Geometry of $\bar{A}_2^{\Sigma}$.

The general set-up is:

$\mathfrak{h}_g$ = Siegel upper ½-plane

Γ_g = $Sp(g, \mathbb{Z})$ acting by linear fractional transformations

$A_g = \mathfrak{h}_g / \Gamma_g$.

$\bar{A}_g^{\Sigma}$ = toroidal compactification asso. to a fan Σ

Here Σ = cone decomposition of $\bar{C}_g$, where

$N = N_g$ = lattice of $g \times g$ symmetric matrices w/ integer coefficients

$C_g \subseteq N_{\mathbb{R}}$ is the open cone of positive definite quadratic forms

$\bar{C}_g$ = closed cone of positive semidefinite quadratic forms w/ rat'l radical

Σ = an infinite collection of cones $\Sigma = \{\sigma_\alpha\}$ s.t.

- $\sigma \in \Sigma$ and $\tau < \sigma \Rightarrow \tau \in \Sigma$
- $\sigma, \tau \in \Sigma \Rightarrow \sigma \cap \tau < \sigma, \tau$

Σ is <u>admissible</u> if

1) $\bar{C} = \bigcup_{\sigma \in \Sigma} \sigma$

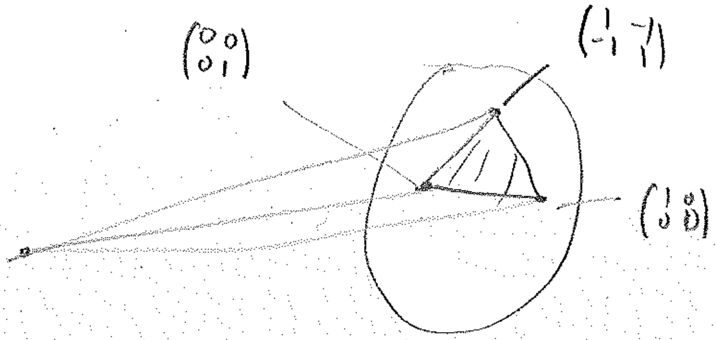
2) action of $GL(N)$ induces an action on Σ (i.e. $\sigma \in \Sigma \Rightarrow g(\sigma) \in \Sigma$)

3) $\# \Sigma / GL(N) < \infty$.

Given a Σ, we get a decomposition $\Sigma_{g'}$ of $\bar{C}_{g'}$ for all $g' \leq g$ by embedding $\bar{C}_{g'} \hookrightarrow \bar{C}_g$ using $q \longmapsto \left(\begin{array}{c|c} q & 0 \\ \hline 0 & 0 \end{array}\right)$

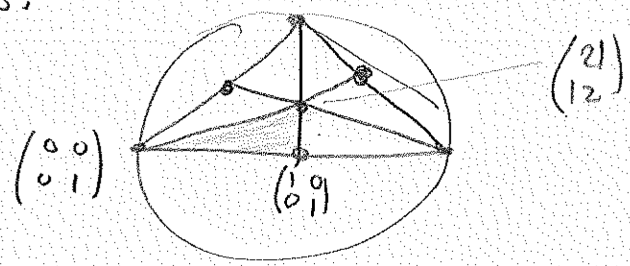
In general, ~~we~~ (i.e. w/ a more general g?), we need to pick a decomposition of each $\bar{C}_{g'}$ and assume a compatibility condition holds.

The standard decomposition of $\overline{C}_2$ is

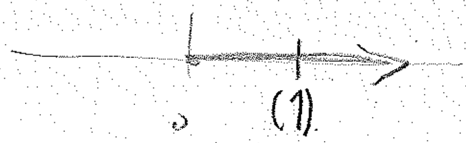


given by
$\sigma_0 = \mathbb{R}_{\geq 0}\begin{pmatrix}0&0\\0&1\end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix}1&0\\0&0\end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$,
its faces, and their translates.

A natural refinement is given by replacing σ_0 by a subdivision into fun. domains:



In both cases, the induced decomposition of $\overline{C}_1$ is



The space $\overline{A}_g^{\Sigma}$ is constructed by constructing 1 partial compactification for each cusp and then glueing the partial compactifications together.
The partial compactifications are constructed by
1.) taking the 1st partial quotient by Λ'
2) forming a partial compactification using Σ
3) forming the 2nd partial quotient by Λ''.

More precisely if F = cusp, then set (3)

$\Lambda(F) = \Gamma_g \cap P(F)$ ← parabolic stabilizing F

$\Lambda'(F) = \Gamma_g \cap$ center of unipotent radical $\quad \Lambda''(F) = \Lambda(F)/\Lambda'(F)$

$g' = \dim F$

$g'' = g - g'$

$V(F)$ = cplx $g' \times g''$ matrices

$\mathbb{H}_F = \mathbb{H}_{g'}$

T_F = cplx torus w/ char grp $N_{g'}$

= symmetric $g' \times g'$ matrices w/ $\neq 0$ entries

Have
$$\begin{array}{ccc} \mathbb{H}_g/\Lambda'(F) & \hookrightarrow & T(F) \times V(F) \times \mathbb{H}_F \\ \| & & \| \\ X(F) & & \mathcal{X}(F) \end{array}$$
by

$$\tau = \left(\begin{array}{c|c} \tau' & z \\ \hline z & \tau'' \end{array}\right) \mapsto (\exp(2\pi i \tau'), z, \tau'')$$

The fan $\Sigma_{g'}$ defines a torus embedding $T(F) \hookrightarrow T_\Sigma(F)$, we set

$\mathcal{X}_\Sigma(F) = T_\Sigma(F) \times V(F) \times \mathbb{H}_F$

$X_\Sigma(F)$ = <u>interior</u> of closure of $X(F)$ in $\mathcal{X}_\Sigma(F)$

<u>Check</u>: Action of $\Lambda''(F)$ on $X(F)$ extends to $X_\Sigma(F)$. The quotient

$$Y_\Sigma(F) = X_\Sigma(F)/\Lambda''(F)$$

is the partial compactification asso. to F.

How does this work for $g=2$?

$\boxed{\text{line } \ell = \mathbb{Q} e_1}$ $\Lambda(\ell) = \left\{ \begin{pmatrix} \epsilon & m & s & n \\ 0 & a & \epsilon(an-bm) & b \\ 0 & 0 & \epsilon & 0 \\ 0 & c & \epsilon(cn-dm) & d \end{pmatrix} : \begin{array}{l} \epsilon \in \{\pm 1\}, \\ s, m, n \in \mathbb{Z} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \end{array} \right\}$

$$\Lambda'(\ell) = \left\{ \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \simeq \mathbb{Z}$$

$$\Lambda''(\ell) = \Lambda(\ell)/\Lambda'(\ell) \simeq \left\{ \begin{pmatrix} \epsilon & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \right\}$$

$\Lambda'(\ell)$ acts by $\begin{pmatrix} \tau' & z \\ z & \tau'' \end{pmatrix} \longmapsto \begin{pmatrix} \tau' + s & z \\ z & \tau'' \end{pmatrix}$.

Get
$$\begin{array}{ccccc} \mathbb{H}_2/\Lambda'(\ell) & \longleftrightarrow & \mathbb{C}^\times \times \mathbb{C} \times \mathbb{H}_1 & \subseteq & \mathbb{C} \times \mathbb{C} \times \mathbb{H}_1 \\ \parallel & & \parallel & & \parallel \\ X(\ell) & & \mathfrak{X}(\ell) & & \mathfrak{X}_\Sigma(\ell) \end{array}$$

$\longleftarrow$ this is true for all Σ because every Σ induces the obvious decomp. of $\overline{C}_1$.

A (lengthy) computation shows that $\Lambda(\ell)$ acts on $\mathbb{H}_2$ by

$$\left(\begin{array}{c|c} \tau' + \epsilon m z + \epsilon s + \dfrac{(\epsilon z + m\tau'' + n)(-cz - \epsilon\{cn-dm\})}{c\tau'' + d} & \dfrac{\epsilon z + m\tau'' + n}{c\tau'' + d} \\ \hline \epsilon z + m\tau'' + n / c\tau'' + d & a\tau'' + b / c\tau'' + d \end{array} \right)$$

So the induced action on $X(\ell) = \{(q', z, \tau'')\}$ is given by

$$\left(q' \cdot \begin{array}{l}\text{complicated expression} \\ \text{involving } z, \tau'' \\ \text{but not } q' \end{array}, \frac{\epsilon z + m\tau'' + n}{c\tau'' + d}, \frac{a\tau'' + b}{c\tau'' + d} \right).$$

This formula ~~extends~~ defines an extended grp action on $X_\Sigma(\ell)$.

We glue $Y_\Sigma(\ell)$ to A_2 as follows.

Have $\mathbb{H}_2 \xrightarrow{\pi} X_\Sigma(\ell)$

$$\begin{pmatrix} \tau' & z \\ z & \tau'' \end{pmatrix} \mapsto (\exp(2\pi i \tau'), z, \tau'').$$

~~the set~~

We define $\sim$ on $\mathbb{H}_2 \sqcup X_\Sigma(\ell)$ to be the equiv rel. generated by requiring:

$x_1 \sim x_2$ if $x_1, x_2 \in \mathbb{H}_2$ are in a common Γ_2-orbit

if $x_1, x_2 \in X_\Sigma(\ell)$ are in a common $\Lambda''(\ell)$-orbit

if $\pi(x_1) = x_2$.

Then $\mathbb{H}_2 \sqcup X_\Sigma(\ell)/\sim$ is Mumford's rnk 1 partial compactification

$= A_2 \sqcup Y_\Sigma(\ell)$.

What get's added?

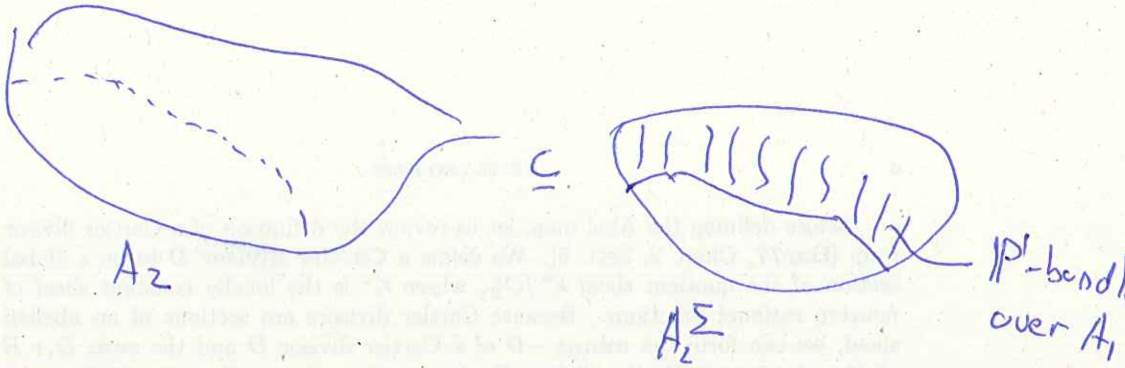
$X_\Sigma(\ell) \supseteq \boxed{\{0\} \times \mathbb{C} \times \mathbb{H}_1}$ — the quotient of this by $\Lambda''(\ell)$ gets added.

$$\begin{array}{ccccc} \mathbb{C} \times \mathbb{H}_1 & & \mathbb{C} \times \mathbb{H}_1/\Lambda''(\ell) & & \mathbb{C}/\mathrm{stab}(\tau) = E_\tau/z \sim -z = \mathbb{P}^1 \\ \downarrow & \text{is equiv} \Rightarrow & \downarrow & & \downarrow \qquad \mathbb{C}/\mathbb{Z}^2 \\ \mathbb{H}_1 & & \mathbb{H}_1/\Gamma_1 = A_1 & \ni & \{\tau\} \end{array}$$

Typical fiber is the quotient by $K = \ker(\Lambda'(\ell) \to \Gamma_1)$, which is an extension

$$\mathbb{Z}^2 \hookrightarrow K \twoheadrightarrow \{\pm 1\}$$

(m,n)



$\boxed{\text{plane: } h = \mathbb{Q}\cdot e_1 + \mathbb{Q}\cdot e_2}$

$$\Lambda(h) = \left\{ \begin{pmatrix} {}^tQ^{-1} & 0 \\ 0 & Q \end{pmatrix} \cdot \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^tQ^{-1} & {}^tQ^{-1}S \\ 0 & Q \end{pmatrix} : \begin{array}{l} Q \in GL_2(\mathbb{Z}) \\ S \in Sym^2(\mathbb{Z}) \end{array} \right\}$$

$$\Lambda'(h) = \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \right\}$$

$$\Lambda''(h) = \Lambda(h)/\Lambda'(h) \simeq GL_2(\mathbb{Z})$$

$$\mathbb{H}_2/\Lambda'(h) \longleftrightarrow (\mathbb{C}^*)^3 \times 0 \times 3\,pts, \quad \begin{pmatrix} \tau' & z \\ z & \tau'' \end{pmatrix} \mapsto \begin{pmatrix} \exp(2\pi i \tau') & \exp(2\pi i z) \\ \exp(2\pi i z) & \exp(2\pi i \tau'') \end{pmatrix}$$

$$\begin{array}{cc} \shortparallel & \cap \\ X''(h) & \mathfrak{X}(h) \end{array}$$

⊛ Now $\mathfrak{X}_\Sigma(h)$ depends on choice of Σ. ⊛

For all choices, we have an action of $\Lambda''(h)$ on $X_\Sigma(h)$ and a map

$$X_\Sigma(\ell) \xrightarrow{\pi} X_\Sigma(h)$$

$$\cup$$

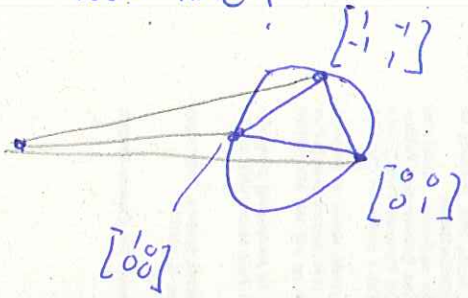
$$(q', z, \tau'') \longmapsto \begin{pmatrix} q' & \exp(2\pi i z) \\ \exp(2\pi i z) & \exp(2\pi i \tau'') \end{pmatrix},$$

~~tile~~ As before, we define

$$\overline{A}_g^{\Sigma} = \underline{\mathbb{H}_2 \sqcup X_\Sigma(\ell) \sqcup X_\Sigma(h)} \big/ \sim,$$ where $\sim$ is the equiv. rel. generated by requiring

What does ~~X~~ $Y_\Sigma(h)$ look like?

- for Σ =



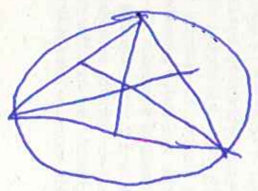
$\mathfrak{X}_\Sigma(h)$ is obtained by gluing together ∞-ly many copies of $\mathbb{C}^3$, one for each top dim'l orbit, along the dense torus.

Up to the grp action, Σ has a unique r-dim'l cone of dim $r = 0, 1, 2, 3$, so

$$\mathfrak{X}_\Sigma(h) = (\mathbb{C}^\times)^3 \sqcup \text{surface} \sqcup \text{curve} \sqcup \text{pt.}$$

$\boxed{\text{Question}}$ 1) Which surface/curve/pt?
What is $X_\Sigma(h) \subseteq \mathfrak{X}(h)$?
What is the complement of
$X_\Sigma(\ell) \to X_\Sigma(h)$?

2) What happens if we refine Σ to



?

(Now there ~~are~~ is more than one $GL_2(\mathbb{Z})$ orbit of top dim'l cones)