Decompositions of the cone of positive semi-definite forms

Recap/Introduction

We are interested in decompositions of $\overline{\Omega}^{rat}$:= rational closure of the cone of pos. def ~~sym~~ matrices $y \in \mathrm{Sym}^2(g,\mathbb{R}) \subseteq \mathbb{R}^{\binom{g+1}{2}}$

$y \in \overline{\Omega}^{rat} \Leftrightarrow \ker y$ rational

We have a $G_{\mathbb{Z}} := GL(g,\mathbb{Z})$-action on $\overline{\Omega}^{rat}$: $u \cdot y := u y {}^t u$

We want decompositions given by a fan $\Sigma = \{\sigma_i\}$ s.t.

- σ_i are rational polyhedral, convex cones
  (Note that we have the lattice $N = \mathrm{Sym}^2(g,\mathbb{Z}) \subseteq \mathbb{R}^{\binom{g+1}{2}}$ giving the rational structure)
- $\forall u \in G_{\mathbb{Z}}, \sigma_i \in \Sigma$: $u \cdot \sigma_i \in \Sigma$ ($G_{\mathbb{Z}}$-invariance of Σ)
- $\#\{\sigma_i\}/G_{\mathbb{Z}} < \infty$ (finiteness of Σ modulo $G_{\mathbb{Z}}$)
- the σ_i cover $\overline{\Omega}^{rat}$: $\bigcup_i \sigma_i = \overline{\Omega}^{rat}$.

admissible decomposition

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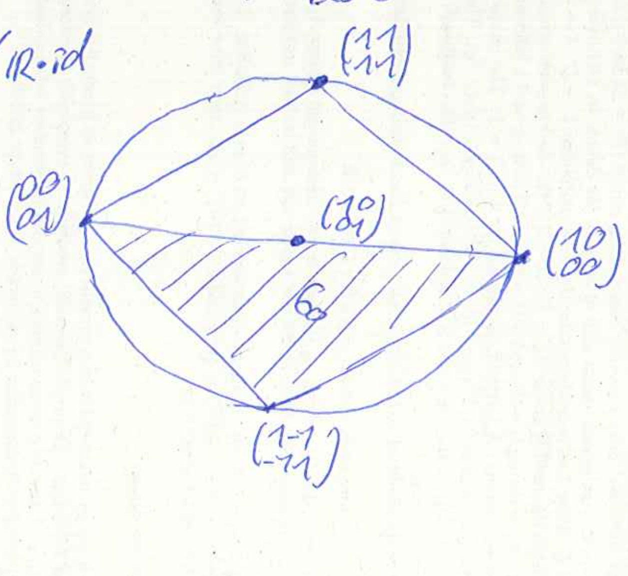
We shall discuss two examples of such decompositions and illustrate them for $g=2$.

$g=2$ We are considering symmetric 2x2 matrices as elements in $\mathbb{R}^3$. We draw a picture in $\mathbb{R}^2 := \mathrm{Sym}^2(2,\mathbb{R})/\mathbb{R}\cdot \mathrm{id}$



By a general result we have

$G_{\mathbb{Z}} \cdot \sigma_0 = \overline{\Omega}^{rat}$

* **Thm** For every admissible decomp. Σ we obtain a compactification A_g^{Σ} together with a morphism $A_g^{\Sigma} \to A_g^{sat}$ which is the identity on A_g.

The perfect cone (or 1st Voronoi) decomposition

Let $y \in \Omega = \mathrm{Sym}^2(g, \mathbb{R})$. We consider y as a norm $\mathbb{R}^g \to \mathbb{R}$.

Set $\mu(y) := \min_{\xi \in \mathbb{Z}^g \setminus \{0\}} y(\xi)$ and $M(y) := \{ \xi \in \mathbb{Z}^g \mid y(\xi) = \mu(y) \}$.

One can show that $M(y)$ is a finite non-empty set of lattice points $\xi \in \mathbb{Z}^g \setminus \{0\}$.

We define the cone assoc. with y: $\sigma(y) := \mathrm{conv}\{ \mathbb{R}_0^+ ({}^t\xi \cdot \xi) \mid \xi \in M(y) \}$, where ${}^t\xi \cdot \xi$ is the rank one form assoc. with $\xi \in \mathbb{Z}^g \setminus \{0\}$.

The <u>perfect cone decomposition</u> is given by $\Sigma_{\mathrm{perf}} := \{ \sigma(y) \mid y \in \Omega \}$

<u>Thm</u> Σ_{perf} is admissible.

A form $y \in \Omega$ is called <u>perfect</u>, if $\dim \sigma(y) = \binom{g+1}{2}$ (= maximal)

Roughly speaking we have: the bigger $\sigma(y)$, the less directions we have to move y in Ω s.t. $s(y)$ stays the same.

$\rightsquigarrow$ y perfect $\Leftrightarrow$ $\{ y' \mid \sigma(y) = \sigma(y') \} = \mathbb{R}^+ \cdot y$ *

For $g=2$ there is (up to the $G_{\mathbb{Z}}$-action) a unique perfect form: $y_0 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$, $\mu(y_0) = 1$

$\mu(y_0)$ is attained for $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\qquad$ $y_0 \leftrightarrow$ quadr. form $x^2 + xy + y^2$

corresponding rank one forms: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $\rightsquigarrow$ $\sigma(y_0) = \sigma_0$ (in our picture)

Furthermore we may look at $y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \sigma(y_1) = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$, $y_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \rightsquigarrow \sigma(y_2) = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rangle$

Thus the perfect cone decomp. for $g=2$ is given by σ_0, its faces (and $G_{\mathbb{Z}}$-translates).

* Extremal rays of a cone $\sigma(y)$ correspond to rank one forms $({}^t\xi \cdot \xi)$, $\xi \in M(y)$

The central cone decomposition

Now let $y \in \overline{\Omega}^{rat}$ ← ! and set $\mu'(y) = \min_{a \in \Omega \cap M} \langle a, y \rangle$.

(Recall that on $\mathrm{Sym}^2(g, \mathbb{R})$ we have a non-deg bilinear form given by $\langle y, y' \rangle := \mathrm{tr}(y \cdot y')$.
Furthermore we defined M to be the dual lattice of N given by matrices with half-integer entries and integers on the diagonal.)

<u>Lemma</u> $\mu'(y) > 0$.

We define $C(a) := \{ y \in \overline{\Omega}^{rat} \mid \langle a, y \rangle = \mu'(y) \}$ and $\Sigma_{centr} = \{ C(a) \mid a \in \Omega \cap M \}$ <u>central cone decomposition</u>.

<u>Thm</u> Σ_{centr} is admissible, $\mu'(y)$ is a polar fct., which immediately yields the projectivity of the assoc. compactification.

A form $a \in \Omega \cap M$ is called <u>central</u>, if $C(a)$ is of max. dimension

(Note that now ~~a~~ the forms a are in a <u>discrete</u> set, so no "moving" possible.)

<u>Fact</u> The central cone ~~decomp.~~ compactification is the normalisation of the blow-up of the Satake compactification along the boundary.

<u>$g=2$</u> We have a unique central form: $a_0 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \rightsquigarrow C(a_0) = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rangle = \sigma_0$

$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $C(a_1) = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$ o o o o.

We obtain the same decomposition as in the case of perfect cones.