Pathologies over p: The Picard Scheme

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Abstract

In these notes, we discuss results due to Igusa and Mumford concerning some surprising behavior of the Picard scheme of a smooth projective variety in positive charateristic. There notes are based on a talk that the author gave in Kiran Kedlaya's STAGE seminar in March 2008.

Contents

1 Introduction

In his paper "On Some Problems in Abstract Algebraic Geometry," [Igu55], Jun-ichi Igusa proved the surprising result that the Picard scheme of a smooth surface over an algebraically closed field of positive characteristic can be non-reduced. In the article, he explicitly constructed such a surface. We give an exposition of the topic in modern scheme-theoretic language. Also discussed are some later computations due to David Mumford. The focus of this note is on the study some special features of the Picard scheme in characteristic p from the perspective of elementary moduli theory. In particular, we do not discuss various connections with p-adic cohomology theories. All of the results in this note are due to Igusa, Mumford, Serre, and others, while all of the errors are due to the author.

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Let's briefly recall the context of Igusa's example. The Picard scheme of a projective variety X is a scheme that parametrizes the line bundles on X. When X is a non-singular curve, the rough features of the Picard scheme is well-understood. The connected components of this scheme are indexed by the integers and a given component parametrizes line bundles of a fixed degree. The identity component is the Jacobian of the curve, an abelian variety of dimension equal to the genus of X.

In [Poi10], Henri Poincaré used transcendental methods to prove a result that implies that an analogous result for the Picard scheme of a smooth projective variety of arbitrary dimension over the complex numbers. The Picard scheme of such a variety is again a disjoint union of smooth projective varieties. It is natural to ask if this result still holds over a field of positive characteristic. In 1955, Igusa proved that this is surprisingly not the case. The purpose of this note is to explore this phenomenon.

2 Notation and Conventions

Let k be an algebraically closed field of characteristic p.

All schemes are implicitly assumed to be locally Noetherian unless we explicitly state otherwise. The term "**variety**" will mean an integral scheme that is separated and of finite type over k.

The term "point" is somewhat ambiguous in algebraic geometry, so let us be precise about what we mean. In this note, a k**-point** of a k-scheme X will be a morphism $\mathrm{Spec}(k) \to X$ over $\mathrm{Spec}(k)$. The set of k-points of a k-scheme that is locally of finite type is in natural bijection with the set of closed points of X. If no other adjectives are attached, then the term "point of X" will mean a morphism $T \to X$ from some unspecified locally finite type k-scheme T to X. We will primarily use this notion of point to prove identies of morphisms using a Yoneda's Lemma argument. We will write $X(T)$ for the set of all T-valued points of X. The reader uncomfortable with formalism will lose little understanding in thinking of points in an intuitive manner.

Another overused term in algebraic geometry is the term "quotient". In this note, we will need to take the quotient of a variety by a finite group. Suppose that X is a k-scheme that is locally of finite type and that G is a finite group of automorphisms of X. We say that a **quotient** of X by G is a pair (Y, π), where Y is a k-scheme that is locally of finite type and $\pi : X \to Y$ is a morphism that satisfies the following conditions:

1. As a topological space, $(|Y|, \pi)$ is the quotient of $|X|$ by G.

2. The morphism $\pi : X \to Y$ is G-invariant and the natural map $\mathscr{O}_Y \to \pi_*(\mathscr{O}_X)^G$ is an isomorphism. Here the superscripted G denotes the subsheaf of G-invariant sections.

3. The pair (Y, π) is universal. In other words, if $f : X \to Z$ is any G-invariant morphism, then there is a unique morphism $\tilde{f} : Y \to Z$ such that $f = \tilde{f} \circ \pi$.

By general formalism, if a quotient exists then it is unique up to a unique isomorphism. The basic existence theorem that we shall need is the following:

Theorem 2.1. *If X is a projective variety over k and G is a finite group of automorphisms, then a quotient (Y, π) of X by the group G exists. Furthermore, the morphism π is finite, surjective, and separable. The k-scheme Y is projective. If the action of G is free, then π is étale.*

A relatively elementary proof of this theorem can be found on page 66 of Mumford's book [Mum70]. Mumford's theorem does explicitly not state that the quotient Y is projective, but this can be proven using a "norm" construction.

3 The Picard Scheme

We now turn our attention to defining the Picard scheme. The **Picard functor** $\mathrm{Pic}_{X/k}$ is the functor from k-schemes to sets that is given by the rule:

$$\mathrm{Pic}_{X/k}(T) = \{\text{line bundles on } X_T\}/\{\text{line bundles on } T\}$$

When we write "line bundles" in this definition, we mean line bundles up to isomorphism. The group of line bundles on T is considered a subset of the group of line bundles on X_T via pullback. One should think of the above set as being "the set of algebraic families of line bundles on X that are parameterized by T modulo the isotrivial families." We should warn the ambitious reader that we are sweeping some non-trivial technical issues under the rug. If we consider smooth varieties over a base that is more complicated than the spectrum of an algebraically closed field, then this definition would need to be modified. These difficulties stem from the fact that a line bundles has non-trivial automorphisms. The construction of the Picard functor is discussed in great detail in Kleiman's article on the Picard scheme [Kle05].

For our purposes, the following existence theorem is more than sufficient:

Theorem 3.1. *Suppose that k is an algebraically closed field and that X/k is a projective variety over k. Then the Picard functor $Pic_{X/k}$ can be represented by a separated scheme that is locally of finite type. Furthermore, this scheme can be written as the disjoint union of open subschemes that are quasi-projective.*

Proof. As stated this result is due to Grothendieck, although many of the ideas are present in the work of Castelnuovo and Matsusaka. Grothendieck's proof can be found in the fifth expose of FGA [Gro95]. A modern exposition of the proof can be found in Kleiman's article [Kle05]. $\square$

We more can be said about the structure of the Picard scheme? Firstly, the Picard scheme has natural group scheme structure coming from the tensor product operation. Secondly, the Picard scheme has an infinite number of connected components. Fix a projective embedding of X. For every integer d, one can show that the locus inside of $\mathrm{Pic}_{X/k}$ that parametrizes degree d line bundles is the union of a finite number of connected components of $\mathrm{Pic}_{X/k}$. In this note, we will not investigate the discrete structure of the Picard scheme and so we make the following definition:

Definition 3.2. Let $P^o_{X/k}$ denote the **identity component** of the Picard scheme.

The focus of our study will be the scheme $P^o_{X/k}$. One can show that the line bundles on X that correspond to points of $P^o_{X/k}$ are precisely the line bundles that are algebraically equivalent to zero.

In general, one can make the following assertions about the structure of $P^o_{X/k}$:

Proposition 3.3. *The following results hold:*

1. *The scheme $P^o_{X/k}$ is a closed and open subgroup scheme of $Pic_{X/k}$ that is irreducible and of finite type.*

2. *If $P^o_{X/k}$ contains an open subscheme that is reduced, then $P^o_{X/k}$ is smooth over k. In particular, if the dimension of the dimension of the Zariski tangent space at a single k-valued point is equal to the combinatorial dimension of $P^o_{X/k}$, then $P^o_{X/k}$ is smooth over k.*

3. *If X is normal (e.g. X is smooth over k), then $P^o_{X/k}$ is projective.*

Proof. The first two results are general facts about k-group schemes that are separated and locally of finite type. To prove that $P^o_{X/k}$ is projective, it is enough to prove that this scheme is proper. This is most easily proved by reducing to proving properness of an appropriate closed subscheme of a Hilbert scheme. Proofs of these results can be found in section three of Kleiman's article [Kle05]. □

Our question "Is $\mathrm{Pic}_{X/k}$ a disjoint union of smooth projective varieties?" can be restated as "Does $P^o_{X/K}$ contain an open subscheme that is reduced?"

Here is a list of results:

- If X is 1-dimensional and the field k is arbitrary, then the Picard scheme is reduced.

  In fact, one can verify directly that $P^o_{X/k}$ is formally smooth over k. Formal smoothness says that every infinitesimal deformation of a line bundle may be extended to every larger infinitesimal base. The obstruction to lifting a given infinitesimal deformation lies in a second coherent cohomology group of X. For dimensional reasons, this group must be zero.

- If X has arbitrary dimension and k is the field of complex numbers, then the Picard scheme is reduced.

  This follows from Poincaré work. Alternatively, it is a theorem of Grothendieck and Cartier that a k-group scheme that is separated and locally of finite type over a field of characteristic zero is automatically reduced. In particular, their result applies to the Picard scheme of X. A third proof can be given using the cohomology computations in the fifth section of this note.

- If X is a smooth projective surface and k is a field of positive characteristic, then $P^o_{X/k}$ can be non-reduced.

  This is the content of Igusa's example.

4 Igusa's Example

In this section, we present Igusa's construction. In order to prove that the Picard scheme of Igusa's surface is non-reduced, it is necessary to able to compute the Zariski tangent space to the Picard scheme. Let us review how to compute the tangent space to the Picard scheme or, more generally, the space of n-jets to the Picard scheme.

Review of Tangent Vectors and Jets: In general, suppose that Y/k is locally of finite type and that $p_0 \in Y(k)$ is a given k-point. The **Zariski tangent space** to Y at p_0, denoted $T_{p_0}(Y)$, is defined as follows. The k-algebra $k_\epsilon := k[\epsilon]/(\epsilon^2)$ admits a natural augmentation $k[\epsilon]/(\epsilon^2) \to k$ given by mapping ϵ to 0. The set of elements of $Y(k_\epsilon)$ that map to $p_0 \in Y(k)$ under the map induced by the augmentation is defined to be the Zariski tangent space $T_{p_0}(Y)$. It naturally has the structure of a k-vector space.

We can generalize this construction by replacing the algebra $k[\epsilon]/(\epsilon^2)$ with $k[\epsilon]/(\epsilon^{n+1})$. In this note, we will call the resulting set the set of **order n jets** to Y at p_0 and denote it by $J^n_{p_0}(Y)$. This is a k-vector space and there are natural maps $J^{n+1}_{p_0}(Y) \to J^n_{p_0}(Y)$.

When Y is equal to the Picard scheme of a projective variety X, then we can compute the space of order n jets in terms of the cohomology of X:

Proposition 4.1. *Suppose that X/k is a projective variety. Then there is a natural identification of the vector space $J^n_e(P^o_{X/k})$ with $H^1(X, 1+\epsilon\,\mathscr{O}_X[\epsilon]/(\epsilon^{n+1})$. Here we are considering $1+\epsilon\,\mathscr{O}_X[\epsilon]/(\epsilon^{n+1})$ as a sheaf of abelian group on the underlying topological space of X.*

Proof. Consider the split exact sequence:

$$0 \to 1+\epsilon\,\mathscr{O}_X[\epsilon]/(\epsilon^{n+1}) \to \mathscr{O}_X[\epsilon]/(\epsilon^{n+1})^* \to \mathscr{O}_X^* \to 0$$

This induces a short exact sequence on first cohomology groups. By chasing cocycles, one can identify this with a short exact sequence:

$$0 \to H^1(X, 1+\epsilon\,\mathscr{O}_X[\epsilon]/(\epsilon^{n+1})) \to \mathrm{Pic}_{X/k}(k[\epsilon]/(\epsilon^{n+1})) \to \mathrm{Pic}_{X/k}(k) \to 0$$

Here $\mathrm{Pic}_{X/k}(k[\epsilon]/(\epsilon^{n+1})) \to \mathrm{Pic}_{X/k}(k)$ is the map induced by the natural augmentation. This completes the proof. □

The sheaf $1+\epsilon\ \mathscr{O}_X[\epsilon]/(\epsilon^{n+1})$ will play an important role in the fourth section of these notes, so it is useful to have some notation for it:

Definition 4.2. Let $V_n(\mathscr{O}_X)$ denote the sheaf $1 + \epsilon\ \mathscr{O}_X[\epsilon]/(\epsilon^{n+1})$. This is a sheaf on X.

We now turn out attention to Igusa's example. Igusa's construction works only when $p = 2$.

For the remainder of this section assume that the characteristic of k is equal to 2.

Igusa's Surface: Let E_0 be an ordinary elliptic curve over k. The subscheme $E_0[2]$ of 2-torsion is non-reduced, but it contains a reduced sub-group scheme of order 2 (the group of "physical" 2-torsion points). Let $t_0 \in E_0[2](k)$ be the non-trivial 2-torsion k-point. We set E equal to the quotient of E_0 by the action of t_0. We will not need to make use of this fact, but when the field k is perfect the Verschiebung map defines an isomorphism between this quotient and the "twist" of E_0 by Frobenius, $E^{(p)}$.

To be totally explicit, one can take k to be the algebraic closure $\bar{\mathbb{F}}_2$ of the field with two elements and E_0 to be the elliptic curve associated to the Weierstrass equation $y^2 + xy + y = x^3 + x + 1$. The non-trivial 2-torsion k-point $t_0 \in E_0(k)$ of this curve is equal to $(1, 1)$. Since E_0 is defined over $\mathbb{F}_p$, the curve E is abstractly isomorphic to E_0.

Set $X_0 = E_0 \times_k E_0$. This is an abelian surface. Igusa's surface is defined to be a quotient of X_0. Define an involution $i : X_0 \to X_0$ by the rule $i(x, y) = (x + t_0, -y)$ and let X be the quotient of X_0 by i. The map i is a fixed-point free involution of a projective variety, so X is a smooth projective surface and the quotient map $\pi : X_0 \to X$ is étale. The surface X is **Igusa's surface**.

Theorem 4.3 (Igusa)**.** *The Picard scheme of Igusa's surface X is non-reduced.*

Proof. There are two parts to this proof. We first prove that the Zariski tangent space to the Picard scheme at the origin is 2-dimensional and then we prove that the combinatorial dimension of $P^o_{X/k}$ is equal to 1. In fact, we show that the reduced subscheme is isomorphic to the elliptic curve $E^\vee$. Here $E^\vee$ is the dual elliptic curve. The dual elliptic curve is naturally isomorphic to E, but it occurs as the dual curve.

The proof is broken up into the two lemmas below.

Lemma 4.4. *The dimension of the Zariski tangent space to $P^o_{X/k}$ at at the origin is* 2*.*

Proof. Shizhang Li provides the following argument.

Let's look at the Hochschild–Serre spectral sequence:

$$E_2^{i,j} = H^i(G, H^j(X_0, \mathcal{O}_{X_0})) \Longrightarrow H^{i+j}(X, \mathcal{O}_X),$$

where G is isomorphic to $\mathbb{Z}/2$ generated by the involution i.

We have: $H^j(X_0, \mathcal{O}_{X_0}) = k$ if $j = 0$ or 2 and $H^1(X_0, \mathcal{O}_{X_0}) = k^2$, and G acts trivially on all of them. Here are two facts concerning these group cohomology groups:

1. $H^i(\mathbb{Z}/2, k) \cong k$ for all $i \geq 0$;

2. $H^2(\mathbb{Z}/2, \mathbb{F}_2) \cong \mathbb{F}_2$, and cupping with the nonzero class (call it α for now) induces 2-periodicity: $H^i(\mathbb{Z}/2, k) \cong H^{i+2}(\mathbb{Z}/2, k)$ (note that we have $k \otimes_{\mathbb{Z}} \mathbb{F}_2 \cong k$).

These are standard facts, let me skip the proof here. Now we see that our Hochschild–Serre spectral sequence has an operator (cupping with α) acting on it. We will utilize this fact. Let's write out the spectral sequence:

$$\begin{tikzcd}
E_2^{0,2} = k \arrow[rr, bend left, "\cup\alpha"] \arrow[drr, "g"] & E_2^{1,2} = k \arrow[drr] & \cdots & & & \\
E_2^{0,1} = k^2 \arrow[rr, bend left, "\cup\alpha"] \arrow[drr, "f"] & E_2^{1,1} = k^2 \arrow[drr] & E_2^{2,1} = k^2 \arrow[drr, "h"] & E_2^{3,1} = k^2 \arrow[drr] & \cdots & \\
E_2^{0,0} = k & E_2^{1,0} = k & E_2^{2,0} = k \arrow[rr, bend left, "\cup\alpha"] & E_2^{3,0} = k & E_2^{4,0} = k & \cdots
\end{tikzcd}$$

Now let's analyze this spectral sequence, we know it converges to the cohomology of $\mathcal{O}_X$. Hence the infinity page shouldn't have anything of degree greater than 2. But for degree reason, everything in the middle row, if it survived after the second page, will survive to the infinity page. Therefore, we conclude $\mathrm{im}(g) = \ker(f)$. The map g cannot be surjective for dimensional reasons, so h is nonzero and thus, again for dimensional reasons, surjective. Since cupping with α induces isomorphisms (we do not check that this is compatible with the differential, but it is clear from the definition), we therefore also conclude that f is surjective. Hence we see that $H^1(X, \mathcal{O}_X)$ is two-dimensional.

The above analysis actually yields that on the infinity page, only 4 terms are left, namely the lower left square. Moreover, the cup product induce an isomorphism between the tensor product of $(0,1)$ and $(1,0)$ terms and the $(1,1)$ term. Concretely, this says that $\wedge^2 H^1(X, \mathcal{O}_X) \cong H^2(X, \mathcal{O}_X)$.

$\square$

Now let us turn out attention to the second lemma. The proof that the reduced subscheme of $P^o_{S/k}$ is isomorphic to $E^\vee$ uses some basic facts about the Albanese variety.

The Albanese Variety: The relationship between the Picard scheme and the Albanese variety is discussed in section 2 of the sixth expose of FGA [Gro95]. We will briefly review the facts that we need. The following discussion holds when the field k is an arbitrary algebraically closed field, or more generally, a perfect field. The reader interested in the Albanese variety in greater generality is advised to consult the literature.

Definition. Let Y/k be a smooth projective variety and $p_0 \in Y(k)$ a distinguished k-point. Given a map $(Y, p_0) \to (A, e)$ from a smooth projective variety to an abelian variety, we say that this map satisfies the **Albanese mapping property** if this map is universal with respect to maps of (Y, p_0) into an abelian variety that take p_0 to the origin. In other words, if $(Y, p_0) \to (B, e)$ is any map from (Y, p_0) into another abelian variety then there is a unique homomorphism $A \to B$ with that makes the following diagram commute:

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & A \arrow[d] \\ & B \end{tikzcd}$$

If $(Y, p_0) \to (A, e)$ satisfies the Albanese property, then we say that A is the **Albanese variety** of X. By general formalism, the Albanese property determines the abelian variety A up to isomorphism so we are justified in calling A *the* Albanese variety.

Under our assumptions, the Albanese variety always exists and can be constructed from the Picard scheme.

Lemma 4.5. *Suppose that Y/k is a smooth projective variety over k. Then the reduced subscheme of $P^o_{Y/k}$ is an abelian variety and the dual of this abelian variety is the Albanese variety of Y.*

Proof. The reduced subscheme of $P^o_{Y/k}$ is a proper, connected group variety and hence an abelian variety. Let A be the dual abelian variety. By general formalism, there is a natural map $Y \to A$. Using the reflexivity relation $(A^\vee)^\vee \cong A$ for abelian varieties and the universal property of the Picard scheme, one can show that this map satisfies the Albanese mapping property. $\square$

We now complete the proof of our main theorem:

Lemma 4.6. *The reduced subscheme of $P^o_{X/k}$ is isomorphic to the elliptic curve $E^\vee$.*

Proof. Set $p_0 \in X(k)$ equal to the image of $(e, e) \in X(k)$ under the quotient map. By the previous lemma, it is enough to prove that E is isomorphic to the Albanese variety of (X, p_0). Let $f : X_0 \to E$ be the map given by the first projection map followed by the quotient map $E_0 \to E$. This map is i-invariant and hence induces a map $f : X \to E$. Under f, the point p_0 is sent to the identity.

Let us prove that f satisfies the Albanese mapping property. Suppose that $(X, p_0) \to (B, e)$ is a map from X into an arbitrary abelian variety. Consider the composition $X_0 \to X \to B$. Call this map g. This map sends the identity to the identity, so, by rigidity, it is a homomorphism. The map satisfies the relation $g(i(x, y)) = g(x, y)$ for all points (x, y) of X. Using the fact that g is a group homomorphism, we can expand out this relation and obtain the two relations:

$$g(t_0, 0) = 0$$

and

$$2g(0, y) = 0 \text{ for all } y$$

The second relation says that $[2] \circ g(0, y)$ is identically zero. The map $[2]$ is an isogeny, so this can only happen if $g(0, y)$ is identically zero. In other words, the kernel of the homomorphism g must contain the closed subgroup:

$$< t_0 > \times_k E_0$$

Thus g factors through the quotient map $X_0 \to E$. This proves that $f : X \to E$ satisfies the Albanese property, completing the proof. $\square$

Remark: In our discussion of the Albanese variety, we have made use of the fact that the reduced subscheme of $P^o_{X/k}$ is a k-group scheme. Over an algebraically closed field, the reduced subscheme of a group scheme is always a subgroup scheme. Over a non-algebraically closed field, this does not need to be true. An example can be found in Waterhouse's book [Wat79]. Such a pathology can never occur for the group scheme $\mathrm{Pic}^0_{X/k}$. It is a general theorem that the reduced subscheme of a connected finite type commutative k-group scheme that is *proper* is always a subgroup scheme. A proof of this can be found in the section 2 of the sixth expose of FGA [Gro95]. On the other hand, one can probably prove that this pathological phenomenon can occur for $\mathrm{Pic}^\tau_{X/k}$ using a theorem of Raynaud (see sections 4.2.3 and 4.2.6 of [Ray79]). Here $\mathrm{Pic}^\tau_{X/k}$ denotes the subscheme of $\mathrm{Pic}_{X/k}$ that parametrizes line bundles that are numerically equivalent to zero. The author does not know what happens if one considers the Picard scheme of a singular projective variety.

$\square$

5 Mumford's Computation

In his book [Mum66], Mumford computes the Zariski tangent space to the reduced subscheme to the Picard scheme:

Proposition 5.1. *Suppose that X/k is a smooth projective variety over k. Then the Zariski tangent space to the reduced subscheme of the Picard scheme at the identity is equal to those elements of the tangent space to the Picard scheme that lift to n-th order jets for all n. In symbols, we have that:*

$$T_e(P^o_{X/k,red}) = \bigcap im(J^n_e(P^o_{X/k}) \to T_e(P^o_{X/k}))$$

Proof. The inclusion of the left-hand term in the right-hand term is automatic since $P^o_{X/k,\mathrm{red}}$ is formally smooth (by virtue of being a reduced group scheme that is locally of finite type over a field). The reverse inclusion is a rather general fact.

Suppose that v lies in the intersection $\bigcap \mathrm{im}(J^n_e(P^o_{X/k}) \to T_e(P^o_{X/k}))$. Considering v as a morphism out of $\mathrm{Spec}(k[\epsilon]/(\epsilon^2))$, we need to show that there is a factorization:

$$\begin{tikzcd} & P^o_{X/k,\mathrm{red}} \arrow[d] \\ \mathrm{Spec}(k[\epsilon]/(\epsilon^2)) \arrow[r, "v"] \arrow[ur, dotted, "\tilde{v}"] & P^o_{X/k} \end{tikzcd}$$

The question is local in nature, so let $\mathscr{O}$ equal the local ring of $P^o_{X/k}$ at e. The morphism v corresponds to a homomorphism $v^* : \mathscr{O} \to k[\epsilon]/(\epsilon^2)$. Since the vector v is based at e, this homomorphism must be given by:

$$v^*(f) = f(e) + D(f)\,\epsilon$$

Here f is an arbitrary element of $\mathscr{O}$ and D is a fixed k-valued derivation on $\mathscr{O}$.

To prove the existence of $\tilde{v}$, it is enough to prove that v^* kills all nilpotent elements of $\mathscr{O}$. Suppose that $f \in \mathscr{O}$ satisfies $f^n = 0$. We have that $f(e) = 0$, so we really just need to prove that $D(f) = 0$.

By hypothesis, the tangent vector v lifts to an n-th order jet j. On the level of algebras, we have that

$$\begin{tikzcd} k[\epsilon]/(\epsilon^{n+1}) \arrow[d] & \mathscr{O} \arrow[l, "j^*"'] \arrow[dl, "v^*"] \\ k[\epsilon]/(\epsilon^2) & \end{tikzcd}$$

Evaluating j^* on f^n, we get that $0 = j^*(f^n) = D(f)\epsilon^n$. This proves that $D(f) = 0$, completing the proof. $\square$

Corollary 5.2. *If X/k is a smooth projective variety, then $P^o_{X/k}$ is a smooth projective variety if and only if the natural map $H^1(X, V_n(\mathscr{O}_X))) \to H^1(X, \mathscr{O}_X)$ is surjective for all positive integers n.*

Relation with Serre's Witt Cohomology: In his Mexico paper [Ser58], Serre developed a p-adic cohomology theory using Witt vectors. Mumford observed that the surjectivity of $H^1(X, V_n(\mathscr{O}_X)) \to H^1(X, \mathscr{O}_X)$ can be restated in terms of Serre's Witt cohomology theory. Let us briefly recall some facts about Witt cohomology theory and sketch the connection to Igusa's work.

Recall that if A is a ring and l is a positive integer, then the **ring of length l Witt vectors**, written $W_l(A)$, is a certain ring whose underlying set is equal to A^l. This ring has the property that the quotient ring $W_l(A)/pW_l(A)$ is isomorphic to A. The Witt vector rings carry several distinguished endomorphism. In this note, we will need to make use of the Frobenius, Verschiebung, and truncation operations. These operations are often denoted by $F : W_l(A) \to W_l(A)$, $V : W_l(A) \to W_{l+1}(A)$, and $R : W_l(A) \to W_{l-1}(A)$ respectively. The Verschiebung and truncation operations can be used to define short exact sequences:

$$0 \to W_{l-m}(A) \xrightarrow{V^m} W_l(A) \xrightarrow{R^{l-m}} W_m(A) \to 0$$

for $l \geq m$. We also have that $W_1(A) = A$. For a more detailed discussion of Witt vectors, we direct the interested reader to section 1 of Serre's article [Ser58].

Now suppose that X is a scheme over an algebraically closed field k of characteristic p. For every open subset U of X, we can consider the ring $W_l(\Gamma(U, \mathscr{O}_X))$. These rings fit together to form a coherent sheaf of rings $W_l(\mathscr{O}_X)$ that we will call the **length l Witt vector sheaf**. Serre's Witt cohomology is defined to be the inverse limit $\varprojlim H^*(X, W_l(\mathscr{O}_X))$. Here the inverse limit is taken with respect to the map $H^*(X, W_{l+1}(\mathscr{O}_X)) \to H^*(X, W_l(\mathscr{O}_X))$ induced by truncation.

We now turn to the relation between this cohomology theory and the geometry of the Picard scheme. The connection is established by the following proposition:

Proposition 5.3. *Let X/k be a smooth projective variety over an algebraically closed field k of characteristic p. The sheaf $V_n(\mathscr{O}_X)$ can be described in terms of the Witt sheaves as follows. If i is an integer that is relatively prime to p and satisfies $1 \leq i \leq n-1$, then let r_i denote least integer r such that $p^r \geq n/i$. Then there is an isomorphism of sheaves of abelian groups:*

$$\prod_{\substack{1 \leq i \leq n \\ (i,p)=1}} W_{r_i}(\mathscr{O}_X) \cong V_n(\mathscr{O}_X)$$

Here the sheaf $W_{r_i}(\mathscr{O}_X)$ is considered as a sheaf of additive groups.

Furthermore, this isomorphism may be chosen so that the natural map $V_n(\mathscr{O}_X) \to \mathscr{O}_X$ is identified with the map $\prod W_{r_i}(\mathscr{O}_X) \to \mathscr{O}_X$ given by truncating $W_{r_1}(\mathscr{O}_X)$ and forgetting about the other factors.

In particular, the scheme $P^o_{X/k}$ is reduced if and only if the natural map $H^1(X, W_l(\mathscr{O}_X)) \to H^1(X, \mathscr{O}_X)$ is surjective for all positive integers l.

Proof. The isomorphism is constructed using the Artin-Hasse exponential. A proof can be found in [Mum66] or chapter 5 of [Ser88]. In [Ser88] Serre works only with the algebraic group $V_n(k)$, but the argument generalizes without difficulty. □

The image of the map $H^1(X, W_l(\mathscr{O}_X)) \to H^1(X, \mathscr{O}_X)$ plays a significant role Serre's theory:

Theorem 5.4. *There is a spectral sequence*

$$E_1^{i,j} = H^{i+j}(X, \mathscr{O}_X) \Rightarrow \varprojlim H^{i+j}(X, W_l(\mathscr{O}_X))$$

The groups of cocycles and coboundaries of the r-page are given by:

$$B_r^{i,j} = \ker(H^{i+j}(X, \mathscr{O}_X) \xrightarrow{V^{r-1}} H^{i+j}(X, W_r(\mathscr{O}_X)))$$

and

$$Z_r^{i,j} = im(H^{i+j}(X, W_r(\mathscr{O}_X)) \xrightarrow{R^{r-1}} H^{i+j}(X, \mathscr{O}_X))$$

Proof. The existence of the spectral sequence follows from a general result about the existence of a spectral sequence associated to an inverse system of coherent sheaves. □

In particular, observe that we have $E_r^{0,1} = Z_r^{0,1} = \mathrm{im}(H^1(X, W_l(\mathscr{O}_X)) \to H^1(X, \mathscr{O}_X))$. Let $\beta_r : E_r^{0,1} \to E_r^{r,2-r}$ be the differential. We call this map the r-th **Bockstein operator**. As a corollary to the previous two results, we have:

Corollary 5.5. *Suppose that X/k is a smooth projective variety over an algebraically closed field k of characteristic $p > 0$. The scheme $P^o_{X/k}$ is reduced if and only if the Bockstein operators β_r are all zero.*

From the form of the spectral sequence, it follows that $\beta_r = 0$ for $r \geq 3$ so one only needs to examine the operators β_1 and β_2. One can restate this fact in a way that does not explicitly mention Witt cohomology:

Corollary 5.6. *Suppose that X/k is a smooth projective variety over an algebraically closed field k of characteristic $p > 0$. The scheme $P^o_{X/k}$ is reduced if and only if the map $J_e^{p^2}(X) \to T_e(X)$ is surjective.*

6 Igusa's Example Revisited

Now suppose that X is the Igusa surface over an algebraically closed field k of characteristic 2. By the last corollary of the previous section, it follows that there are first order deformations of the trivial line bundle on X that do not lift to deformations of order 4. Two questions natural follow-up questions are: "which tangent vectors lift to arbitrary order?" and "if v is a tangent vector that does not lift to fourth order, can we lift v to second order?"

The proof of Lemma 4.4 suggests an answer to the first question. The spectral sequence describes $H^1(X, \mathscr{O}_X)$ as containing $H^1(G, H^0(X_0, \mathscr{O}_{X_0}))$ with quotient

$$H^1(X, \mathscr{O}_X)/H^1(G, H^0(X_0, \mathscr{O}_{X_0})) = \ker(H^1(X_0, \mathscr{O}_{X_0})^G \to H^2(G, H^0(X_0, \mathscr{O}_{X_0})).$$

A reasonable guess is that $H^1(G, H^0(X_0, \mathscr{O}_{X_0})) \subset H^1(X, \mathscr{O}_X)$ is the subspace of liftable tangent vectors and all vectors fail to life to order 2. Lemma 4.6 describes the reduced subscheme of the Picard scheme is the image of $E^\vee to P^o_{X/k}$ under the map induced by $X_0 = E_0 \times_k E_0 \to E$, and the subspace of liftable tangent vectors is the image of the tangent space to $E^\vee$, so one could try to verify the guess using this description.

In Igusa's article, he observes that it is possible to generalize his construction of the Igusa surface to construction smooth projective varieties over fields of odd characteristic whose Picard scheme is non-reduced. Let $p > 0$ be an odd prime. Consider the plane curve defined by the equation

$$Z^p - X^{p-1}Z + Z^{p-2}Y^2 = 0$$

The Jacobian of the normalization of this curve admits an automorphism of order p. By imitating the construction of the Igusa surface, one can construct a smooth projective variety of dimension $p-1$ over $\bar{\mathbb{F}}_p$ whose Picard scheme is non-reduced.

In his Mexico paper [Ser58], Serre constructed further example as group quotients of complete intersections in projective space. To be specific, the source of his examples is the following proposition:

Proposition 6.1. *Suppose that k is an algebraically closed field of characteristic p. Let G be a finite group and $n \geq 1$ an integer. Then there exists an n-dimensional projective variety Y/k that is a complete intersection in some projective space and a fixed-point free action of G on Y. For $p \geq 5$, when $n = 2$ and $G = \mathbb{Z}/p\mathbb{Z}$ we can take Y to be a surface in $\mathbb{P}^3$.*

Proof. This is proposition 15 in [Ser58]. □

Using the Serre-Hochshild spectral sequence, one can construct further example of smooth projective varieties whose Picard scheme is non-reduced by taking G equal to an abelian group of p-power order. For example, the proposition implies that there is a smooth surface Y in $\mathbb{P}^3$ that admits a fixed-point free action of $G := \mathbb{Z}/p\mathbb{Z}$. The quotient surface $X = Y/G$ has the property that $\beta_1 \neq 0$.

Real-world examples of smooth projective surfaces of general type whose Picard scheme is non-reduced appeared as a by-product of the thesis work of Junecue Suh. In his thesis [Suh07], he finds examples of quaternionic and unitary Shimura surfaces whose mod p reduction is a smooth projective surface with non-reduced Picard scheme.

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