.949\\ 0.952\\ \end{array}$	0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.942 0.955 0.954 0.955 0.954 0.950 0.945 rical lower $-n^{-1/3}$ 0.952 0.952	0.905 led Sym 0 0.950 0.955 0.954 0.956 0.949 ed Equa 0 0.949 0.953 0.954 0.954 0.956 0.949 r One-S 0 0.999 1.000 1.000 1.000 1.000 1.000	$\begin{array}{c} 0.907\\ \text{imetric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{il-tailed}\\ n^{-1}\\ 0.955\\ 0.954\\ 0.955\\ 0.951\\ \text{ided Co}\\ n^{-1}\\ 0.999\\ 0.998\\ 1.000\\ 1.000\\ \text{ided Co}\\ n^{-1}\\ 0.909\\ 0.905\\ \end{array}$	$\begin{array}{c} 0.948\\ 0.948\\ \hline\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \hline\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \hline\\ \text{verage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.993\\ 0.997\\ 0.998\\ \hline\\ \text{verage Fre}\\ n^{-1/1.5}\\ 0.946\\ 0.946\\ \hline\end{array}$	$\begin{array}{c} 0.952\\ \hline 0.952\\ \hline requent \\ n^{-1/2}\\ 0.831\\ 0.827\\ 0.846\\ 0.833\\ 0.822\\ \hline 0.823\\ 0.823\\ 0.823\\ 0.831\\ 0.820\\ \hline equencie \\ n^{-1/2}\\ 0.832\\ 0.841\\ 0.844\\ 0.846\\ 0.835\\ \hline equencie \\ n^{-1/2}\\ 0.961\\ 0.949\\ \hline \end{array}$	0.952 0.952 0.955 0.946 0.955 0.946 0.953 0.947 0.951 0.954 0.954 0.953 0.947 0.951 0.951 0.951 0.951 0.951 0.951 0.945 0.948 s for $\epsilon_n$ $n^{-1/3}$ 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.955 0.946 0.953 0.947 0.954 0.955 0.946 0.953 0.947 0.954 0.955 0.946 0.953 0.954 0.955 0.946 0.953 0.954 0.955 0.946 0.953 0.955 0.946 0.953 0.954 0.955 0.947 0.951 0.954 0.955 0.947 0.951 0.954 0.955 0.947 0.951 0.955 0.947 0.951 0.954 0.955 0.947 0.951 0.951 0.954 0.955 0.947 0.951 0.955 0.947 0.951 0.955 0.955 0.955 0.955 0.955 0.955 0.955 0.955 0.955 0.955 0.955 0.945 0.955 0.945 0.955 0.945 0.955 0.945 0.945 0.955 0.945 0.955 0.945 0.945 0.955 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.945 0.946 0.955 0.945 0.945 0.955 0.946 0.955 0.945 0.955 0.955 0.946 0.955 0.945 0.955 0.946 0.955 0.956 0.957 0.956 0.95	$\begin{array}{c} 0.948\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ 0.952\\ 0.952\\ = n^{-1}\\ n^{-1/6}\\ 0.947\\ 0.952\\ \end{array}$	$\begin{array}{c} 0.947\\ -^{1} using \\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ -^{1} using \\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.943\\ 0.947\\ using \\ \mathbb{Z}_{n}\\ n^{-1/10}\\ 0.937\\ 0.945\\ 0.945\\ 0.952\\ 0.953\\ using \\ \mathbb{Z}_{n}\\ n^{-1/10}\\ 0.959\\ 0.950\\ \end{array}$	$\begin{array}{c} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.949\\ 0.949\\ 0.943\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.956\\ 0.955\\ \end{array}$
$n = 10000$ $n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 8000$ $n = 10000$ $n = 4000$ $n = 4000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 8000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 4000$ $n = 6000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline 0.956\\ \hline 0.956\\ 0.943\\ 0.950\\ 0.943\\ 0.952\\ \hline 0.943\\ 0.952\\ 0.952\\ 0.951\\ 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\\ 0.950\\ 0.956\\ 0.944\\ 0.953\\ \hline Table\\ -2\\ 0.945\\ 0.945\\ 0.946\\ 0.944\\ \end{array}$	$\begin{array}{c} 0.946\\ 0.946\\ \vdots \ \mathrm{Empirica}\\ -n^{-1/6}\\ 0.953\\ 0.952\\ 0.953\\ 0.953\\ 0.951\\ \end{array}$ $\begin{array}{c} \mathrm{Empirica}\\ -n^{-1/6}\\ 0.952\\ 0.951\\ 0.952\\ 0.951\\ \end{array}$ $\begin{array}{c} 15: \ \mathrm{Empir}\\ -n^{-1/6}\\ 0.953\\ 0.949\\ 0.955\\ 0.950\\ 0.956\\ \end{array}$ $\begin{array}{c} 16: \ \mathrm{Empir}\\ -n^{-1/6}\\ 0.949\\ 0.952\\ 0.947\\ \end{array}$	$\begin{array}{c} 0.951\\ 0.951\\ al\ Two-Sid\\ -n^{-1/3}\\ 0.952\\ 0.952\\ 0.943\\ 0.950\\ 0.949\\ l\ Two-Sid\\ -n^{-1/3}\\ 0.953\\ 0.951\\ 0.943\\ 0.949\\ 0.948\\ cical\ upper\\ -n^{-1/3}\\ 0.942\\ 0.955\\ 0.954\\ 0.955\\ 0.954\\ 0.950\\ 0.945\\ rical\ lower\\ -n^{-1/3}\\ 0.952\\ 0.952\\ 0.939\\ \end{array}$	0.905 led Sym 0 0.950 0.955 0.954 0.956 0.949 ed Equa 0 0.949 0.953 0.954 0.956 0.949 0.956 0.999 0.999 0.903 0.905	$\begin{array}{c} 0.907\\ \text{metric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{d}.955\\ 0.951\\ \text{d}.955\\ 0.954\\ 0.955\\ 0.951\\ \text{ided Cc}\\ n^{-1}\\ 0.999\\ 0.998\\ 1.000\\ 1.000\\ \text{ded Co}\\ n^{-1}\\ 0.909\\ 0.905\\ 0.908\\ \end{array}$	$\begin{array}{c} 0.948\\ 0.948\\ \hline\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \hline\\ \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \hline\\ \text{verage Fr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.993\\ 0.997\\ 0.998\\ \hline\\ \text{verage Fre}\\ n^{-1/1.5}\\ 0.946\\ 0.946\\ 0.939\\ \hline\end{array}$	$\begin{array}{c} 0.952\\ \hline 0.952\\ \hline requent \\ n^{-1/2}\\ 0.831\\ 0.827\\ 0.846\\ 0.833\\ 0.822\\ \hline e \ Freques \\ n^{-1/2}\\ 0.823\\ 0.819\\ 0.839\\ 0.831\\ 0.820\\ \hline equencie \\ n^{-1/2}\\ 0.832\\ 0.841\\ 0.844\\ 0.835\\ \hline equencie \\ n^{-1/2}\\ 0.961\\ 0.949\\ 0.953\\ \hline \end{array}$	0.952 0.952 0.955 0.946 0.955 0.946 0.953 0.947 0.951 0.954 0.954 0.953 0.947 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.945 0.948 s for $\epsilon_n$ $n^{-1/3}$ 0.950 0.946 0.955	$\begin{array}{c} 0.948\\ \hline\\ \epsilon_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \hline\\ \epsilon_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ \hline\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ 0.952\\ 0.952\\ \hline\\ = n^{-1}\\ n^{-1/6}\\ 0.947\\ 0.952\\ 0.942\\ \end{array}$	$\begin{array}{c} 0.947\\ -^{1} using \\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ -^{1} using \\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.947\\ using \\ \\ n^{-1/10}\\ 0.937\\ 0.945\\ 0.945\\ 0.952\\ 0.953\\ using \\ \\ n^{-1/10}\\ 0.959\\ 0.950\\ 0$	$\begin{array}{c} 0.957\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.949\\ 0.943\\ 0.953\\ = N(0,\hat{\sigma})\\ 2\\ 0.956\\ 0.955\\ 0.950\\ \end{array}$
$n = 10000$ $n = 10000$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ $n = 4000$ $n = 6000$ $n = 10000$ $\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 4000$ $n = 8000$ $n = 4000$ $n = 8000$	$\begin{array}{c} 0.951\\ 0.956\\ \hline 0.956\\ \hline 0.956\\ \hline 0.956\\ 0.943\\ 0.953\\ 0.952\\ \hline 0.952\\ 0.952\\ 0.952\\ 0.952\\ 0.952\\ 0.953\\ \hline 0.943\\ 0.953\\ \hline Table\\ -2\\ 0.948\\ 0.950\\ 0.956\\ 0.944\\ 0.953\\ \hline Table\\ -2\\ 0.945\\ 0.945\\ 0.946\\ 0.944\\ 0.951\\ \end{array}$	$\begin{array}{l} 0.946\\ \hline 0.946\\ \hline 0.946\\ \hline 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.953\\ 0.951\\ \hline \\ \text{Empirical}\\ -n^{-1/6}\\ 0.952\\ 0.951\\ \hline 0.952\\ 0.951\\ \hline \\ 15: \text{ Empir}\\ -n^{-1/6}\\ 0.953\\ 0.949\\ 0.955\\ 0.950\\ 0.956\\ \hline \\ 16: \text{ Empir}\\ -n^{-1/6}\\ 0.949\\ 0.952\\ 0.947\\ 0.953\\ \hline \end{array}$	0.951 0.951 al Two-Sid $-n^{-1/3}$ 0.952 0.952 0.943 0.950 0.949 l Two-Sid $-n^{-1/3}$ 0.953 0.951 0.943 0.949 0.948 cical upper $-n^{-1/3}$ 0.942 0.955 0.954 0.955 0.954 0.955 0.954 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.952 0.955 0.9	0.905 led Sym 0 0.950 0.955 0.954 0.956 0.949 ed Equa 0 0.949 0.953 0.954 0.956 0.949 0.956 0.949 1.000 0.903 0.905 0.910	$\begin{array}{c} 0.907\\ \text{metric}\\ n^{-1}\\ 0.955\\ 0.956\\ 0.954\\ 0.956\\ 0.951\\ \text{d} \\ \text{d} \\ n^{-1}\\ 0.955\\ 0.951\\ \text{d} \\ 0.954\\ 0.955\\ 0.951\\ \text{ided Cco}\\ n^{-1}\\ 0.999\\ 0.998\\ 1.000\\ 1.000\\ \text{ded Cco}\\ n^{-1}\\ 0.909\\ 0.905\\ 0.908\\ 0.909\\ 0.905\\ 0.908\\ 0.909\\ \end{array}$	$\begin{array}{c} 0.948\\ 0.948\\ \hline \text{Coverage}\\ n^{-1/1.5}\\ 0.972\\ 0.976\\ 0.972\\ 0.971\\ 0.973\\ \hline \text{Coverage}\\ n^{-1/1.5}\\ 0.961\\ 0.971\\ 0.965\\ 0.967\\ 0.971\\ \hline \text{werage Frr}\\ n^{-1/1.5}\\ 0.988\\ 0.996\\ 0.993\\ 0.997\\ 0.998\\ \hline \text{werage Fre}\\ n^{-1/1.5}\\ 0.946\\ 0.939\\ 0.940\\ \hline \end{array}$	$\begin{array}{c} 0.952\\ \hline 0.952\\ \hline \\ Frequen\\ n^{-1/2}\\ 0.831\\ 0.827\\ 0.846\\ 0.833\\ 0.822\\ \hline \\ e \ \\ Freques\\ n^{-1/2}\\ 0.823\\ 0.819\\ 0.839\\ 0.831\\ 0.820\\ \hline \\ equencie\\ n^{-1/2}\\ 0.832\\ 0.841\\ 0.844\\ 0.846\\ 0.835\\ \hline \\ equencie\\ n^{-1/2}\\ 0.961\\ 0.949\\ 0.953\\ 0.941\\ \hline \end{array}$	0.952 0.952 0.955 0.946 0.953 0.947 0.951 0.951 0.954 0.954 0.953 0.947 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.951 0.948 s for $\epsilon_n$ $n^{-1/3}$ 0.951 0.951 0.951 0.951 0.955 0.945 0.945 0.955 0.946 0.955 0.955 0.946 0.955 0.95	$\begin{array}{c} 0.948\\ \hline\\ e_n = n^{-1/6}\\ 0.938\\ 0.956\\ 0.946\\ 0.947\\ 0.955\\ \hline\\ e_n = n\\ n^{-1/6}\\ 0.939\\ 0.956\\ 0.946\\ 0.948\\ 0.955\\ \hline\\ = n^{-1}\\ n^{-1/6}\\ 0.940\\ 0.953\\ 0.952\\ 0.952\\ 0.952\\ \hline\\ = n^{-1}\\ n^{-1/6}\\ 0.947\\ 0.952\\ 0.942\\ 0.950\\ \hline\end{array}$	$\begin{array}{c} 0.947\\ -^1 using 2\\ n^{-1/10}\\ 0.949\\ 0.953\\ 0.943\\ 0.943\\ 0.944\\ 0.947\\ -^1 using\\ n^{-1/10}\\ 0.948\\ 0.951\\ 0.943\\ 0.943\\ 0.943\\ 0.947\\ using \mathbb{Z}_n\\ n^{-1/10}\\ 0.937\\ 0.945\\ 0.945\\ 0.952\\ 0.953\\ using \mathbb{Z}_n\\ n^{-1/10}\\ 0.959\\ 0.950\\ 0.950\\ 0.947\\ \end{array}$	$\begin{array}{c} 0.957\\ \hline 0.957\\ \hline \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.952\\ 0.957\\ 0.949\\ 0.953\\ \hline \mathbb{Z}_n = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.951\\ 0.957\\ 0.949\\ 0.953\\ \hline = N(0,\hat{\sigma})\\ 2\\ 0.961\\ 0.949\\ 0.943\\ 0.953\\ \hline = N(0,\hat{\sigma})\\ 2\\ 0.956\\ 0.955\\ 0.950\\ 0.954\\ \hline \end{array}$

Monte Carlo Simulations for the Second Order Directional Delta Method Number of Monte Carlo Simulations is 2000

	Table 1: R	Rejection I	Frequencie	s using And	drews ar	nd Soare	es, $\epsilon_n =$	$\sqrt{\log(n)}$	$\sqrt{n}$ , an	$d \mathbb{Z}_n = d$	$N(0,\hat{\sigma})$	
$ heta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	
n = 2000	1.000	0.778	0.144	0.082	0.068	0.051	0.056	0.041	0.017	0.000	0.000	
n = 4000	1.000	0.871	0.170	0.074	0.060	0.056	0.049	0.046	0.015	0.000	0.000	
n = 6000	1.000	0.928	0.147	0.076	0.064	0.052	0.057	0.042	0.013	0.000	0.000	
n = 8000	1.000	0.946	0.160	0.069	0.055	0.049	0.052	0.043	0.012	0.000	0.000	
n = 10000	1.000	0.957	0.160	0.070	0.049	0.057	0.055	0.038	0.009	0.000	0.000	
<b>Т 1 0 Р</b>	· .·		·	· · · · ·	1		1 D	. ,.	/1		- 177	$\mathbf{M}(0, \mathbf{\hat{0}})$
Table 2: Re	Jection Fre $-1/6$	equencies $1 - \frac{1}{3}$	1 sing Derr	vative of Ai -1/1.5	$\frac{1}{-1}$	irst Or	der Der	1vative, $\epsilon_i$	$n = \sqrt{l0} - \frac{1}{2}$	$\frac{g(n)}{\sqrt{r}}$	$i, and \mathbb{Z}_{2}$	$n = N(0,\sigma)$
$\theta_n$	$-n^{-1/2}$	$-n^{-1/2}$	$-n^{-n}$	$-n^{-1/10}$	$-n^{-1}$	0	$n^{-1}$	n 1/110	$n - \frac{1}{2}$	$n^{-1/2}$	$n^{-1/6}$	
n = 2000	1.000	0.969	0.258	0.083	0.048	0.048	0.043	0.025	0.003	0.000	0.000	
n = 4000	1.000	0.987	0.260	0.078	0.050	0.043	0.046	0.033	0.004	0.000	0.000	
n = 6000	1.000	0.995	0.257	0.077	0.057	0.046	0.042	0.024	0.002	0.000	0.000	
n = 8000	1.000	0.998	0.257	0.080	0.043	0.042	0.036	0.025	0.003	0.000	0.000	
n = 10000	1.000	0.999	0.267	0.072	0.050	0.050	0.050	0.026	0.003	0.000	0.000	
Table 3:	Rejection	Frequenci	es using N	Numerical S	econd C	Drder De	erivative	$e 1. \epsilon_n =$	$\sqrt{log(n)}$	$\sqrt{n}$ , ar	ad $\mathbb{Z}_n =$	$N(0,\hat{\sigma})$
$\theta_{m}$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	- (0,0)
n = 2000	1.000	0.845	0.167	0.092	0.075	0.061	0.068	0.051	0.019	0.000	0.000	
n = 4000	1.000	0.925	0.186	0.088	0.010 0.067	0.061	0.059	0.001 0.051	0.017	0.000	0.000	
n = 6000	1.000	0.959	0.100 0.173	0.086	0.001 0.073	0.062	0.067	0.049	0.017	0.000	0.000	
n = 8000 n = 8000	1.000	0.900 0.973	0.175	0.080	0.010	0.002	0.060	0.046	0.011	0.000	0.000	
n = 0000 n = 10000	1.000	0.910	0.170	0.000	0.001	0.002 0.067	0.000	0.040 0.043	0.014	0.000	0.000	
<i>n</i> = 10000	1.000	0.001	0.100	0.010	0.001	0.001	0.001	0.010	0.011	0.000	0.000	
Table 4:	Rejection	Frequenci	les using N	Numerical S	econd C	Order De	erivative	$e 2, \epsilon_n =$	$\sqrt{log(n)}$	$\sqrt{n}$ , and	d $\mathbb{Z}_n =$	$N(0,\hat{\sigma})$
$ heta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	
n = 2000	1.000	0.969	0.261	0.089	0.054	0.051	0.050	0.030	0.004	0.000	0.000	
n = 4000	1.000	0.987	0.262	0.083	0.053	0.048	0.049	0.035	0.005	0.000	0.000	
n = 6000	1.000	0.995	0.257	0.079	0.060	0.049	0.045	0.025	0.003	0.000	0.000	
n = 8000	1.000	0.998	0.259	0.080	0.044	0.043	0.037	0.028	0.004	0.000	0.000	
n = 10000	1.000	0.999	0.266	0.071	0.051	0.055	0.050	0.029	0.003	0.000	0.000	
	T-1-1- 1	. D.:			A1	1 C		1/6	õ 1 77	N/O	<u>^</u> )	
ρ	Table $\frac{5}{m^{-1/6}}$	5: Rejection $m^{-1/3}$	on Frequence $n^{-1/2}$	ncies using $m^{-1/1.5}$	And rew $m^{-1}$	s and S	oares, $\epsilon_{i}$	$n = n^{-1/6}$	$^{3}$ , and $\mathbb{Z}$	n = N(0)	$,\hat{\sigma})_{m^{-1/6}}$	
$\theta_n$	Table $\frac{3}{-n^{-1/6}}$	5: Rejectio $-n^{-1/3}$	on Frequence $-n^{-1/2}$	ncies using $-n^{-1/1.5}$	Andrew $-n^{-1}$	s and S $0$	oares, $\epsilon_i$ $n^{-1}$	$n = n^{-1/6}$ $n^{-1/1.5}$	$\overset{3}{,}  ext{ and } \mathbb{Z}$	$n = N(0)$ $n^{-1/3}$	$(\hat{\sigma})$ $n^{-1/6}$	
$\theta_n$ n = 2000 m = 4000	Table $\frac{1}{2}$ - $n^{-1/6}$ 1.000	5: Rejectio $-n^{-1/3}$ 0.957 0.081	on Frequen $-n^{-1/2}$ 0.233 0.244	ncies using $-n^{-1/1.5}$ 0.087 0.084	Andrew $-n^{-1}$ 0.056 0.052	s and S 0 0.049 0.050	oares, $\epsilon_n^{-1}$ 0.051	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.025	$n^{5}, \text{ and } \mathbb{Z}$ $n^{-1/2}$ 0.006 0.006	n = N(0) $n^{-1/3}$ 0.000 0.000	$(\hat{\sigma}) = n^{-1/6} = 0.000$	
$\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$	Table $\frac{1}{2}$ - $n^{-1/6}$ 1.000 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.903	$\begin{array}{c} \text{on Freque} \\ -n^{-1/2} \\ 0.233 \\ 0.244 \\ 0.236 \end{array}$	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076	Andrew $-n^{-1}$ 0.056 0.052 0.061	s and S 0 0.049 0.050 0.051	oares, $\epsilon_{n}^{-1}$ 0.051 0.051 0.046	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028	<sup>5</sup> , and $\mathbb{Z}$ $n^{-1/2}$ 0.006 0.006 0.004	n = N(0) $n^{-1/3}$ 0.000 0.000 0.000	$(\hat{\sigma}) = n^{-1/6} = 0.000 = 0.000 = 0.000$	
$\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$	Table $\begin{cases} -n^{-1/6} \\ 1.000 \\ 1.000 \\ 1.000 \\ 1.000 \\ 1.000 \end{cases}$	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997	$\begin{array}{c} \text{on Frequer} \\ -n^{-1/2} \\ 0.233 \\ 0.244 \\ 0.236 \\ 0.241 \end{array}$	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.070	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043	s and S 0 0.049 0.050 0.051 0.045	oares, $\epsilon_{n}^{-1}$ 0.051 0.051 0.046 0.040	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028	<sup>5</sup> , and $\mathbb{Z}$ $n^{-1/2}$ 0.006 0.004 0.004	n = N(0) $n^{-1/3}$ 0.000 0.000 0.000 0.000	$(\hat{\sigma})_{n^{-1/6}}^{n^{-1/6}}$ 0.000 0.000 0.000 0.000	
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000	Table $\frac{3}{-n^{-1/6}}$ 1.000 1.000 1.000 1.000 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999	$\begin{array}{c} \text{on Frequer} \\ -n^{-1/2} \\ 0.233 \\ 0.244 \\ 0.236 \\ 0.241 \\ 0.252 \end{array}$	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051	s and S 0 0.049 0.050 0.051 0.045 0.054	oares, $\epsilon_n^{-1}$ 0.051 0.046 0.040 0.048	$n = n^{-1/6}$ $n^{-1/1.5}$ $0.032$ $0.035$ $0.028$ $0.028$ $0.031$	<sup>5</sup> , and $\mathbb{Z}$ $n^{-1/2}$ 0.006 0.006 0.004 0.004 0.004	$n = N(0)$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $0.000$ $0.000$	$(\hat{\sigma})_{n^{-1/6}}^{n^{-1/6}}$ 0.000 0.000 0.000 0.000 0.000	
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000	Table $\frac{1}{2}$ - $n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999	$\begin{array}{c} \text{on Frequer} \\ -n^{-1/2} \\ 0.233 \\ 0.244 \\ 0.236 \\ 0.241 \\ 0.252 \end{array}$	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051	s and S 0 0.049 0.050 0.051 0.045 0.054	$\begin{array}{c} \text{oares, } \epsilon_{i} \\ n^{-1} \\ 0.051 \\ 0.051 \\ 0.046 \\ 0.040 \\ 0.048 \end{array}$	$n = n^{-1/6}$ $n^{-1/1.5}$ $0.032$ $0.035$ $0.028$ $0.028$ $0.028$ $0.031$	<sup>5</sup> , and $\mathbb{Z}$ $n^{-1/2}$ 0.006 0.006 0.004 0.004 0.004	$n = N(0) \\ n^{-1/3} \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 $	$(\hat{\sigma}) = n^{-1/6} = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.0000 = 0.00000 = 0.0000 = 0.0000 = 0.00000 = 0.0000000 = $	
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table 6:	Table $\frac{1}{2}$ - $n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection	5: Rejectio $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First	$\begin{array}{c} \text{oares, } \epsilon_{i} \\ n^{-1} \\ 0.051 \\ 0.051 \\ 0.046 \\ 0.040 \\ 0.048 \\ \text{Order } \end{array}$	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative	5, and $\mathbb{Z}_{n^{-1/2}}^{n^{-1/2}}$ 0.006 0.006 0.004 0.004 0.004 e, $\epsilon_n = \eta$	$n = N(0)$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $0.000$ $n^{-1/6}, an$	$(\hat{\sigma})_{n^{-1/6}}^{n^{-1/6}}$ $(\hat{\sigma})_{0.000}^{n^{-0.000}}$ $(\hat{\sigma})_{0.000}^{n^{-0.000}}$ $(\hat{\sigma})_{0.000}^{n^{-0.000}}$ and $\mathbb{Z}_{n} =$	$N(0,\hat{\sigma})$
$\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ Table 6: $\theta_n$	Table 5 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt $-n^{-1}$	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$	$n = n^{-1/6}$ $n^{-1/1.5}$ $0.032$ $0.035$ $0.028$ $0.028$ $0.031$ Derivative $n^{-1/1.5}$	5, and $\mathbb{Z}_{n^{-1/2}}^{2}$ 0.006 0.006 0.004 0.004 0.004 e, $\epsilon_n = r_{n^{-1/2}}^{2}$	$n = N(0)$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $0.000$ $n^{-1/6}, \text{an}$ $n^{-1/3}$	$(\hat{\sigma})_{n^{-1/6}}^{n^{-1/6}}$ $(\hat{\sigma})_{0.000}^{n^{-0.000}}$ $(\hat{\sigma})_{0.000}^{n^{-0.000}}$ $(\hat{\sigma})_{0.000}^{n^{-0.000}}$ $\hat{\sigma}_{0.000}^{n^{-0.000}}$ $\hat{\sigma}_{0.000}^{n^{-0.000}}$	$N(0,\hat{\sigma})$
$\theta_n$ $n = 2000$ $n = 4000$ $n = 6000$ $n = 8000$ $n = 10000$ Table 6: $\theta_n$ $n = 2000$	Table 3 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt $-n^{-1}$ 0.055	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030	<sup>3</sup> , and $\mathbb{Z}$ $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 e, $\epsilon_n = r$ $n^{-1/2}$ 0.004	$n = N(0)$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $0.000$ $n^{-1/6}, \text{ an }$ $n^{-1/3}$ $0.000$	$(\hat{\sigma})_{n^{-1/6}}$ $(\hat{\sigma})_{n^{-1/6}}$ $(\hat{\sigma})_{n^{-1/6}}$ $(\hat{\sigma})_{n^{-1/6}}$ $(\hat{\sigma})_{n^{-1/6}}$ $(\hat{\sigma})_{n^{-1/6}}$ $(\hat{\sigma})_{n^{-1/6}}$	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000	Table 5 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequencia $-n^{-1/3}$ 0.969 0.987	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt $-n^{-1}$ 0.055 0.053	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050	$\begin{array}{c} \text{oares, } \epsilon_n \\ n^{-1} \\ 0.051 \\ 0.051 \\ 0.046 \\ 0.040 \\ 0.048 \\ \text{Order I} \\ n^{-1} \\ 0.050 \\ 0.050 \end{array}$	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035	5, and $\mathbb{Z}_{n^{-1/2}}^{2}$ 0.006 0.006 0.004 0.004 0.004 e, $\epsilon_n = r_{n^{-1/2}}^{2}$ 0.004 0.005	$n = N(0)$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $n^{-1/6}, \text{au}$ $n^{-1/3}$ $0.000$ $0.000$	$\begin{array}{l} (\hat{\sigma}) \\ n^{-1/6} \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ \text{nd } \mathbb{Z}_n = \\ n^{-1/6} \\ 0.000 \\ 0.000 \end{array}$	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000 n = 6000	Table $\frac{1}{2}$ $-n^{-1/6}$ 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000 1.000 1.000 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt $-n^{-1}$ 0.055 0.053 0.061	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051	$\begin{array}{c} \text{oares, } \epsilon_n \\ n^{-1} \\ 0.051 \\ 0.051 \\ 0.046 \\ 0.040 \\ 0.048 \\ \text{Order I} \\ n^{-1} \\ 0.050 \\ 0.050 \\ 0.046 \end{array}$	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026	5, and $\mathbb{Z}_{n^{-1/2}}^{2}$ 0.006 0.006 0.004 0.004 0.004 0.004 e, $\epsilon_{n} = r_{n^{-1/2}}^{2}$ 0.004 0.005 0.003	$n = N(0)$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $n^{-1/6}, \text{ an }$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$	$\begin{array}{l} (\hat{\sigma}) \\ n^{-1/6} \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ nd \mathbb{Z}_n = \\ n^{-1/6} \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{array}$	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000	Table $\frac{1}{2}$ $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.998	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analytt $-n^{-1}$ 0.055 0.053 0.061 0.045	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.051 0.043	$\begin{array}{c} \text{oares, } \epsilon_n \\ n^{-1} \\ 0.051 \\ 0.051 \\ 0.046 \\ 0.040 \\ 0.048 \end{array}$ $\begin{array}{c} \text{Order 1} \\ n^{-1} \\ 0.050 \\ 0.050 \\ 0.046 \\ 0.040 \end{array}$	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028	5, and $\mathbb{Z}_{n^{-1/2}}^{-1/2}$ 0.006 0.006 0.004 0.004 0.004 e, $\epsilon_n = r_{n^{-1/2}}^{-1/2}$ 0.004 0.005 0.003 0.004	$n = N(0)$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $n^{-1/6}, \text{ an }$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $0.000$	$(\hat{\sigma})_{n}^{-1/6}$ $(\hat{\sigma})_{0.000}^{0.000}$ $(\hat{\sigma})_{0.000}^{0.000}$ $(\hat{\sigma})_{0.000}^{0.000}$ $(\hat{\sigma})_{0.000}^{0.000}$ $(\hat{\sigma})_{0.000}^{0.000}$ $(\hat{\sigma})_{0.000}^{0.000}$	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000	Table 3 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative or $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056	$\begin{array}{c} \text{oares, } \epsilon_n \\ n^{-1} \\ 0.051 \\ 0.051 \\ 0.046 \\ 0.040 \\ 0.048 \end{array}$ $\begin{array}{c} \text{Order 1} \\ n^{-1} \\ 0.050 \\ 0.050 \\ 0.046 \\ 0.040 \\ 0.050 \end{array}$	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030	5, and $\mathbb{Z}_{n^{-1/2}}^{-1/2}$ 0.006 0.004 0.004 0.004 e, $\epsilon_n = r_{n^{-1/2}}^{-1/2}$ 0.004 0.005 0.003 0.004 0.003	$n = N(0)$ $n^{-1/3}$ 0.000 0.000 0.000 0.000 0.000 n^{-1/6}, an n^{-1/3} 0.000 0.000 0.000 0.000 0.000 0.000 0.000	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{0.000}_{0.000}_{0.000}^{0.000}_{0.000}^{0.000}_{0.000}^{0.000}_{0.000}_{0.000}^{0.000}^{0.000}_{0.000}^{0.000}_{0.000}^{0.000}^{0.000}_{0.000}^{0.000}^{0.000}_{0.000}^{0.000}^{0.000}_{0.000}^{0.000}^{0.000}_{0.000}^{0.000}^{0.000}_{0.000}^{0.000}^{0.000}_{0.000}^{0.000}^{0.000}^{0.000}^{0.000}^{0.000}_{0.000}^{$	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000	Table 3 $-n^{-1/6}$ 1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000           1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequencia $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequenci	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative or $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 Derivative or one of the second se	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 0.052	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 add 0.043	oares, $\epsilon_n^{n-1}$ 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.050 0.046 0.040 0.040 0.050	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 time 1	5, and $\mathbb{Z}$ $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 e, $\epsilon_n = r$ $n^{-1/2}$ 0.004 0.005 0.003 0.004 0.003 $n^{-1/2}$	$n = N(0)$ $n^{-1/3}$ 0.000 0.000 0.000 0.000 0.000 $n^{-1/6}, \text{ arr}$ $n^{-1/3}$ 0.000	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{0.000}_{0.000}_{0.000}^{0.000}_{0.000}_{0.000}^{0.000}_{0.000}_{0.000}_{0.000}^{0.000}_{0.000}_{0.000}^{0.000}_{0.000}_{0.000}^{0.000}_{0.000}$	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table	Table 3 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequence $m^{-1/3}$	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $m^{-1/2}$	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $n^{-1/1.5}$	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Order	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order I $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 attive 1, $\epsilon_n$ $n^{-1/1.5}$	5, and $\mathbb{Z}_{n^{-1/2}}^{n^{-1/2}}$ 0.006 0.004 0.004 0.004 0.004 e, $\epsilon_n = n^{n^{-1/2}}$ 0.004 0.005 0.003 0.004 0.003 e, $n^{-1/2}$	$n = N(0) \\ n^{-1/3} \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ n^{-1/6}, \text{ an} \\ n^{-1/3} \\ 0.000 \\ $	$(\hat{\sigma})_{n}^{-1/6}$ $(\hat{\sigma})_{n}^{-1/6}$ $(\hat{\sigma})_{n}^{-1/6}$ $(\hat{\sigma})_{n}^{-0.00}$ $(\hat{\sigma})_{n}^{-0.00}$ $(\hat{\sigma})_{n}^{-1/6}$ $(\hat{\sigma})_{n}^{-1/6}$ $(\hat{\sigma})_{n}^{-1/6}$ $(\hat{\sigma})_{n}^{-1/6}$ $(\hat{\sigma})_{n}^{-1/6}$ $(\hat{\sigma})_{n}^{-1/6}$	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table $\theta_n$ $\pi = 2000$	Table 3 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequenci $-n^{-1/3}$ 0.957	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.224	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $-n^{-1/1.5}$ 0.088	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analytt $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Order 0 0.050	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 etive 1, $\epsilon_n$ $n^{-1/1.5}$ 0.022	5, and Z $n^{-1/2}$ 0.006 0.006 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.003 0.004 0.003 0.004 0.003 0.004 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.006 0.003 0.004 0.005 0.006 0.005 0.006 0.006 0.005 0.006 0.005 0.006 0.006 0.005 0.006 0.005 0.006 0.005 0.006 0.006 0.005 0.006 0.006 0.005 0.006	$n = N(0)$ $n^{-1/3}$ 0.000 0.000 0.000 0.000 0.000 n^{-1/6}, an n^{-1/3} 0.000 0.000 0.000 0.000 0.000 0.000 0.000 6, and Z n^{-1/3} 0.000	$\begin{aligned} & (\hat{\sigma}) \\ & n^{-1/6} \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \\ & 0.000 \end{aligned}$	$N(0,\hat{\sigma})$
$ \begin{array}{l} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \mbox{Table 6:} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \mbox{Table} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ \mbox{Table} \\ \end{array} $	Table 3 $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.995 0.995 0.998 0.999 ion Frequenci $-n^{-1/3}$ 0.957 0.951	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.264 0.259 0.261 0.267	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative or $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $-n^{-1/1.5}$ 0.088 0.085	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analytt $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secom $-n^{-1}$ 0.057 0.052	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Order 0 0.050 0.050	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.051 0.050	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 attive 1, \epsilon_n^{-1/1.5} 0.032 0.032 0.032	5, and $\mathbb{Z}$ $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 e, $\epsilon_n = r$ $n^{-1/2}$ 0.004 0.005 0.003 0.004 0.003 0.004 0.003 $n^{-1/2}$ 0.004 0.005 0.003 0.004 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.003 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.006	$n = N(0)$ $n^{-1/3}$ 0.000 0.000 0.000 0.000 0.000 n^{-1/6}, an n^{-1/3} 0.000 0.0	$\hat{\sigma}, \hat{\sigma}) = \frac{n^{-1/6}}{n^{-1/6}} \\ 0.000 $	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000 n = 8000 n = 10000 Table $\theta_n$ n = 2000 n = 2000 n = 4000 n = 6000 n = 6000	Table 3 $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequenci $-n^{-1/3}$ 0.957 0.981 0.902	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.266 0.267	ncies using $-n^{-1/1.5}$ 0.087 0.087 0.076 0.079 0.072 Derivative or $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $-n^{-1/1.5}$ 0.088 0.085 0.075	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057 0.052 0.062	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.050 0.050 0.050 0.050 0.050 0.051	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.050 0.046	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 ative 1, \epsilon_n^{-1/1.5} 0.032 0.035 0.028 0.032 0.035 0.028	5, and $\mathbb{Z}$ $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 e, $\epsilon_n = r$ $n^{-1/2}$ 0.004 0.005 0.003 0.004 0.003 0.004 0.003 $\epsilon_n = n^{-1/2}$ 0.006 0.004 0.005 0.003 0.004 0.005 0.004 0.005 0.006 0.004 0.005 0.006 0.004 0.005 0.005 0.006 0.004 0.005 0.003 0.004 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.004 0.004 0.005 0.004 0.004 0.004 0.005 0.004 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.005 0.004 0.006	$n = N(0) \\ n^{-1/3} \\ 0.000 $	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-1/6}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-1/6}_{0.000}^{-1/6}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{-0.000$	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000 n = 6000 n = 2000 n = 4000 n = 4000 n = 2000 n = 4000 n = 2000 n = 4000 n = 8000 n = 2000 n = 8000 n = 2000 n = 8000 n = 8000	Table 8 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequenci $-n^{-1/3}$ 0.957 0.957 0.981 0.993 0.007	on Frequer $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.261 0.267 encies usin $-n^{-1/2}$ 0.266 0.261 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.267 $-n^{-1/2}$ 0.234 0.246 0.236 0.242	ncies using $-n^{-1/1.5}$ 0.087 0.087 0.079 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $-n^{-1/1.5}$ 0.088 0.085 0.077 0.085 0.077 0.070	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyte $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057 0.057 0.052 0.062 0.042	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Order 0 0.050 0.050 0.050 0.050 0.050 0.051 0.045 0.050 0.050 0.051 0.045 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.051 0.043 0.050 0.	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.051 0.046 0.040 0.050 0.050 0.050 0.050 0.050 0.050 0.046 0.040 0.040 0.050 0.050 0.046 0.040 0.050 0.050 0.050 0.046 0.040 0.040 0.050 0.046 0.040 0.040 0.040 0.040 0.040 0.040 0.040 0.040 0.040 0.040 0.040 0.040 0.040 0.040 0.040 0.050 0.040 0.040 0.050 0.040 0.040 0.040 0.050 0.040 0.040 0.050 0.040 0.050 0.040 0.050 0.040 0.040 0.050 0.050 0.040 0.050 0.040 0.050 0.050 0.040 0.050 0.040 0.050 0.050 0.050 0.040 0.050	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 etive 1, $\epsilon_n$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.03 0.035 0.028 0.035 0.028 0.035 0.028 0.03 0.035 0.028 0.03 0.035 0.028 0.03 0.035 0.028 0.03 0.03 0.035 0.028 0.03 0.03 0.035 0.028 0.03 0.03 0.03 0.03 0.03 0.03 0.03 0.0	5, and Z $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 e, $\epsilon_n = r$ $n^{-1/2}$ 0.004 0.005 0.003 0.004 0.003 0.004 0.006 0.006 0.004 0.006 0.004 0.006 0.004 0.004 0.006 0.004 0.004 0.006 0.004 0.005 0.006 0.004 0.005 0.006 0.004 0.004 0.005 0.006 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.006 0.004 0.005 0.003 0.004 0.005 0.004 0.005 0.003 0.004 0.005 0.006 0.004 0.005 0.004 0.005 0.006 0.	$n = N(0)$ $n^{-1/3}$ 0.000 0.000 0.000 0.000 0.000 n^{-1/6}, an n^{-1/3} 0.000 0.000 0.000 0.000 0.000 0.000 6, and Z n^{-1/3} 0.000 0.0	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-1/6}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-1/6}_{0.000}^{-1/6}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{-0.000}^{-0.0000}_{-0.000}^{-0.000}_{-0.000}^{-0.000}_{-0.0$	$N(0,\hat{\sigma})$
$\theta_n$ n = 2000 n = 4000 n = 6000 n = 8000 n = 10000 Table 6: $\theta_n$ n = 2000 n = 4000 n = 6000 n = 10000 n = 6000 n = 6000 n = 8000 n = 8000 n = 8000 n = 8000 n = 8000 n = 8000 n = 8000	Table 3 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequencia $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequencia $-n^{-1/3}$ 0.957 0.957 0.981 0.993 0.997 0.990	on Frequer $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 0.234 0.234 0.236 0.242 0.259 0.262 0.264 0.259 0.267 0.267 0.267 0.259 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.267 0.262 0.262 0.267 0.262 0.262 0.267 0.262 0.252 0.25	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numericc $-n^{-1/1.5}$ 0.088 0.085 0.077 0.088 0.085 0.077 0.079 0.072	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyte $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057 0.052 0.062 0.062 0.063 0.051	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Order 0 0.050 0.050 0.050 0.050 0.051 0.043 0.050 0.050 0.050 0.051 0.050 0.055 0.	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order I $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.052 0.046 0.040 0.040	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 etive 1, $\epsilon_n$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.032 0.035 0.028 0.0	5, and Z $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 e, $\epsilon_n = r$ $n^{-1/2}$ 0.004 0.005 0.003 0.004 0.003 $n^{-1/2}$ 0.004 0.003 $n^{-1/2}$ 0.004 0.005 0.003 0.004 0.006 0.004 0.006 0.004 0.004 0.006 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.006 0.004 0.004 0.004 0.005 0.003 0.004 0.004 0.005 0.004 0.006 0.006 0.006 0.006 0.006 0.006 0.006 0.006 0.006 0.006 0.006 0.006 0.006 0.006 0.004 0.006 0.006 0.004 0.006 0.004 0.006 0.006 0.004 0.006 0.004 0.004 0.006 0.004	$n = N(0)$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $n^{-1/6}, \text{ an}$ $n^{-1/3}$ $0.000$	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$	$N(0,\hat{\sigma})$
$\begin{array}{l} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table 6:} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ n = 6000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \end{array}$	Table 3 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequencia $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequencia $-n^{-1/3}$ 0.957 0.998 0.997 0.991 0.993 0.997 0.999	on Frequer $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.234 0.246 0.236 0.242 0.253	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numerice $-n^{-1/1.5}$ 0.088 0.085 0.077 0.085 0.072	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analytt $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057 0.052 0.062 0.063 0.051	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Order 0 0.050 0.050 0.050 0.050 0.050 0.051 0.050 0.050 0.051 0.050 0.050 0.050 0.051 0.050 0.054	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.052 0.046 0.040 0.048	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 etive 1, $\epsilon_n$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.031	$\begin{array}{c} \text{3. and } \mathbb{Z} \\ n^{-1/2} \\ 0.006 \\ 0.006 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.005 \\ 0.003 \\ 0.004 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.006 \\ 0.006 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \end{array}$	$n = N(0)$ $n^{-1/3}$ 0.000 0.000 0.000 0.000 0.000 n^{-1/6}, an n^{-1/3} 0.000 0.0	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-1/6}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{-0.000}^{-0.000}_$	$N(0, \hat{\sigma})$
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table 6:} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table} \\ \end{array}$	Table 8 $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequenci $-n^{-1/3}$ 0.957 0.957 0.981 0.993 0.997 0.999 ion Frequenci	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.264 0.267 encies usin $-n^{-1/2}$ 0.264 0.267 encies usin $-n^{-1/2}$ 0.262 0.263 encies usin $-n^{-1/2}$ 0.263 0.264 0.259 0.261 0.265 encies usin $-n^{-1/2}$ 0.262 0.263 encies usin $-n^{-1/2}$ 0.264 0.265 encies usin $-n^{-1/2}$ 0.265 encies usin $-n^{-1/2}$ 0.265 encies usin $-n^{-1/2}$ 0.265 encies usin $-n^{-1/2}$ 0.267 encies usin $-n^{-1/2}$ 0.268 0.267 encies usin $-n^{-1/2}$ 0.268 0.267 encies usin $-n^{-1/2}$ 0.268 0.267 encies usin $-n^{-1/2}$ 0.268 0.268 0.267 encies usin $-n^{-1/2}$ 0.268 0.268 0.268 0.269 0.261 0.267 encies usin $-n^{-1/2}$ 0.268 0.2588 0.2588 0.2588 0.2588 0.2588 0.2588 0.2588 0.25	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative or $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $-n^{-1/1.5}$ 0.088 0.085 0.077 0.088 0.085 0.077 0.072 ng Numeric	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyte $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057 0.052 0.062 0.062 0.043 0.051 al Secon	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Order 0 0.050 0.051 0.050 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.050 0.051 0.050 0.054 0.054 0.050 0.054	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.052 0.046 0.040 0.052 0.046 0.040 0.048 r Deriva r Deriva	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 ative 1, \epsilon_n^{-1/1.5} 0.032 0.035 0.028 0.032 0.028 0.028 0.028 0.031 ative 2, \epsilon_n ative 2, \	$\begin{array}{l} \text{and } \mathbb{Z} \\ n^{-1/2} \\ 0.006 \\ 0.006 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.005 \\ 0.003 \\ 0.004 \\ 0.003 \\ 0.004 \\ 0.003 \\ n^{-1/2} \\ 0.006 \\ 0.006 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ 0.004 \\ n^{-1/2} \end{array}$	$n = N(0) \\ n^{-1/3} \\ 0.000 $	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-1/6}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-1/6}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{0.000}^{-0.000}_{-$	$N(0,\hat{\sigma})$ $(0,\hat{\sigma})$
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table 6:} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 10000 \\ \\ \text{Table} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table} \\ \theta_n \end{array}$	Table 8 $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequenci $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 ion Frequenci $-n^{-1/3}$	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.234 0.246 0.236 0.242 0.236 0.242 0.253 encies usin $-n^{-1/2}$	ncies using $-n^{-1/1.5}$ 0.087 0.087 0.079 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $-n^{-1/1.5}$ 0.088 0.085 0.077 0.079 0.072 ng Numeric $-n^{-1/1.5}$	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyte $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057 0.052 0.062 0.043 0.051 al Secon $-n^{-1}$	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Orde: 0 0.050 0.051 0.050 0.050 0.051 0.043 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.050 0.051 0.050 0.050 0.051 0.050 0.050 0.050 0.050 0.051 0.050 0.050 0.050 0.050 0.050 0.051 0.050 0.050 0.050 0.050 0.050 0.051 0.050 0.050 0.050 0.050 0.050 0.051 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.050 0.051 0.045 0.050 0.050 0.050 0.051 0.050 0.050 0.051 0.045 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.054 0.054 0.054 0.050 0.054 0.050 0.054 0.050 0.054 0.050 0.054 0.050 0.054 0.050 0.054 0.050 0.050 0.054 0.050 0.050 0.054 0.050 0.050 0.054 0.050 0.050 0.054 0.050 0.050 0.050 0.054 0.050 0.050 0.050 0.054 0.050 0.050 0.050 0.054 0.050 0.	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.052 0.046 0.040 0.048 r Deriva $n^{-1}$	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 ative 1, \epsilon_n^{-1/1.5} 0.032 0.035 0.028 0.031 ative 2, \epsilon_n^{-1/1.5} 0.031 ative 2, \epsilon_n^{-1/1.5} 0.032 0.031 ative 2, \epsilon_n^{-1/1.5} 0.031	5, and Z $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.003 0.004 0.003 0.004 0.006 0.006 0.006 0.004 0.006 0.004 0.005 0.004	$n = N(0) \\ n^{-1/3} \\ 0.000 $	$\begin{aligned} & (\hat{\sigma}) \\ & n^{-1/6} \\ & 0.000 $	$N(0,\hat{\sigma})$ $(0,\hat{\sigma})$
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table 6:} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table} \\ \theta_n \\ n = 2000 \\ n = 8000 \\ n = 10000 \\ \\ \\ \text{Table} \\ \theta_n \\ n = 2000 \\ \end{array}$	Table 8 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequenci $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 ion Frequenci $-n^{-1/3}$ 0.957 0.969	on Frequer $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.234 0.236 0.242 0.236 0.242 0.253 encies usin $-n^{-1/2}$ 0.262	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $-n^{-1/1.5}$ 0.088 0.085 0.077 0.079 0.072 ng Numeric $-n^{-1/1.5}$ 0.088 0.079 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.092 0.091 0.091 0.091 0.092 0.091 0.092 0.091 0.092 0.091 0.092 0.091 0.091 0.092 0.091 0.092 0.091 0.091 0.092 0.091 0.091 0.091 0.091 0.091 0.091 0.091 0.091 0.091 0.091 0.091	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyte $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057 0.052 0.062 0.043 0.051 al Secon $-n^{-1}$ 0.055	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Order 0 0.050 0.051 0.050 0.050 0.051 0.045 0.051 0.045 0.050 0.051 0.050 0.051 0.050 0.050 0.051 0.050 0.050 0.050 0.051 0.050 0.051 0.045 0.050 0.050 0.052	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.052 0.046 0.040 0.048 r Deriva $n^{-1}$ 0.052 0.046 0.040 0.052 0.046 0.040 0.052 0.046 0.040 0.052 0.055 0.055 0.055 0.055 0.055 0.055 0.050 0.050 0.050 0.050 0.050 0.050 0.046 0.040 0.040 0.040 0.040 0.050 0.050 0.046 0.040 0.040 0.040 0.050 0.050 0.050 0.050 0.050 0.055 0.046 0.046 0.040 0.055	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 etive 1, \epsilon_n^{-1/1.5} 0.032 0.035 0.028 0.031 etive 2, \epsilon_n^{-1/1.5} 0.030 etive 2, \epsilon_n^{-1/1.5} etive	5, and Z $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.006 0.004 0.006 0.004 0.006 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.004 0.005 0.003 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.006 0.004 0.004 0.004 0.005 0.003 0.004 0.004 0.005 0.004 0.004 0.005 0.004 0.004 0.005 0.004 0.004 0.006 0.006 0.006 0.004 0.004 0.004 0.006 0.004	$n = N(0) = N(0) = 0$ $n^{-1/3} = 0.0000 = 0.000 = 0.0000 = 0.000 = 0.000 = 0.000 = 0.000 = 0.000 = 0$	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$	$N(0,\hat{\sigma})$ $(0,\hat{\sigma})$
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table 6:} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 8000 \\ n = 10000 \\ \\ \end{array}$	Table 8 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequencia $-n^{-1/3}$ 0.969 0.987 0.995 0.998 0.999 ion Frequencia $-n^{-1/3}$ 0.969 0.993 0.997 0.993 0.997 0.999 ion Frequencia $-n^{-1/3}$ 0.957 0.993 0.993 0.997 0.993 0.993 0.997 0.993 0.993 0.993 0.997 0.993 0.993 0.997 0.993 0.993 0.997 0.993 0.993 0.993 0.993 0.993 0.993 0.993 0.993 0.993 0.993 0.993 0.995 0.993	on Frequer $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.234 0.246 0.236 0.242 0.253 encies usin $-n^{-1/2}$ 0.262 0.264	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numericc $-n^{-1/1.5}$ 0.088 0.085 0.077 0.079 0.072 ng Numericc $-n^{-1/1.5}$ 0.091 0.085	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyte $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057 0.052 0.062 0.043 0.051 al Secon $-n^{-1}$ 0.055 0.053	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Order 0 0.050 0.051 0.050 0.050 0.051 0.045 0.045 0.051 0.045 0.050 0.051 0.050 0.050 0.051 0.050 0.050 0.051 0.050 0.050 0.050 0.050 0.051 0.050 0.050 0.050 0.050 0.050 0.051 0.045 0.050 0.050 0.050 0.050 0.051 0.045 0.050 0.050 0.050 0.050 0.051 0.050 0.	oares, $\epsilon_n^{n-1}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.052 0.046 0.040 0.048 r Deriva $n^{-1}$ 0.052 0.046 0.040 0.050 0.050 0.050 0.050 0.052 0.046 0.040 0.050 0.050 0.0550 0.055	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 etive 1, \epsilon_n^{-1/1.5} 0.032 0.035 0.028 0.031 etive 2, \epsilon_n^{-1/1.5} 0.030 0.035	5, and Z $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.006 0.006 0.006 0.006 0.004 0.006 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.004 0.004 0.005 0.004 0.005 0.003 0.004 0.004 0.005 0.003 0.004 0.004 0.005 0.003 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.006 0.004 0.005 0.003 0.004 0.006 0.004 0.004 0.005 0.003 0.004 0.006 0.004 0.005 0.004 0.006 0.004 0.005 0.004 0.006 0.006 0.004 0.006 0.004 0.004 0.004 0.005 0.004 0.006 0.004 0.004 0.004 0.005 0.004 0.006 0.004 0.004 0.005 0.004 0.004 0.004 0.005 0.004 0.005 0.004 0.004 0.004 0.005 0.005	$n = N(0) = N(0) = 0$ $n^{-1/3} = 0.0000 = 0.000 = 0.0000 = 0.000 = 0.000 = 0.000 = 0.000 = 0.000 = 0$	$ \begin{aligned} & (\hat{\sigma}) \\ & n^{-1/6} \\ & 0.000$	$N(0,\hat{\sigma})$ $(0,\hat{\sigma})$
$\begin{array}{c} \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table 6:} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 6000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table} \\ \theta_n \\ n = 2000 \\ n = 4000 \\ n = 8000 \\ n = 10000 \\ \\ \text{Table} \\ \theta_n \\ n = 2000 \\ n = 6000 \\ n = 6000 \\ \end{array}$	Table 8 $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequencia $-n^{-1/3}$ 0.969 0.995 0.998 0.999 ion Frequencia $-n^{-1/3}$ 0.957 0.993 0.993 0.993 0.997 0.993 0.999 ion Frequencia $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 ion Frequencia 0.993 0.997 0.993 0.993 0.997 0.999 ion Frequencia $-n^{-1/3}$ 0.969 0.995 0.995	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.262 0.264 0.236 0.242 0.236 0.242 0.253 encies usin $-n^{-1/2}$ 0.262 0.264 0.259	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $-n^{-1/1.5}$ 0.088 0.085 0.079 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.085 0.079 0.072	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyt $-n^{-1}$ 0.055 0.053 0.061 0.045 0.052 al Secon $-n^{-1}$ 0.057 0.052 0.062 0.043 0.051 al Secon $-n^{-1}$ 0.055 0.053 0.061	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.056 ad Orde: 0 0.050 0.051 0.051 0.045 0.051 0.052 0.051 0.051 0.052 0.051 0.052 0.051 0.050 0.051 0.052 0.052 0.050 0.052 0.050 0.052 0.050 0.052 0.050 0.052 0.050 0.052 0.050 0.052 0.050 0.050 0.052 0.050 0.050 0.052 0.050 0.050 0.052 0.050 0.050 0.050 0.052 0.050 0.050 0.050 0.050 0.050 0.052 0.050 0.	oares, $\epsilon_n^{n-1}$ 0.051 0.046 0.040 0.048 Order I $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.052 0.046 0.040 0.048 r Deriva $n^{-1}$ 0.050 0.052 0.046 0.040 0.048	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 attive 1, \epsilon_n^{-1/1.5} 0.032 0.035 0.028 0.031 attive 2, \epsilon_n^{-1/1.5} 0.030 0.035 0.026	5, and Z $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.003 0.004 0.003 $n^{-1/2}$ 0.006 0.003 0.004 0.006 0.006 0.004 0.006 0.004 0.006 0.004 0.004 0.004 0.005 0.006 0.004 0.005 0.004 0.004 0.005 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.005 0.004 0.005 0.004 0.005 0.003 0.003 0.004 0.005 0.003 0.003 0.004 0.005 0.003 0.005 0.003 0.005 0.003 0.005 0.003 0.005 0.003 0.005 0.003 0.005 0.003 0.005 0.003 0.005 0.003 0.005 0.0	$n = N(0)$ $n^{-1/3}$ $0.000$ $0.000$ $0.000$ $0.000$ $0.000$ $n^{-1/6}, \text{ an}$ $n^{-1/3}$ $0.000$	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$	$N(0, \hat{\sigma})$ $(0, \hat{\sigma})$
$\begin{array}{c} \theta_n \\n=2000 \\n=4000 \\n=6000 \\n=8000 \\n=10000 \\\text{Table 6:} \\\theta_n \\n=2000 \\n=4000 \\n=6000 \\n=8000 \\n=10000 \\\text{Table} \\\theta_n \\n=2000 \\n=4000 \\n=8000 \\n=10000 \\\text{Table} \\\theta_n \\n=2000 \\n=4000 \\n=8000 \\n=6000 \\n=8000 \\n=800 \\n=800 \\n=8000 \\n=800 \\n=8000 \\n=8000 \\n=8000 \\n=8000 \\n=8000 \\n=8000 \\n=$	Table 8 $-n^{-1/6}$ 1.000 1.000 1.000 1.000 1.000 Rejection $-n^{-1/6}$ 1.000	5: Rejection $-n^{-1/3}$ 0.957 0.981 0.993 0.997 0.999 Frequencii $-n^{-1/3}$ 0.969 0.995 0.998 0.999 ion Frequenci $-n^{-1/3}$ 0.957 0.981 0.993 0.993 0.997 0.999 ion Frequenci $-n^{-1/3}$ 0.993 0.997 0.999 ion Frequenci $-n^{-1/3}$ 0.969 0.987 0.995 0.995 0.998	on Frequen $-n^{-1/2}$ 0.233 0.244 0.236 0.241 0.252 es using D $-n^{-1/2}$ 0.262 0.264 0.259 0.261 0.267 encies usin $-n^{-1/2}$ 0.234 0.236 0.246 0.236 0.242 0.253 encies usin $-n^{-1/2}$ 0.262 0.264 0.259 0.261	ncies using $-n^{-1/1.5}$ 0.087 0.084 0.076 0.079 0.072 Derivative of $-n^{-1/1.5}$ 0.091 0.085 0.079 0.081 0.072 ng Numeric $-n^{-1/1.5}$ 0.088 0.085 0.077 0.079 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.079 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.085 0.079 0.072 ng Numeric $-n^{-1/1.5}$ 0.091 0.085 0.079 0.085 0.079 0.081	Andrew $-n^{-1}$ 0.056 0.052 0.061 0.043 0.051 f Analyte $-n^{-1}$ 0.055 0.053 0.061 0.045 0.057 0.052 0.062 0.062 0.043 0.051 al Secon $-n^{-1}$ 0.055 0.053 0.061 0.055 0.053 0.061 0.045	s and S 0 0.049 0.050 0.051 0.045 0.054 ic First 0 0.052 0.050 0.051 0.043 0.050 0.050 0.051 0.045 0.051 0.052 0.051 0.054 ad Order 0 0.052 0.050 0.051 0.052 0.051 0.052 0.051 0.052 0.051 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.050 0.051 0.052 0.051 0.052 0.051 0.052 0.051 0.052 0.051 0.052 0.051 0.052 0.051 0.052 0.051 0.051 0.052 0.051 0.052 0.051 0.052 0.052 0.052 0.052 0.050 0.052 0.050 0.052 0.050 0.051 0.052 0.050 0.051 0.052 0.050 0.051 0.052 0.050 0.051 0.052 0.050 0.051 0.052 0.050 0.051 0.050 0.052 0.051 0.052 0.051 0.051 0.051 0.051 0.051 0.051 0.051 0.051 0.051 0.051 0.043 0.051 0.043 0.043 0.043 0.051 0.043 0.043 0.043 0.051 0.043 0.051 0.043 0.051 0.043 0.051 0.043 0.051 0.043 0.051 0.043 0.051 0.043 0.051 0.043 0.051 0.	oares, $\epsilon_n^{n^{-1}}$ 0.051 0.051 0.046 0.040 0.048 Order 1 $n^{-1}$ 0.050 0.050 0.046 0.040 0.050 r Deriva $n^{-1}$ 0.051 0.052 0.046 0.040 0.048 r Deriva $n^{-1}$ 0.050 0.046 0.040 0.050 0.046 0.040 0.051 0.051 0.050 0.046 0.040 0.050 0.046 0.040 0.050 0.046 0.040 0.050 0.046 0.040 0.050 0.050 0.046 0.040 0.050 0.050 0.046 0.040 0.050 0.050 0.046 0.040 0.050 0.050 0.046 0.040 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.046 0.0550 0.0550 0.046 0.0550 0.046 0.0400 0.0550 0.046 0.0400 0.0550 0.046 0.0400 0.0550 0.046 0.0400 0.0550 0.046 0.0550 0.0550 0.046 0.0550 0.0550 0.0550 0.0550 0.0550 0.0550 0.0460 0.0550 0.0550 0.0460 0.0550 0.0460 0.0550 0.0460 0.0400 0.0460 0.0400 0.0460 0.0400 0.0460 0.0400 0.0460 0.0400 0.0400 0.0460 0.04000 0.04000 0.04000 0.04000 0.04000 0.040000000000	$n = n^{-1/6}$ $n^{-1/1.5}$ 0.032 0.035 0.028 0.028 0.028 0.031 Derivative $n^{-1/1.5}$ 0.030 0.035 0.026 0.028 0.030 attive 1, \epsilon_n^{-1/1.5} 0.032 0.035 0.028 0.031 attive 2, \epsilon_n^{-1/1.5} 0.030 0.035 0.028 0.031 attive 2, \epsilon_n^{-1/1.5} 0.030 0.035 0.026 0.028 0.028 0.028 0.028 0.031 attive 2, \epsilon_n^{-1/1.5} 0.030 0.035 0.026 0.028 0.028 0.028 0.028 0.028 0.028 0.028 0.028 0.028 0.028 0.028 0.028 0.028 0.031 0.028 0.031 0.028 0.031 0.035 0.032 0.031 0.035 0.032 0.031 0.035 0.032 0.031 0.035 0.032 0.031 0.035 0.032 0.031 0.035 0.032 0.031 0.035 0.031 0.035 0	5, and Z $n^{-1/2}$ 0.006 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.006 0.006 0.006 0.006 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.004 0.005 0.003 0.004 0.005 0.003 0.004 0.006 0.004	$n = N(0) \\ n^{-1/3} \\ 0.000 $	$(\hat{\sigma})_{n}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$ $0.000_{0.000}^{-1/6}$	$N(0, \hat{\sigma})$ $(0, \hat{\sigma})$

	Table 9	9: Rejectio	on Frequei	ncies using	Andrew	s and S	oares, $\epsilon$	$n = n^{-1/3}$	$^{3}$ , and $\mathbb{Z}$	n = N(0)	$(\hat{\sigma})$	
$ heta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	
n = 2000	1.000	0.857	0.162	0.080	0.061	0.048	0.051	0.037	0.013	0.000	0.000	
n = 4000	1.000	0.942	0.191	0.076	0.057	0.053	0.050	0.042	0.009	0.000	0.000	
n = 6000	1.000	0.970	0.184	0.076	0.061	0.051	0.051	0.035	0.006	0.000	0.000	
n = 8000	1.000	0.984	0.184	0.070	0.053	0.047	0.046	0.035	0.009	0.000	0.000	
n = 10000	1.000	0.991	0.190	0.067	0.051	0.054	0.049	0.033	0.004	0.000	0.000	
Table 10:	Rejection	Frequenci	les using I	Derivative of	of Analy	tic First	Order	Derivativ	e, $\epsilon_n =$	$n^{-1/3}, a$	and $\mathbb{Z}_n = \mathbb{Z}_n$	$N(0,\hat{\sigma})$
$ heta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	
n = 2000	1.000	0.969	0.262	0.092	0.054	0.052	0.050	0.030	0.004	0.000	0.000	
n = 4000	1.000	0.987	0.264	0.085	0.053	0.049	0.050	0.035	0.005	0.000	0.000	
n = 6000	1.000	0.995	0.259	0.079	0.061	0.051	0.046	0.026	0.003	0.000	0.000	
n = 8000	1.000	0.998	0.261	0.081	0.045	0.043	0.040	0.028	0.004	0.000	0.000	
n = 10000	1.000	0.999	0.267	0.073	0.052	0.056	0.050	0.030	0.003	0.000	0.000	
Table	11: Rejec	tion Frequ	encies usi	ng Numerio	cal Seco	nd Orde	er Deriv	ative 1, $\epsilon_i$	$n = n^{-1}$	$^{/3}$ , and	$\mathbb{Z}_n = N(0$	$,\hat{\sigma})$
$ heta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	. ,
n = 2000	1.000	0.888	0.181	0.090	0.065	0.054	0.059	0.043	0.015	0.000	0.000	
n = 4000	1.000	0.956	0.207	0.080	0.062	0.058	0.052	0.046	0.011	0.000	0.000	
n = 6000	1.000	0.976	0.193	0.080	0.066	0.055	0.053	0.042	0.007	0.000	0.000	
n = 8000	1.000	0.986	0.196	0.076	0.056	0.052	0.048	0.036	0.010	0.000	0.000	
n = 10000	1.000	0.995	0.203	0.072	0.054	0.058	0.051	0.037	0.004	0.000	0.000	
Table	12: Reject	tion Frequ	encies usi	ng Numeria	cal Seco	nd Orde	er Deriv	ative 2. $\epsilon$	$n = n^{-1}$	$^{/3}$ . and	$\mathbb{Z}_m = N(0)$	$(\hat{\sigma})$
$\theta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	n = n $n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	,0)
n = 2000	1.000	0.969	0.261	0.090	0.054	0.051	0.050	0.031	0.004	0.000	0.000	
n = 4000	1.000	0.987	0.263	0.084	0.053	0.049	0.049	0.035	0.005	0.000	0.000	
n = 6000	1.000	0.995	0.257	0.079	0.060	0.050	0.046	0.026	0.003	0.000	0.000	
n = 8000	1.000	0.998	0.260	0.080	0.045	0.043	0.038	0.028	0.004	0.000	0.000	
n = 10000	1.000	0.999	0.266	0.071	0.052	0.056	0.050	0.030	0.003	0.000	0.000	
	Table 1	3: Rejection	on Freque	encies using	Andrew	vs and S	boares.	$f_{m} = n^{-1/2}$	$^2$ and $\overline{2}$	$Z_m = N($	$(0, \hat{\sigma})$	
$\theta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	
n = 2000	0.017	0.025	0.056	0.087	0.081	0.084	0.081	0.080	0.072	0.000	0.000	
n = 4000	0.010	0.023	0.058	0.071	0.085	0.076	0.069	0.081	0.056	0.000	0.000	
n = 6000	0.018	0.030	0.054	0.072	0.081	0.077	0.083	0.073	0.064	0.000	0.000	
n = 8000	0.013	0.022	0.059	0.071	0.077	0.065	0.084	0.086	0.055	0.000	0.000	
n = 10000	0.016	0.025	0.062	0.085	0.080	0.081	0.081	0.077	0.055	0.000	0.000	
Table 14:	Rejection	Frequenci	es using I	Derivative o	of Analv	tic First	Order	Derivativ	e. $\epsilon_n =$	$n^{-1/2}$ . a	and $\mathbb{Z}_n = $	$N(0,\hat{\sigma})$
$\theta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	
n = 2000	1.000	0.932	0.051	0.004	0.002	0.002	0.001	0.000	0.000	0.000	0.000	
n = 4000	1.000	0.978	0.051	0.006	0.002	0.001	0.003	0.001	0.000	0.000	0.000	
n = 6000	1.000	0.990	0.055	0.004	0.002	0.002	0.001	0.000	0.000	0.000	0.000	
n = 8000	1.000	0.997	0.051	0.004	0.002	0.001	0.000	0.001	0.000	0.000	0.000	
n = 10000	1.000	0.997	0.058	0.004	0.003	0.001	0.002	0.001	0.000	0.000	0.000	
Table	15: Rejec	tion Frequ	encies usi	ng Numerio	cal Seco	nd Orde	er Deriv	ative 1, $\epsilon_i$	$n = n^{-1}$	$^{/2}$ , and	$\mathbb{Z}_n = N(0)$	$,\hat{\sigma})$
$ heta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	) - )
n = 2000	1.000	0.412	0.124	0.136	0.120	0.118	0.117	0.107	0.082	0.000	0.000	
n = 4000	1.000	0.525	0.139	0.126	0.119	0.113	0.104	0.114	0.065	0.000	0.000	
n = 6000	1.000	0.573	0.128	0.121	0.128	0.116	0.111	0.101	0.072	0.000	0.000	
n = 8000	1.000	0.632	0.142	0.123	0.115	0.103	0.117	0.114	0.064	0.000	0.000	
n = 10000	1.000	0.644	0.132	0.130	0.117	0.112	0.116	0.103	0.062	0.000	0.000	
Table	16: Rejec	tion Frequ	encies usi	ng Numerio	cal Seco	nd Orde	er Deriv	ative 2. $\epsilon$	$n = n^{-1}$	$^{/2}$ . and	$\mathbb{Z}_n = N(0)$	$(\hat{\sigma})$
$\theta_n$	$-n^{-1/6}$	$-n^{-1/3}$	$-n^{-1/2}$	$-n^{-1/1.5}$	$-n^{-1}$	0	$n^{-1}$	$n^{-1/1.5}$	$n^{-1/2}$	$n^{-1/3}$	$n^{-1/6}$	, ~ <b>)</b>
n = 2000	1.000	0.969	0.260	0.087	0.057	0.056	0.050	0.031	0.013	0.000	0.000	
n = 4000	1.000	0.987	0.260	0.079	0.054	0.051	0.050	0.038	0.011	0.000	0.000	
n = 6000	1.000	0.995	0.255	0.079	0.059	0.051	0.045	0.028	0.010	0.000	0.000	
n = 8000	1.000	0.998	0.256	0.080	0.049	0.043	0.039	0.033	0.011	0.000	0.000	
n = 10000	1.000	0.999	0.266	0.072	0.049	0.056	0.051	0.029	0.008	0.000	0.000	