Extension Criteria

Jesse Kass (based on explanation by Klaus Hulek)

References: 1) Namikawa's book Toroidal Compactifications of Siegel Space. (esp. pages 77-84)

2) Namikawa's thesis On the canonical map from the moduli space of stable curves to the Igusa monoidal transform

2) Namikawa's paper A new compactification of the Siegel space and degeneration of abelian varieties.

4) Alexeev + Brunyate's paper Extending the Torelli map to toroidal compactifications of Siegel space.

Want to explain a criteria for the Torelli map

$$t: M_g \longrightarrow A_g$$

to extend to a map

$$\bar{t}: \bar{M}_g \longrightarrow \bar{A}_g^{\Sigma}$$

from $\bar{M}_g$ to a given toroidal compactification of A_g.

To begin, let us observe that the analogous question for the Satake compactification A_g^* has a very nice answer:

$\boxed{\text{Thm}}$ The Torelli map extends to a map

$$t^*: \bar{M}_g \longrightarrow A_g^*.$$

This is a consequence of a general extension thm for arithmetic quotients of symmetric spaces. Recall

A_g is the quotient of $D = \mathbb{H}_g$ (a bounded symmetric domain)

by $\Gamma = Sp(2g, \mathbb{Z})$ (an arithmetic group)

The Satake compactification is defined to be the quotient of D^* = rational closure by Γ.

We have the following general extension theorem of Borel/Kobayashi-Ochiai:

$\boxed{\text{Extension Thm}}$ Let D = bounded symmetric domain, Γ = arithmetic group acting on D.

Then every holomorphic map

$$f: (\Delta^\times)^a \times \Delta^b \longrightarrow D/\Gamma$$

extends to a regular map

$$f^*: \Delta^a \times \Delta^b \longrightarrow (D/\Gamma)^*.$$

The proof is an argument w/ cplx geometry, Siegel sets, arithmetic grps,...

The Extension Theorem for the Satake compactification immediately implies that the Torelli map extends. Indeed, the problem of showing that t extends is local on $\overline{M}_g$, and given $[X] \in \overline{M}_g$, there is a neighborhood of $[X]$ of the form $(\Delta^a \times \Delta^b)$/finite group. Here a = # of nodes of X, $b = 3g-3-a$, and $\Delta^a \times \Delta^b$ is constructed so $\exists$ a miniversal family of curves

$$\begin{array}{c} X_t \\ \downarrow \\ \Delta^a \times \Delta^b \end{array}$$

s.t. $\Delta \times \dots \times \Delta \times \{0\} \times \Delta \times \dots \times \Delta \times \Delta^b$ (ith position) is the locus where the ith node $p_i \in X_0$ remains a node.

Locally the Torelli map lifts to a map $t: (\Delta^\times)^a \times \Delta^b \longrightarrow A_g$. This lifted map extends by the Extension theorem. The extended map is invariant under the action of the relevant finite group, so we get the desired extension

$$\overline{M}_g \supseteq \frac{\Delta^a \times \Delta^b}{\text{finite group}} \longrightarrow A_g^*.$$

$\boxed{\text{Remark}}$ The fact that the Torelli map lifts from $\Delta^a \times \Delta^b$/finite group to $\Delta^a \times \Delta^b$ is essentially the assertion that the ~~map~~ ~~Torelli~~ $t: \overline{M}_g$ = coarse moduli space of stable curves $\longrightarrow A_g^*$ lifts to a map out of the moduli stack. In this write-up, we will not treat stack-theoretic issues carefully, but it would be nice to have such a write-up as most of the references we are using were written ~~before stacks came into~~ when the theory of stacks was less well-developed.

Remark Namikawa first proved the existence of t^*. He did not use the Borel / Kobayashi-Ochiai Extension Theorem, but rather proved the result by computing directly with period matrices.

His work shows that if $[X] \in \overline{M}_g$, then $t^*([X]) \in A_h \subseteq A_g^*$ is the product of the Jacobians of the connected components of the normalization $\tilde{X}$ (= max'l abelian quotient of the generalized Jacobian of X).

The Torelli map does not always extend to a map into $\bar{A}_g^{\Sigma}$, but there is a useful combinatorial criteria for the map to extend.

Recall: N_g = lattice of $g \times g$ integral symmetric matrices
= lattice of integral quadratic forms on $\mathbb{Z}^g$

M_g = lattice of $g \times g$ ½-integral symmetric matrices
= lattice of $\underbrace{\text{½-integral}}_{(= \text{integer valued})}$ quadratic forms on $\mathbb{Z}^g$

X = stable curve

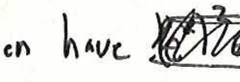
Γ_X = dual graph

$N(\Gamma)$ = lattice of integral quadratic forms on $H_1(\Gamma_X, \mathbb{Z})$

$M(\Gamma)$ = lattice of ½-integral quadratic forms on $H_1(\Gamma, \mathbb{Z})$

If e is an edge of Γ_X, we write e^* for the functional $e^*: H_1(\Gamma_X, \mathbb{Z}) \to \mathbb{Z}$ that sends a chain to the coefficient of e.

We then have $(e^*)^2 \in N(\Gamma)$.

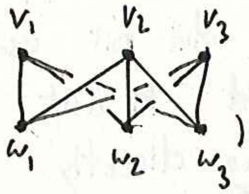
Example $\Gamma =$ 

Write $(v,w)_1$, $(v,w)_2$, $(v,w)_3$ for the 3 edges oriented from v to w.

$H_1(\Gamma, \mathbb{Z}) = \langle b_1 = (vw)_1 + (wv)_2,\ b_2 = (vw)_1 + (wv)_3 \rangle$

~~$\langle (vw)_1 + (wv)_2,\ (vw)_1 + (wv)_2 \rangle$~~

The functionals $(vw)^*_1$, $(vw)^*_2$, $(vw)^*_3$ are $X_1+X_2, -X_1, -X_2$ respectively. The quadratic form $((vw)^*_1)^2$, $((vw)^*_2)^2$, and $((vw)^*_3)^2$ are thus $(X_1+X_2)^2, X_1^2, X_2^2$.

Example $\Gamma =$ 

$$H_1(\Gamma, \mathbb{Z}) = \langle b_1 = v_1w_1 + w_1v_2 + v_2w_2 + w_1v_1,$$
$$b_2 = v_3w_2 + w_2v_1 + v_1w_1 + w_1v_3,$$
$$b_3 = v_3w_3 + w_3v_2 + v_2w_2 + w_2v_3,$$
$$b_4 = w_3v_1 + v_1w_2 + w_2v_3 + v_3w_3 \rangle$$

The functionals e^* are:

$(v_1w_1)^* = X_1 + X_2,$
$(v_1w_2)^* = X_1 + -X_2 + \quad +X_4,$
$(v_1w_3)^* = \quad X_2.$
$(v_2w_1)^* = -X_1$
$(v_2w_2)^* = X_1 + \quad X_3$
$(v_2w_3)^* = \quad -X_3$
$(v_3w_1)^* = \quad -X_2$
$(v_3w_2)^* = \quad X_2 - X_3 - X_4$
$(v_3w_3)^* = \quad X_3 + X_4$

Quadratic forms are:
$X_1^2, X_2^2, X_3^2, X_4^2, (X_1+X_2)^2, (X_3+X_4)^2, (X_1+X_3)^2, (X_1-X_2+X_4)^2, (X_2-X_3-X_4)^2$

We can now state the Extension Criteria!

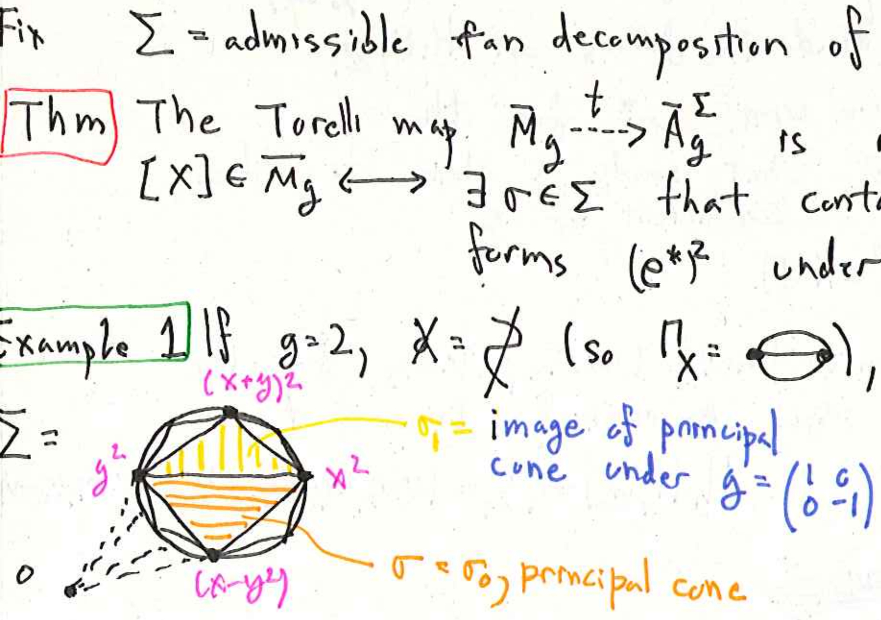
Given X, fix a surjection $s: \mathbb{Z}^g \to H_1(\Gamma_X, \mathbb{Z})$. We get $N(s): N(\Gamma_X) \to N_g$ given by $q \mapsto q \circ s$.

~~Extension Criteria~~ ~~The Torelli map is regular on~~
~~For a given~~

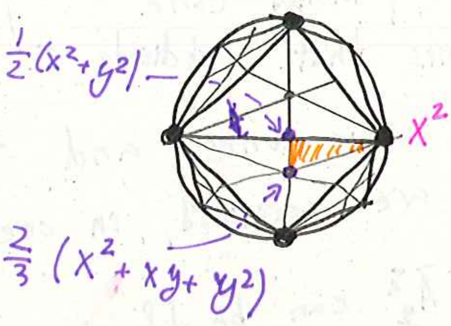
Fix Σ = admissible fan decomposition of $\overline{C}_g \subseteq (N_g)_{\mathbb{R}}$.

Thm The Torelli map $\overline{M}_g \overset{t}{\dashrightarrow} \overline{A}_g^{\Sigma}$ is regular on a neighborhood of $[X] \in \overline{M}_g \iff \exists\, \sigma \in \Sigma$ that contains the images of the forms $(e^*)^2$ under $N(s)$.

Example 1 If $g=2$, $X = $ (so $\Gamma_X = $), and Σ is the standard fan, then t is regular on a neighborhood of X. Indeed, the forms $(e^*)^2$ map to the vertices of σ_1.

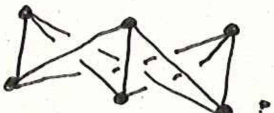
$\Sigma =$ 

The Torelli map $t: M_g \dashrightarrow \bar{A}_g^{\Sigma}$ is NOT regular on a neighborhood of X if we take Σ to be the barycenter subdivision of the standard fan decomposition.



Example 2

Say $g=4$ and X = curve w/ rat'l components and dual graph Γ_X =



Here we have a few natural choices for Σ.
We have the central = the perfect cone decomposition. This ~~[illegible]~~ consists of the

σ = ~~our~~ principal cone = cone spanned by $X_1^2, X_2^2, X_3^2, X_4^2, (X_1-X_2)^2, (X_1-X_3)^2, (X_1-X_4)^2, (X_2-X_3)^2, (X_2-X_4)^2, (X_3-X_4)^2$

= cone spanned by the squares of ~~[illegible]~~ the duals of ~~a~~ lattice pts minimizing $\frac{1}{2}\left(X_1^2+\dots+X_n^2+(X_1+\dots+X_n)^2\right)$

= cone asso. to A_4 root system

and

τ = cone spanned by $X_1^2, X_2^2, X_3^2, X_4^2, (X_1-X_3)^2, (X_1-X_4)^2, (X_2-X_3)^2, (X_2-X_4)^2, (X_3-X_4)^2, (X_1+X_2-X_3)^2, (X_1+X_2-X_4)^2, (X_1+X_2-X_3-X_4)^2$

= cone dual to ray spanned by perfect form $\frac{1}{2}\left\{(X_1-X_2)^2+X_3^2+X_4^2+(X_1+X_2+X_3+X_4)^2\right\}$

= cone asso. to the D_4 root system

together with faces and translates.

The forms $(e_i^*)^2$ do <u>not</u> lie in the principal cone or any of its translates! But t is regular on a neighborhood of $[X]$.

<u>Check:</u> the forms $(e^*)^2$ are all contained in some translate of τ.

~~(ii)~~ Alternatively, we could take $\Sigma = 2^{nd}$ Voronoi decomposition. This decomposition

Consists of (8)

σ = the principal cone
and two cones that subdivide τ. ~~they are~~

The cones are obtained by Subdividing τ by adding the central ray $\frac{1}{3}\cdot(x_1^2+x_2^2+x_3^2+x_4^2+(x_1-x_3)^2+(x_2-x_3)^2+(x_1-x_4)^2+\dots+(x_1+x_2-x_3-x_4)^2)$

$\boxed{\text{Exercise:}}$ Find the two cones and then prove that the $(e^*)^2$-s are contained in one of them.

$\boxed{\text{Remark}}$ Both $\overline{M}_g$ and $\overline{A}_g^{\Sigma}$ can be defined as fine stacks or coarse schemes. ~~Mumford~~ Persumably the criteria applies to the map on stacks, but I don't know a reference for this!! ~~Most~~ All the references I looked ~~at~~ criteria for D/Γ where Γ is a neat (→ torsion-free) arithmetic group, but Γ is not neat. only discussed the
It would be nice to have the details written down.

$\boxed{\text{Remark}}$ One strange feature of criteria is that one side of the equivalence (but not the other) depends on the choice of a surjection $s: \mathbb{Z}^g \twoheadrightarrow H_1(\Gamma_X, \mathbb{Z})$. The reader may easily check that if the criteria is fulfilled for one s then it is fulfilled for all s. (Use the GL-equivariance of Σ).

The surjection arises naturally in the proof as follows. First, at one point we will pick a rat'l boundary component representing a given cusp. ~~Second~~ Second, to analyze a monodromy action, we will pick a splitting of the natural surjection $H_1(X, \mathbb{Z}) \twoheadrightarrow H_1(\Gamma_X, \mathbb{Z})$.

How to prove the criteria? Given X, set $a = \#$ of nodes and $b = 3g-3-a$.
We can find a family of stable curves $\begin{matrix} X_t \\ \downarrow \\ \Delta^a \times \Delta^b \end{matrix}$ s.t. 1) the fiber over $(0,0)$ is X
2) the locus $\Delta \times \dots \times \underset{i^{\text{th}} \text{ position}}{\{0\}} \times \dots \times \Delta \times \Delta^b$ is the locus where the node $P_i \in X$ remains a node
3) the classifying map defines an open immersion

$$\frac{\Delta^a \times \Delta^b}{\text{finite grp}} \hookrightarrow \overline{M}_g.$$

So we are in the same situaton as when we were studying the map into A_g^*. The general problem ~~as~~ of extending a map $(\Delta^*)^a \times \Delta^b \to D/\Gamma$ to a map $\Delta^a \times \Delta^b \longrightarrow \overline{(D/\Gamma)}^{\Sigma}$ is analyzed in

- Ash-Mumford-Rapoport-Tai's book <u>Smooth Compactifications of Locally Symmetric Spaces</u>

in Chapter <u>III</u>, Section 7 (except they only treat the case where Γ is neat). First, ~~the~~ ~~on~~ there is a lift

$$\begin{tikzcd}
\mathbb{Z}^a = \pi_1((\Delta^*)^a \times \Delta^b, *) \arrow[r, "\varphi"] & \Gamma \\
\mathbb{H}_1^a \times \Delta^b \arrow[r, "\tilde{t}"] \arrow[d, "\exp"] & D \arrow[d] \\
(\Delta^*)^a \times \Delta^b \arrow[r, "t"] & D/\Gamma
\end{tikzcd}$$

(This existence of the lift follows from covering space theory when Γ is neat, and ~~probably~~ ~~follows~~ the lift probably exists when t lifts to a map into the quotient stack $[D/\Gamma]$.)

Proposition 7.1 of AMRT states:

Prop There is an ext.

$$\begin{tikzcd}
(\mathbb{H}_1^*)^a \times \Delta^b \arrow[r, "(\tilde{t})^*"] & D^* = \text{rat'l closure} \\
(\mathbb{H}_1)^a \times \Delta^b \arrow[r, "\tilde{t}"] \arrow[u, hook] & D \arrow[u, hook]
\end{tikzcd}$$

$\boxed{\mathbb{H}_1^* = \mathbb{H}_1 \cup \{i\infty\}.}$

Set $p = (\tilde{t})^*(i\infty, \ldots, i\infty, 0, \ldots, 0)$. The point lies in a unique rat'l boundary component F. Asso to F are

$P(F)$ = asso. parabolic subgroup stabilizing $F = \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & 1 & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & {}^t u^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & n \\ {}^t m & 1 & {}^t n & 0 \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & B'' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

for F = std. rat'l bounding component

$\cup$

$U(F)$ = center of unipotent radical

= ~~upha~~ vector space

$= (N_{g''})_{\mathbb{R}}$ w/ $g'' = g - \dim F$

$\cup$

$\overline{C}(F)$ = rat'l closure of $C(F)$

$\cup$

$C(F)$ = self-dual open curve = positive define forms

The cone $\overline{C}(F)$ gets a fan decomposition $\Sigma(F)$ by pulling back ~~a~~ Σ via an embedding $\overline{C}(F) \hookrightarrow \overline{C}$.

Set $\Lambda(F) = P(F) \cap \Gamma$, $\Lambda'(F) = U(F) \cap \Gamma$, $\gamma_1, \dots, \gamma_a$ = image of ~~std.~~ standard generators of $\pi_1((\Delta^\times)^a_{\phi} \times \Delta^b, *)$

$\overset{?}{=} N_{g''}$ for std choice

Check: $\gamma_1, \dots, \gamma_a \in \Lambda'(F)$, so we have an induced map

$$f^0 : (\Delta^\times)^a \times \Delta^b \longrightarrow D/\Lambda'(F) = X(F).$$

Recall

$$\begin{array}{ccccccccc} X(F) & \hookrightarrow & \mathfrak{X}(F) & = & T(F) & \times & V(F) & \times & \mathbb{H}(F) \\ \cap & & \cap & & \uparrow & & \uparrow & & \uparrow \\ & & & & \text{cplx torus} & & \text{cplx vector space} & & \text{Siegel } \tfrac{1}{2}\text{-plane} \\ X_\Sigma(F) & \hookrightarrow & \mathfrak{X}_\Sigma(F) & = & T_\Sigma(F) & \times & V(F) & \times & \mathbb{H}(F) \\ & & & & \uparrow & & & & \\ & & & & \text{infinite toric variety} & & & & \end{array}$$

The space $\mathfrak{X}_\Sigma(F)$ has a natural action of $\Lambda(F)$ and $Y_\Sigma(F) := X_\Sigma(F)/\Lambda(F)$ is a neighborhood of $\bar{A}^\Sigma_g$. ~~the have~~

The following result shouldn't be too surprising:

$\boxed{\text{Thm (1/2 of Thm 7.2 in AMRT)}}$ $f : (\Delta^\times)^a \times \Delta^b \longrightarrow D/\Gamma$ extends to $\bar{f} : \Delta^a \times \Delta^b \longrightarrow (D/\Gamma)^\Sigma$ $\longleftrightarrow$ $f^0 : (\Delta^\times)^a \times \Delta^b \longrightarrow X(F) = D/\Lambda'(F)$ extends to $\bar{f}^0 : \Delta^a \times \Delta^b \longrightarrow X_\Sigma(F)$.

The map f^0 is essentially a map of trivial torus bundles, and ~~and~~ $\bar{f}^0$ is essentially a map of trivial toric bundles, so we'd expect to be able to analyze this problem combinatorially.

In fact, we have $\gamma_i \in \bar{C}(F) \subseteq U(F)$ and

$\boxed{\text{Thm}}$ f^0 extends to $\bar{f}^0 \longleftrightarrow \exists\, \sigma \in \Sigma(F)$ s.t. all the γ_i's lie in σ.

We can choose F = std. ~~std~~ boundary component, in which case $U(F)$ is identified w/ $(N_{g''})_{\mathbb{R}}$; $\Sigma(F)$ is the pullback of Σ under the embedding $U(F) \hookrightarrow (N_g)_{\mathbb{R}}$, $q \mapsto \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

To ~~prove~~ establish the criteria, we need to compute the elements $\gamma_1, \dots, \gamma_g$.

The map $\varphi: \mathbb{Z}^a = \pi_1((\Delta^\times)^a \times \Delta^b, *) \longrightarrow Sp(2g, \mathbb{Z})$ has the following natural description: given $\gamma \in \pi_1((\Delta^\times)^a \times \Delta^b, *)$ monodromy induces an automorphism $H_1(X_*, \mathbb{Z}) \to H_1(X_*, \mathbb{Z})$ that preserves the intersection pairing. If we fix a basis for $H_1(X_*, \mathbb{Z})$, then we can represent this automorphism by an element of $Sp(2g, \mathbb{Z})$, and this element is $\varphi(\gamma)$. (Note: Since π_1 is abelian, $\varphi(\gamma)$ is independent of our choice of basis.)

We can compute $\varphi(\gamma)$ using the Picard-Lefschetz Formula:

$\boxed{\text{Picard-Lefschetz}}$ Let $X_t \longrightarrow \Delta$ be a family of curves with

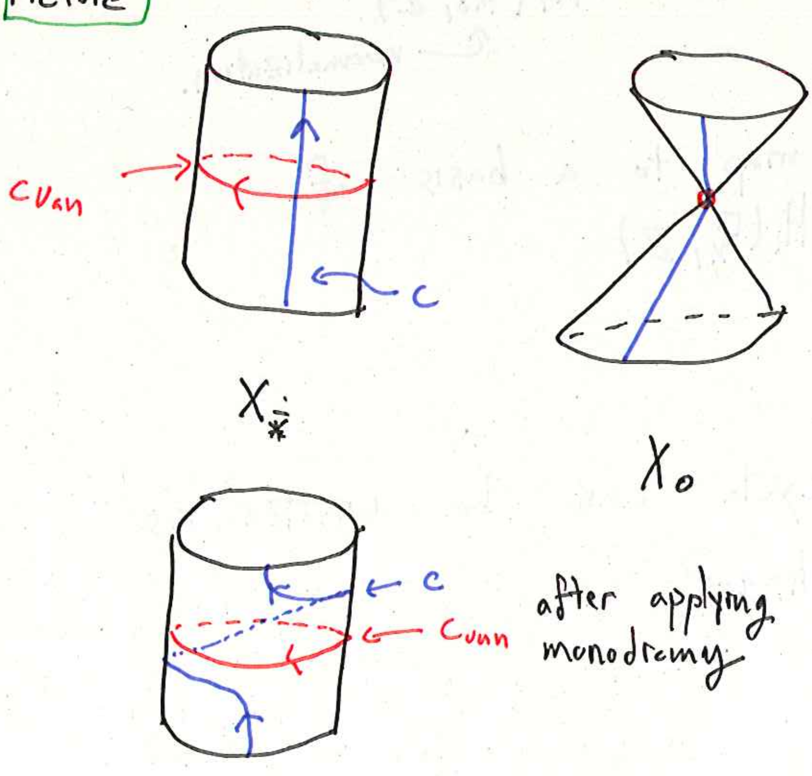
- total space X_t smooth;
- X_0 a curve with 1 node;
- X_{t_0} smooth for $t_0 \neq 0$.

~~Then $\exists$ $c_{van} \in H_1(X_*)$~~ Fix a base point $* \in \Delta^\times$. Then $\exists$ $c_{van} \in H_1(X_*, \mathbb{Z})$ ("vanishing cycle") s.t. the monodromy operator $H_1 \longrightarrow H_1$ given by the action of the std. generator is

$$c \longmapsto c + \langle c, c_{van} \rangle \cdot c_{van}$$

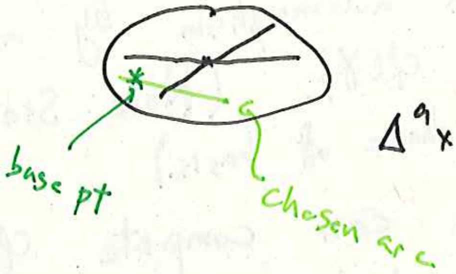
"transvection across line $\mathbb{R} \cdot c_{van}$"

$\boxed{\text{Picture}}$



The ele. γ_i has the following description. Pick a general arc $\Delta \longrightarrow \Delta^a \times \Delta^b$ that passes through $*$ and meets $\Delta \times \dots \times 0 \times \dots \times \Delta \times \Delta^b$ transversely at 0:

$$\Delta \times \dots \times \underset{\substack{\uparrow \\ i^{\text{th}} \text{ position}}}{0} \times \dots \times \Delta \times \Delta^b$$



Consider the restriction $X_t|_\Delta$ of the miniversal family. The total space might be singular, but we can resolve singularities $Y_t \to X_t|_\Delta$ w/o changing the monodromy. Then γ_i = transvection accross the line spanned by the vanishing cycle associated to $Y_t \to \Delta$.

We can compute these vanishing cycles using a particularly nice basis for $H_1(X_*, \mathbb{Z})$. Pick a family of cycles on $X_t \longrightarrow (\Delta^*)^a \times \Delta^b$ s.t.

a) $\{A_1(t), \dots, A_g(t), B_1(t), \dots, B_g(t)\}$ is a standard basis on every fiber;

b) $\lim_{t \to 0} A_1(t), \dots, A_{g'}(t), B_1(t), \dots, B_{g'}(t)$ = standard basis for $H_1(\tilde{X}_0, \mathbb{Z})$ (g' = genus of $\tilde{X}_0$) ← normalization.

c) $\lim_{t \to 0} B_{g'+1}(t), \dots, B_g(t)$ map to a basis of $H_1(\Gamma_X, \mathbb{Z})$

d) $\lim_{t \to 0} A_{g'+1}(t), \dots, A_g(t) = 0$.

By (d), every vanishing cycle can be written as $C_{van} = c_1 A_{g'+1} + \dots + c_h A_g$ w/ $h = g - g'$.

So if the vanishing cycle asso. to the i^{th} node p_i is $c_{van} = \sum c_i \cdot A_{i+g'}$, then γ_i acts by fixing $A_1, \dots, A_g$ and $B_1, \dots, B_{g'}$ and sending

$$B_{g'+1} \longmapsto B_{g'+1} + (B_{g'+1}, c_{van}) \cdot c_{van}$$
$$= B_{g'+1} + c_1 c_1 \cdot A_{g'+1} + c_1 c_2 A_{g'+2} + \dots + c_1 c_n A_g$$
$$\vdots$$
$$B_g \longmapsto B_g + (B_g, c_{van}) \cdot c_{van}$$
$$= B_g + c_n c_1 A_{g'+1} + \dots + c_n c_n A_g.$$

In other words, γ_i is given by the matrix

$$\left(\begin{array}{c|cccc} \mathrm{id} & c_1c_1 & c_2c_1 & \dots & c_nc_1 \\ & c_1c_2 & c_2c_2 & & c_nc_2 \\ & \vdots & \vdots & & \vdots \\ & c_1c_n & c_2c_n & \dots & c_nc_n \\ \hline 0 & & \mathrm{id} & & \end{array}\right)$$

this is the matrix of the quadratic form $(c_1 x_1 + \dots + c_n x_n)^2$.

So to verify the criteria, one needs to check: the functional $e_i^* : H_1(\Gamma_X, \mathbb{Z}) \to \mathbb{Z}$ ~~is the functional~~ satisfies

$$\boxed{}\, e_i^*(B_j(o)) = \langle B_j, c_{van} \rangle$$

maybe there should be a sign here.

vanishing cycle asso. to p_i

Last remark In general, the quadratic forms $(e^*)^2$ can be difficult to describe. However the form has a simple description when Γ_X is planar. When planar, $\exists$ a basis $b_1, \dots, b_n$ for $H_1(X, \mathbb{Z})$ s.t. every functional e^* can be written as

$$e^* = 0$$
or
$$e^* = x_i \text{ for some } i$$
or
$$e^* = x_i - x_j \text{ for some } i \neq j.$$

So these forms lie in the principal cone!