Toroidal compactifications of $\mathcal{A}_g$

$\mathbb{H}_g := \{\tau \in M(g\times g, \mathbb{C}) : \tau = {}^t\tau,\ \operatorname{Im}\tau > 0\}$
Siegel upper half space

$\tau \in \mathbb{H}_g \longrightarrow X_\tau := \mathbb{C}^g / \Lambda_\tau$ complex torus

$\Lambda_\tau :=$ lattice in $\mathbb{C}^g$ spanned by columns of τ and $E = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

$\Rightarrow X_\tau$ p.p.a.v.

Group action to get rid of choice of basis:

$Sp(2g, \mathbb{Z}) := \{\gamma \in GL(2g, \mathbb{Z}) : \gamma J_E {}^t\gamma = J_E\}$
acting on $\mathbb{H}_g$ with Möbius transformations $J_E = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (A\tau + BE)(C\tau + DE)^{-1}E$$

$\mathcal{A}_g := Sp(2g, \mathbb{Z}) \backslash \mathbb{H}_g$ coarse moduli space of p.p.a.v.

$\dim_{\mathbb{C}} \mathcal{A}_g = \frac{g(g+1)}{2}$, dimension of vector space of symmetric matrices
($\operatorname{Im}\tau > 0$ is an open condition, $Sp(2g, \mathbb{Z})$ is a discrete group)

Remarks

- Can treat other polarisations by replacing $E = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ with $E = \begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_g \end{pmatrix}$

- Can treat abelian varieties with level structures ($\hat{=}$ symplectic basis of $X[n]$) by replacing $Sp(2g, \mathbb{Z})$ with arithmetic subgroup

$n \in \mathbb{N}$ $\Gamma_g(n) := \ker(Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/n))$.

$n \geq 3 \implies \Gamma_g(n)$ torsion free

$\implies$ moduli space $\Gamma_g(n) \backslash \mathbb{H}_g$ of p.p.a.v. with level n structure is fine

Toy model: $g=1$

$Sp(2,\mathbb{Z}) = SL(2,\mathbb{Z})$

$$j: \overset{\circ}{X}(1) \xrightarrow{\sim} \mathbb{C}$$
$$\overset{\circ}{X}(1) = SL(2,\mathbb{Z})\backslash \mathbb{H}$$

Obvious compactification: $X(1) = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$, but no explanation why.

rational closure

$\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ has $SL(2,\mathbb{Z})$-action, $\mathbb{Q} \cup \{i\infty\}$ is one orbit,

$X(1) = SL(2,\mathbb{Z})\backslash \overline{\mathbb{H}}$

How to glue:

$P(i\infty) = \{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$ stabiliser of $i\infty$

$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot \tau = \tau + n$

$$\begin{array}{ccl} \mathbb{H} & \longrightarrow & P(i\infty)\backslash \mathbb{H} = D_1^* = \{z \in \mathbb{C} : 0 < |z| < 1\} \\ \tau & \longmapsto & t = e^{2\pi i \tau} \end{array}$$

$D_1^* \subset D_1$ partial compactification,

$X(1) = \overset{\circ}{X}(1) \cup_{D_\varepsilon^*} D_\varepsilon$, ε small enough so that $D_\varepsilon^* \to SL(2,\mathbb{Z})\backslash \mathbb{H}$ injective

Can generalise to arithmetic subgroups
$\Gamma \subset SL(2,\mathbb{Z})$ at expense of:

- $\mathbb{Q} \cup \{i\infty\}$ may consist of several Γ-orbits (always finitely many)
- each such orbit maps to $i\infty$ via $SL(2,\mathbb{Z})$
- replace $P(i\infty)$ by a subgroup

$\longrightarrow$ compactify $X^\circ(\Gamma) = \Gamma\backslash\mathbb{H}$ to $X(\Gamma)$, adding a finite number of cusps

Satake compactification of $\mathcal{A}_g$

$$\mathbb{H}_g = \{\tau \in M(g,\mathbb{C}) : {}^t\tau = \tau,\ \operatorname{Im}\tau > 0\}$$

$$\downarrow \wr \qquad \begin{array}{c}\tau \\ \downarrow \\ (\tau - iE)(\tau + iE)^{-1}\end{array} \qquad \text{Cayley transformation}$$

$$\mathbb{D}_g = \{Z \in M(g,\mathbb{C}) : {}^tZ = Z,\ Z\bar{Z} < E\}$$

bounded symmetric domain
$(Z \mapsto -Z)$

$$\mathbb{D}_g \subset \overline{\mathbb{D}_g} \subset \operatorname{Sym}(g,\mathbb{C})$$
$$\overline{\mathbb{D}_g} = \coprod_\alpha F_\alpha$$

$\coprod_\alpha F_\alpha$ boundary components

($p, q \in \overline{\mathbb{D}_g}$, $p \sim q \Longleftrightarrow$ ex. $\gamma_i : \mathbb{D} \to \overline{\mathbb{D}_g}$
with $\gamma_1(0) = p$,
$\gamma_m(0) = q$
$\gamma_i(\mathbb{D}) \cap \gamma_{i+1}(\mathbb{D}) \neq \emptyset$)

$Sp(2g,\mathbb{R})$-action on $\mathbb{D}_g$ extends to $\overline{\mathbb{D}_g}$, preserving boundary components

$$\overline{\mathbb{D}_g} = \{Z \in M(g,\mathbb{C}) : {}^tZ = Z,\ Z\bar{Z} \leq E\}$$

$$\{F \subset \overline{\mathbb{D}_g} \text{ boundary component}\} \xleftrightarrow{\sim} \{U \subset \mathbb{R}^{2g} \text{ real subspace}\}$$

$$\{Z \in \overline{\mathbb{D}_g} : U_{\mathbb{C}} = W \cap \bar{W}\} \longleftrightarrow U$$

$W \subset \mathbb{C}^{2g}$ spanned by columns of $\begin{pmatrix} Z+E \\ i(Z-E)\end{pmatrix}$

$U \triangleq$ radical of symmetric form Z

$U = 0 \Leftrightarrow$ full rank

Special components: $g' \le g$

$$F_{g'} := \left\{ \begin{pmatrix} z' & 0 \\ 0 & E_{g-g'} \end{pmatrix} : z' \in D_{g'} \right\} \simeq D_{g'}$$

$(\stackrel{\wedge}{=} \dim U = g'' := g - g')$

$$\overline{D_g} = \bigcup_{0 \le g' \le g} Sp(2g, \mathbb{R}) \cdot F_{g'}$$

$$D_g := \{ Z \in \mathrm{Sym}(g, \mathbb{C}) : Z\bar{Z} < E \} \underset{\text{Cayley}}{\simeq} \mathbb{H}_g$$

$\downarrow$ with $Sp(2g, \mathbb{R})$ - action

$$\bar{D}_g = \coprod_\alpha F_\alpha \quad \text{action respects boundary comp.}$$

$$Z \in \bar{D}_g \implies U(Z) := \ker\left(\mathbb{R}^{2g} \to \mathbb{C}^g, v \mapsto v\begin{pmatrix} i(E+Z) \\ E-Z \end{pmatrix}\right)$$

isotropic subspace of $(\mathbb{R}^{2g}, J)$

$$\implies \left\{ \begin{array}{l} \text{boundary} \\ \text{components } F_\alpha \end{array} \right\} \overset{1:1}{\longleftrightarrow} \{ U \subset \mathbb{R}^{2g} \text{ isotropic} \}$$

and $U(Z) \neq 0 \iff Z \in \bar{D}_g \setminus D_g$

F_α rational $\underset{\text{def}}{\iff}$ $P(F_\alpha) := \{ M \in Sp(2g, \mathbb{R}) : MF_\alpha = F_\alpha \}$ rational

$\iff$ $U(F)$ rational subspace

$$D_g^{\text{rat}} := \coprod_{\text{rational}} F_\alpha$$

$\mathcal{A}(\Gamma) := \Gamma \backslash \mathcal{D}_g \qquad \Gamma \subset Sp(2g, \mathbb{Q})$ arithmetic

Want to compactify $\mathcal{A}(\Gamma)$ in direction of a single cusp $F = F(U)$.

$P(F) \subset Sp(2g, \mathbb{R})$ stabiliser of F
parabolic subgroup

$$1 \longrightarrow P'(F) \longrightarrow P(F) \longrightarrow P''(F) \longrightarrow 1$$

$\| F = F_{g'} \qquad \| F = F_{g'}$

$$\left\{ \begin{pmatrix} E_{g'} & 0 & 0 & 0 \\ 0 & E_{g''} & 0 & b \\ 0 & 0 & E_{g'} & 0 \\ 0 & 0 & 0 & E_{g''} \end{pmatrix} : b = {}^t b \right\} \hookrightarrow \left\{ \begin{pmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} : \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in Sp(2g', \mathbb{R}), \ u \in GL(g'', \mathbb{R}) \right\}$$

centre of unipotent radical $W(F)$ of $P(F)$

$$W(F_{g'}) = \left\{ \begin{pmatrix} E_{g'} & 0 & 0 & n \\ {}^t m & E_{g''} & {}^t n & b \\ 0 & 0 & E_{g'} & -m \\ 0 & 0 & 0 & E_{g''} \end{pmatrix} : \begin{matrix} {}^t n m + b \\ \| \\ {}^t m n + {}^t b \end{matrix} \right\}$$

$\Rightarrow P'(F) \simeq Sym(g', \mathbb{R}), \quad g' = \dim U(F)$

$\Lambda(F) := P'(F) \cap \Gamma$ lattice of maximal rank in $P'(F)$

- $F = F(U)$, $U \subset \mathbb{R}^{2g}$

$\Rightarrow \overline{F} = \{ z \in \overline{D_g} : \overline{W} \cap \overline{W} \supset U_{\mathbb{C}} \}$

Hence: F, F' boundary components

$F < F' \overset{}{\underset{def}{\Longleftrightarrow}} F \subset \overline{F'} \Leftrightarrow U \supset U'$

- $F' < F'' \Longrightarrow$ ex. $M \in Sp(2g, \mathbb{R})$ s.th. $M \cdot F' = F_{g_1}$, $M \cdot F'' = F_{g_2}$

(same for longer chains)

- $F \subset \overline{D_g}$ is <u>rational</u> if
  (i) $P(F)$ defined over $\mathbb{Q}$
  $\Leftrightarrow$ (ii) $F = F(U)$, U defined over $\mathbb{Q}$
  $\Leftrightarrow$ (iii) $\exists M \in Sp(2g, \mathbb{Q}) : M \cdot F = F_{g'}$.

$$D^{rat} := \coprod_{F \text{ rational}} F \subset \overline{D_g}$$

involved definition of topology
(Satake topology $\neq$ cylindrical topology, but homeomorphic on quotient)

Thm: $\Gamma - Sp(2g, \mathbb{Q})$ arithmetic.

- $\Gamma \curvearrowright D^{rat}$ properly discontinuously
- $A_{g,\Gamma}^{sat} = \Gamma \backslash D^{rat}$ compact, with $A_g - A_g^{sat}$ open, dense
- $\Gamma \backslash D^{rat}$ projective alg. variety

Algebraic description.

$$A_{g,\Gamma}^{sat} = \mathrm{Proj}\left(\bigoplus_r M_r(\Gamma)\right)$$

$\hat{=}$ modular forms as homogeneous coordinate functions

Set-theoretically:

$$A_g^{sat} = A_g \amalg A_{g-1} \amalg \dots \amalg A_1 \amalg A_0$$

Moduli interpretation: (only one flag of isotropic subspaces for $Sp(2g, \mathbb{Z})$)

A_k = semiabelian varieties with split k-dim. abelian part

$F - \overline{D}_g$ boundary component

$P(F) := \{ M \in Sp(2g, \mathbb{R}) : MF = F \}$
parabolic subgroup w.r.t. F

$W(F) :=$ unipotent radical of $P(F)$

$U(F) :=$ centre of $W(F)$

$$P_{g'} = P(F_{g'}) = \left\{ \begin{pmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} : \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in Sp(2g', \mathbb{R}),\ u \in GL(g'', \mathbb{R}) \right\}$$

$$W_{g'} = W(F_{g'}) = \left\{ \begin{pmatrix} E_{g'} & 0 & 0 & n \\ {}^t m & E_{g''} & {}^t n & b \\ 0 & 0 & E_{g'} & -m \\ 0 & 0 & 0 & E_{g''} \end{pmatrix} : {}^t n m + b = {}^t m n + {}^t b \right\}$$

$$U_{g'} = U(F_{g'}) = \left\{ \begin{pmatrix} E_{g'} & 0 & 0 & 0 \\ 0 & E_{g''} & 0 & b \\ 0 & 0 & E_{g'} & 0 \\ 0 & 0 & 0 & E_{g''} \end{pmatrix} : {}^t b = b \right\}$$

Intrinsic description:

A non-degenerate skew-symmetric form on $\mathbb{R}^{2g}$ ("symplectic"), naturally extends to $\mathbb{C}^{2g}$,

$F(x,y) := i A(x, \bar{y})$ hermitian form

$$G/K = D_g := \{ W \in \mathrm{Grass}(g, \mathbb{C}^{2g}) : A|_W = 0, \ F|_W > 0 \}$$

$\supset$ open

$$G_{\mathbb{C}}/B = D_g^c := \{ W \in \mathrm{Grass}(g, \mathbb{C}^{2g}) : A|_W = 0 \} \quad \text{Lagrangian Grassmannian}$$

$\supset$ closed

$$\mathrm{Grass}(g, \mathbb{C}^{2g})$$

$$G = Sp(2g, \mathbb{R})$$
$$K = (SO(iF)) \cong U(g)$$
$$G_{\mathbb{C}} = Sp(2g, \mathbb{C})$$
$$B$$

$\overline{D_g} \subset D_g^c$ topological closure

$\cup$

$$F(K) := \{ W \in \mathrm{Grass}(g, \mathbb{C}^{2g}) : A|_W = 0, \ F|_W \geq 0, \ W \cap \overline{W} = U_{\mathbb{C}} \}$$

Get a torus bundle associated to F:

$$\begin{array}{c} \mathcal{X}(F) \\ \Big\downarrow T = \Lambda_{\mathbb{C}}/\Lambda \simeq (\mathbb{C}^\times)^{g'} \\ F \times V(F) \end{array}$$

$V(F) := W(F)/P'(F)$
abelian, affine
$\Rightarrow$ vector space

Compactifying the cusp at F:

(1) $X(F) := \Lambda(F) \backslash D_g \subset \mathcal{X}(F)$ open

$$\begin{array}{c} X(F) \\ \Big\downarrow (\mathbb{C}^\times)^{g'} \\ \mathcal{U} \subset \mathbb{C}^{\frac{1}{2}g(g+1)-g'} \end{array}$$

(2) choose complete fan in $P'(F) \simeq \mathrm{Sym}(g';\mathbb{R})$, and use it to construct a trivial bundle

$$\begin{array}{c} X_\Sigma(F) \\ \Big\downarrow X(\Sigma) \text{ proper toric variety} \\ \mathcal{U} \end{array}$$

(3) Σ compatible with action of $(\overset{P''(F)}{\overset{\shortparallel}{P(F) \cap \Gamma}})/\Lambda(F)$
$\Rightarrow$ action on $\mathcal{X}(F)$ extends to $\mathcal{X}_\Sigma(F)$

(4) $X_\Sigma(F) :=$ int (closure of $X(F)$ in $\mathcal{X}_\Sigma(F)$)
$Y_\Sigma(F) := P''(F) \backslash X_\Sigma(F)$ partial compactification

Toric varieties

- Demazure 1970: full description of $Aut(X(\Sigma))$ in terms of Σ (X proper, smooth), → classification of alg. subgroups of the Cremona groups
- Miyake, Oda 1973; Kempf, Knudson, Mumford, Saint-Donat 1973: algebraic varieties with full torus actions
- Satake 1973; Ash, Mumford, Rapoport, Tai 1975: toroidal compactifications of domains

- toric varieties are defined in purely combinatorial terms (cones & fans in lattices) ⇒ algebraic geometry ↔ convex geometry
- nice testing case, but toric varieties are special: they are "very rational", e.g. are rational varieties, all singular cohomology is algebraic, toric singularities are always rational.
- link to symplectic geometry via moment maps and moment polytopes

Toric varieties as torus embeddings

Def: Normal variety X is <u>toric</u> if

- $T \subset X$ dense, open torus
- $T \curvearrowright X$, T acts on X extending action on T

Ex: $X = \mathbb{A}^2 \supset T = (\mathbb{C}^*)^2$ coordinate-wise action

orbits: $O_0 = \{(x_1, x_2) : x_1 \neq 0, x_2 \neq 0\} = T$

$O_1 = \{(0, x_2) : x_2 \neq 0\} \cong \mathbb{C}^*$

$O_{1'} = \{(x_1, 0) : x_1 \neq 0\} \cong \mathbb{C}^*$

$O_2 = \{(0,0)\}$ fixed point

closures: $\overline{O}_2 = O_2$

$\overline{O}_1 = O_1 \cup O_2 \cong \mathbb{A}^1$ T-invariant divisor

$\overline{O}_0 = O_0 \cup O_1 \cup O_{1'} \cup O_2$

Ex: $X = \mathbb{P}^2 \supset T = \{(u_0 : u_1 : u_2) : u_i \neq 0\} \cong (\mathbb{C}^*)^2$

orbits: T open, dense orbit

$\{(u_0 : 0 : 1) : u_0 \neq 0\}$ etc., three orbits of dimension 1

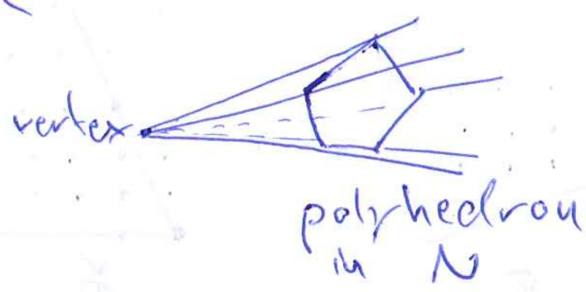
$(1:0:0), (0:1:0), (0:0:1)$ three fixed points

Toric varieties from fans of cones

Affine toric varieties described by (convex rational polyhedral) cones

$\sigma \subset \mathbb{R}^n = N \otimes_{\mathbb{Z}} \mathbb{R}$

$N \cong \mathbb{Z}^n$ lattice $= \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{C}^*, T)$



$\rightsquigarrow$

$\check{\sigma} := \{ u \in M_{\mathbb{R}} : \langle u, v \rangle \geq 0 \ \forall v \in \sigma \}$ dual cone, again convex, rational polyhedral

$M_{\mathbb{R}} = N^*_{\mathbb{R}}$

$\rightsquigarrow$

$\mathbb{C}[\check{\sigma} \cap M]$ semigroup ring (lattice point = monomial)

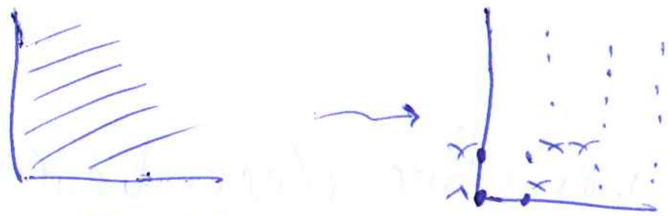
$\rightsquigarrow$

$X(\sigma) := \mathrm{Spec}\, \mathbb{C}[\check{\sigma} \cap M]$ affine variety associated to σ

$\mathbb{C}[\check{\sigma} \cap M] \subset \mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$\Rightarrow \mathrm{Spec}\, \mathbb{C}[M] = T \hookrightarrow X(\sigma)$ dense

Ex:



$\rightsquigarrow \mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}[x,y]$

$\rightsquigarrow X(\sigma) = \mathbb{A}^2$

Ex:

$\rightsquigarrow$ semigroup generators for $\check{\sigma} \cap M$ are x, xy, xy^2, xy^3

$\langle (1,3), (3,-1) \rangle = 0$

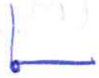
$\Longrightarrow \mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}[x, xy, xy^2, xy^3]$

$\Longrightarrow X(\sigma)$ cone over rational quartic $\mathbb{P}^1 \hookrightarrow \mathbb{P}^4$

Rem:  $\hat{=}\ \mathbb{A}^2$ has subcones (faces)

 $\hat{=}\ T$,

 $\hat{=}\ \{0\} \times \mathbb{C}^*$  $\hat{=}\ \mathbb{A}^1 \times \mathbb{C}^*$

 $\hat{=}\ \mathbb{C}^* \times \{0\}$  $\hat{=}\ \mathbb{C}^* \times \mathbb{A}^1$

 $\hat{=}\ \{(0,0)\}$

$\Longrightarrow$ set of cones (fan) encodes orbit structure of torus action

<u>Facts:</u>

- $$\tau \subset \sigma \iff \check{\tau} \supset \check{\sigma}$$
$$\iff \mathbb{C}[M] \supset \mathbb{C}[\check{\tau} \cap M] \supset \mathbb{C}[\check{\sigma} \cap M]$$
$$\iff \begin{array}{ccc} X(\tau) & \longrightarrow & X(\sigma) \\ \downarrow & & \downarrow \\ T & = & T \end{array}$$

- $\tau \prec \sigma$ face $\iff X(\tau) \hookrightarrow X(\sigma)$ open

<u>Cor:</u> Can patch affine toric varieties $X(\sigma)$ together for fan $\Sigma \ni \sigma$.
$\rightarrow X(\Sigma)$.

Sumihiro 1974:

$$\{\text{finite fans}\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{toric varieties}\\ \text{of finite type}\end{array}\right\}$$
$$\Sigma \longmapsto X(\Sigma)$$

(Follows from: $T \curvearrowright X$ algebraic torus action, X normal of finite type

$\Longrightarrow X$ is covered by affine T-invariant subsets.)

From torus embedding to fan:

$\Sigma \xrightarrow{\sim} \{T\text{-orbits in } X(\Sigma)\}$

$\sigma \longmapsto O_\sigma$

- $\tau < \sigma \iff \overline{O_\tau} \supset O_\sigma$
- $\dim O_\sigma + \dim \sigma = \dim T$
- $O_{\{0\}} = T$

Dictionary algebraic geometry $\leftrightarrow$ combinatorics:

- $X(\Sigma)$ nonsingular
  $\iff$ all $\sigma \in \Sigma$ generated by parts of $\mathbb{Z}$-bases for N
- $X(\Sigma)$ proper $\iff$ Σ finite, $N_{\mathbb{R}} = \bigcup_{\sigma \in \Sigma} \sigma$
- $X(\Sigma)$ quasi-projective
  $\iff$ ex. strictly convex support function for Σ

Reduction theory of positive quadratic forms

Namikawa §8

- $\mathcal{Y}_g = \{ y \in M(g,\mathbb{R}) : {}^t y = y \} \circlearrowleft G := GL(g,\mathbb{R})$:
  $u \cdot y := u y {}^t u$

$$\cup\!\!\vert$$

$\overline{\Omega} := \overline{\mathcal{Y}_g^+} = \{ y \in \mathcal{Y}_g : y \geq 0 \}$ = closure of G-orbit of $E_g = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

$$\cup\!\!\vert$$

$\Omega := \mathcal{Y}_g^+ = \{ y \in \mathcal{Y}_g : y > 0 \}$ = G-orbit of $E_g = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

$\mathcal{Y}_g \times \mathcal{Y}_g \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{R}$, $(y_1, y_2) \mapsto \mathrm{tr}(y_1 y_2)$

non-degenerate symmetric bilinear form

- $\overline{\Omega}$ self-dual for $\langle\cdot,\cdot\rangle$:
  $\overline{\Omega} = \{ y \in \mathcal{Y}_g : \langle y, y' \rangle \geq 0 \ \forall y' \in \overline{\Omega} \}$

- boundary components:
  $\overline{\Omega} = \coprod_{W \subset \mathbb{R}^g} \Omega(W)$ with $\Omega(W) := \{ y \in \overline{\Omega} : \ker y = W \}$ prescribed radical of the quadratic form

  $$\cup\!\!\vert$$

  $\Omega \triangleq W = 0$

special components:

$\Omega_{g''} := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y'' \end{pmatrix} : y'' \in \mathcal{Y}_{g''}^+ \right\} = \Omega(W_{g''})$,

$W_{g''} = \{ (\underbrace{*, \dots, *}_{g'}, \underbrace{0, \dots, 0}_{g''}) \}$

$\dim W = g' \Leftrightarrow \exists_{u \in G} \ u\,\Omega(W)\,{}^t u = \Omega_{g''}$

- lattices in $\mathcal{Y}_g$:

$N = \Lambda_g := \mathcal{Y}_g \cap M(g, \mathbb{Z})$ integer matrices

$M = \Lambda_g' := \{ \gamma \in \mathcal{Y}_g : 2\gamma \in \Lambda_g, \gamma_{ii} \in \mathbb{Z} \}$ half-integer matrices

dual lattices in $\mathcal{Y}_g$:

$\Lambda_g' = \{ \gamma' \in \mathcal{Y}_g : \langle \gamma', \gamma \rangle \in \mathbb{Z} \ \forall \gamma \in \Lambda_g \}$,

$\Lambda_g = \{ \gamma \in \mathcal{Y}_g : \langle \gamma', \gamma \rangle \in \mathbb{Z} \ \forall \gamma' \in \Lambda_g' \}$

E.g. for $g = 2$:

$$\operatorname{tr}\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} a' & \frac{1}{2}b' \\ \frac{1}{2}b' & c' \end{pmatrix} \right) = \operatorname{tr} \begin{pmatrix} aa' + \frac{1}{2}bb' & \\ \cdot & \frac{1}{2}bb' + cc' \end{pmatrix}$$
$$= aa' + bb' + cc' \in \mathbb{Z}$$

Λ_g, Λ_g' preserved by $G_{\mathbb{Z}} = GL(g, \mathbb{Z})$.

- $\Omega(W)$ <u>rational</u> $\Leftrightarrow$ W defined over $\mathbb{Q}$
$\Leftrightarrow \exists u \in G_{\mathbb{Q}} : u \Omega(W)^t u = \text{"}\Omega_g\text{"}$
$\Leftrightarrow \exists u \in G_{\mathbb{Z}} : u \Omega(W)^t u = \text{"}\Omega_g\text{"}$

$$\Omega^{rat} := \coprod_{W \text{ rational}} \Omega(W)$$ <u>rational closure of Ω</u>

Toroidal compactification

$\triangleq$ fan Σ in $N_{\mathbb{R}}$ s.t. "admissible"

- $\operatorname{supp} \Sigma = \cancel{N_{\mathbb{R}}}\ \bar{C}$ (compactness)
- $GL(g,\mathbb{Z})$ preserves Σ
- $GL(g,\mathbb{Z})$ acts on Σ with finitely many orbits

$$N = \Lambda_g, \quad M = \Lambda_g', \qquad N_{\mathbb{R}} = \Lambda_{\mathbb{R}} = \mathcal{Y}_g$$

$$C = \Omega = \{ y \in N_{\mathbb{R}} : y^2 > 0 \} \quad \text{open cone}$$

$$\begin{array}{ccc} C & = & \Omega \\ \| & & ? \\ \bar{C} & = & \Omega^{\mathrm{rat}} \end{array}$$

Central cone decomposition

$\gamma \in \Omega^{rat} \Rightarrow \mu(\gamma) := \min_{a \in \Omega \cap \Lambda'_g} \mathrm{tr}(a\gamma) > 0$

$a \in \Omega \cap \Lambda'_g \Rightarrow c(a) := \{\gamma \in \Omega^{rat} : \mu(\gamma) = \mathrm{tr}(a\gamma)\}$

(cone ~~of this finite set~~)

$\Sigma_{cent} := \{c(a) : a \in \Omega \cap \Lambda'_g\}$

Fact: $\gamma \in \Omega^{rat} \Rightarrow \min_{u \in GL(g,\mathbb{Z})} \mathrm{tr}(u a {}^t u \gamma) > 0$

and attained with some u

Wrong with $\gamma \in \overline{\Omega} \setminus \Omega^{rat}$!

$c > 0 \rightarrow \{\alpha \in \Omega : \mathrm{tr}(\alpha\gamma) < c\}$ bounded

$\Rightarrow \{\alpha \in \Lambda \cap \Omega : \mathrm{tr}(\alpha\gamma) < c\}$ finite.

$a \in \Omega \cap \Lambda_g^1$ <u>central</u> $\Longleftrightarrow$ $c(a)$ of maximal dimension

Alternative description:

$Q := \mathrm{Conv}(C \cap (M \setminus 0))$ $\quad C = \Omega$

infinite polyhedron, $\quad M = \Lambda_g^1$

faces are finite polytopes,

vertices = "central quadratic forms"

Σ_{cent} = dual fan of Q

Perfect cone decomposition

$y \in \Omega \rightarrow \mu(y) := \min_{0 \neq v \in \mathbb{Z}^g} \underbrace{v y {}^t v}_{\operatorname{tr}({}^t v v\, y)}$ minimal integer norm of the form y

$$M(y) := \{ v \in \mathbb{Z}^g : v y {}^t v = \mu(y) \}$$

vectors which realise the minimum

Fact: $M(y) \neq \emptyset$ finite

($C \in \mathbb{R}_{>0} \Rightarrow \{ \alpha \in \Omega : \operatorname{tr}(\alpha y) < C \} - \mathcal{Y}_g$ bounded

$\xRightarrow{\Lambda \cap \Omega \text{ discrete in } \Omega}$ $\{ \alpha \in \Lambda \cap \Omega : \operatorname{tr}(\alpha y) < C \}$ finite

$$\sigma(y) := \sum_{v \in M(y)} \mathbb{R}_{\geq 0} {}^t v v \quad \text{— cone in } \Omega = \Lambda_{\mathbb{R}},$$

with generators (rays) ${}^t v v$

$$\Sigma_{perf} := \{ \sigma(y) : y \in \Omega \}$$

$\gamma \in \Omega$ perfect form

$\Leftrightarrow$ $\sigma(\gamma)$ has maximal dimension

$\Leftrightarrow$ $\sigma^\vee$ generated by a quadratic function determined (up to multiple) by its minimal non-zero integral vectors ("perfect form")

$\Leftrightarrow$ $\sigma(\gamma) = \sigma(\gamma') \Leftrightarrow \gamma = \lambda\gamma', \lambda \in \mathbb{R}_{>0}$

Other description:

$\Sigma_{perf} = \{$ cones over faces of $\underbrace{\mathrm{Conv}(N \cap (\overline{\Omega}^{\mathrm{rat}} \setminus 0))}_{\text{polytope}}\}$

fan polytope $\rightsquigarrow$ projective

$\Gamma = Sp(g, \mathbb{Z})$

Σ admissible fan for $\overline{Y_g^+}$:

- each $\sigma \in \Sigma$ generated by a finite number of integral, positive, semi-definite matrices
- $u \in GL(g, \mathbb{Z})$, $\sigma \in \Sigma$ $\Rightarrow$ $u \sigma {}^t u \in \Sigma$
- $g' < g$, $Y_{g'} \hookrightarrow Y_g$, $x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$
  $\Rightarrow \bigcup_{\sigma \in \Sigma} \sigma = \bigcup_{u \in GL(g,\mathbb{Z})} u \left(\coprod_{g' \leq g} Y_{g'}^+ \right) {}^t u$
- Σ mod $GL(g, \mathbb{Z})$ finite

Build $(\Gamma \backslash \mathbb{H}_g)^{\Sigma}$ from this, with stratification $\{X(\bar\sigma)\}_{\bar\sigma \in \Sigma / GL(g,\mathbb{Z})}$ s.t.

- $X_{\bar i} \subset X_{\bar j} \Leftrightarrow$ ex. $\sigma_i > \sigma_j$, $\bar\sigma_i = \bar\imath$, $\bar\sigma_j = \bar\jmath$
- $\dim X_{\bar i} + \dim_{\sigma_i} = \dim \mathbb{H}_g$ $(= g(g+1)/2)$
- $\sigma_i = \{0\} \Leftrightarrow X_{\bar i} = \Gamma \backslash \mathbb{H}_g$
-

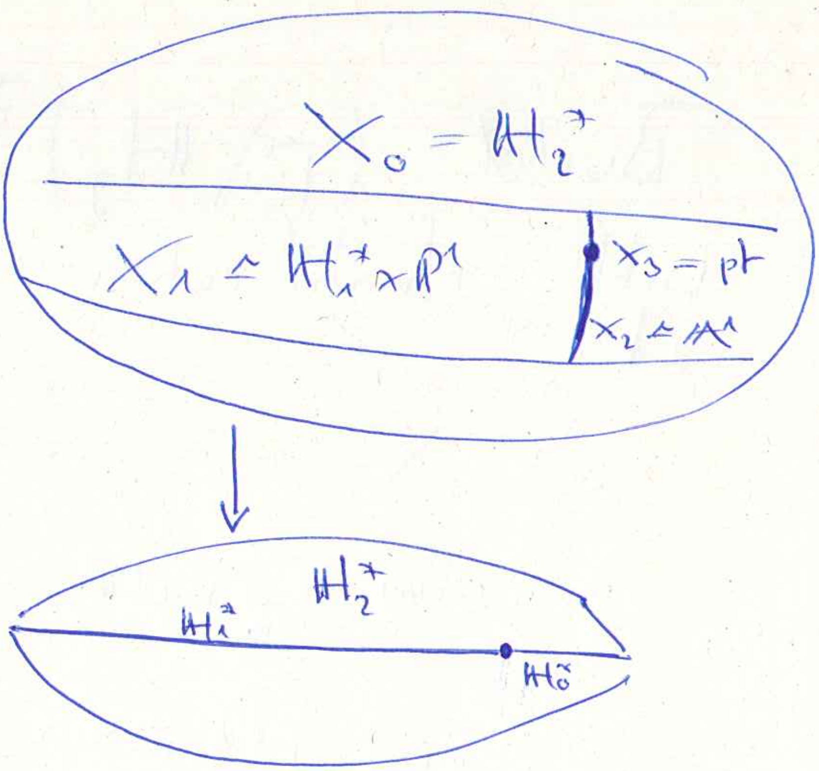
Example: $g = 2$

- $\sigma_0 = \{0\}$
- $\sigma_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \geq 0 \right\}$
- $\sigma_2 = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \geq 0 \right\}$
- $\sigma_3 = \left\{ \begin{pmatrix} \lambda_1+\lambda_3 & -\lambda_3 \\ -\lambda_3 & \lambda_2+\lambda_3 \end{pmatrix} : \lambda_i \geq 0 \right\}$

$$\Sigma = GL(2,\mathbb{Z}) \cdot \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$$

$$\Sigma / GL(2,\mathbb{Z}) = \{\sigma_{\bar 0}, \sigma_{\bar 1}, \sigma_{\bar 2}, \sigma_{\bar 3}\}$$

$$\begin{array}{c} (\Gamma \backslash \mathbb{H}_2)^{\Sigma} \\ \Big\downarrow \\ (\Gamma \backslash \mathbb{H}_2)^{sat} \end{array}$$



$\dim \mathbb{H}_2^* = 3$
$\dim \mathbb{H}_1^* = 1$

$\mathbb{H}_i^* := \Gamma \backslash \mathbb{H}_i$

$\underline{g=2}$ (Legendre)

$$F := \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} : 0 \leq 2c \leq a \leq b \right\}$$

$$= \left\{ (b-a)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (a-2c)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right\}$$

fundamental domain in Ω for $GL(2,\mathbb{Z})$-action

$$\sigma_0 = \left\{ \begin{pmatrix} \lambda_1+\lambda_3 & -\lambda_3 \\ -\lambda_3 & \lambda_2+\lambda_3 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\}$$

$$= \left\{ \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\},$$ (principal) $\underline{\text{cone}}$

$$GL(2,\mathbb{Z}) \cdot \sigma_0 = \Omega^{rat}$$

$$\mathrm{Iso}(\sigma_0) = \{ u \in GL(2,\mathbb{Z}) : u\sigma_0 {}^t u = \sigma_0 \}$$

$$= \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$

extension of S_3 by $\{\pm 1\}$

($S_3 \curvearrowright \{\lambda_1, \lambda_2, \lambda_3\}$ coordinates for $\mathcal{Y}_2$ w.r.t. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and S_3 preserves σ_0)

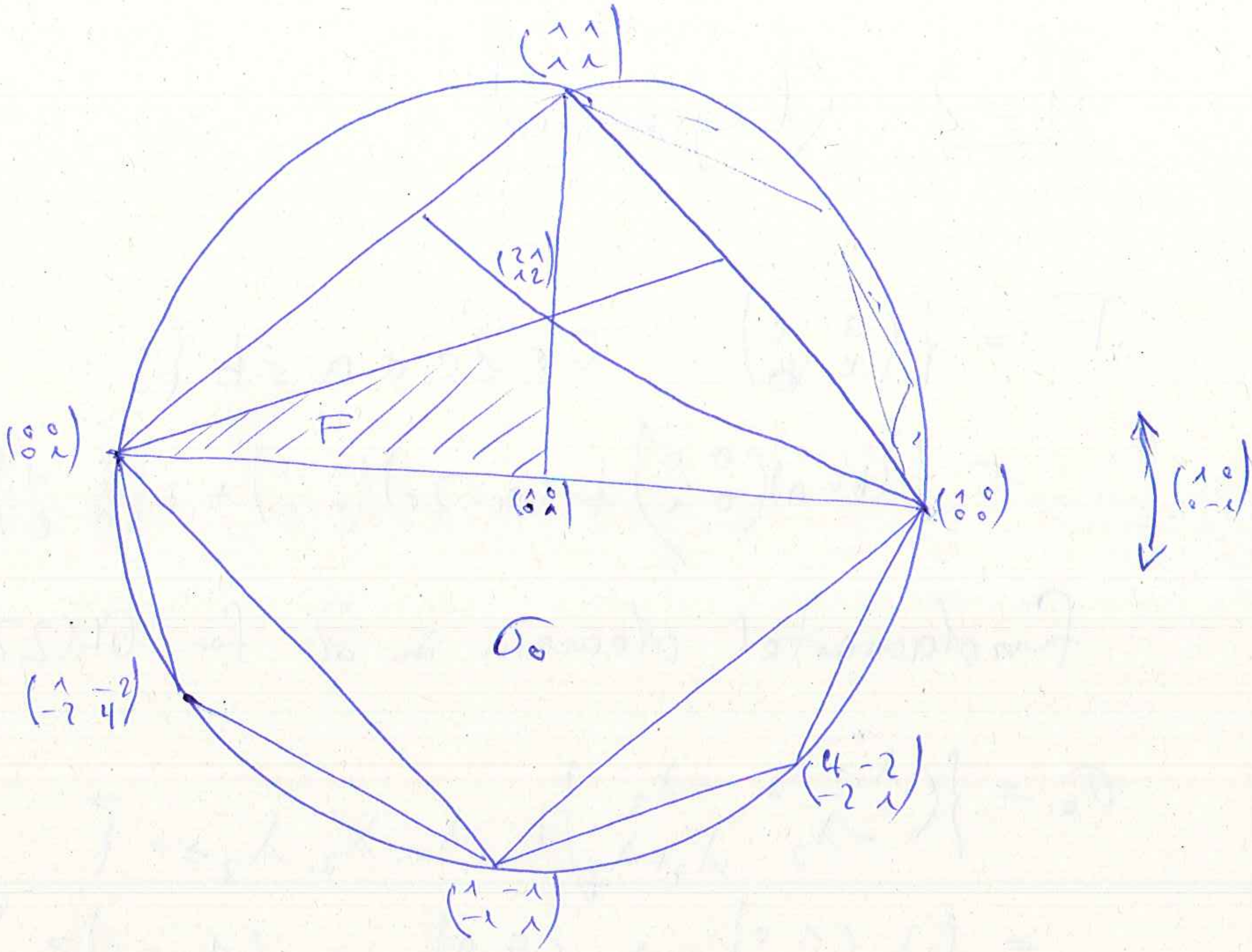
$$\Omega = \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} : a > 0,\ ab > c^2 \right\}$$

$$\cap$$

$$\Omega^{rat}$$

$$\cap$$

$$\overline{\Omega} = \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} : a \geq 0,\ ab \geq c^2 \right\}$$



Ω and $\bar{\Omega}$, up to scalars