Notes on Dieudonné Modules

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The goal of Dieudonné theory is to "classify" all finite commutative p-groups over a perfect field of characteristic p. The following notes are an overview of the fundamentals of Dieudonné theory over a field (rather than a more general base). A good reference for the material covered in these notes is Fontaine's book *Groupes p-divisibles sur les corps locaux* [2].

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1 Frobenius

Suppose that S is a $\mathbb{F}_p$-scheme and X is a S-scheme. Recall that on S there is the **absolute Frobenius morphism** $F_S : S \to S$. As a map of topological spaces, the morphism F_S is the identity. The absolute Frobenius morphism acts on sections by sending a section t to t^p. This morphism is functorial in S.

We let $X^{(p)}$ denote the scheme defined by the cartesian diagram:

$$\begin{array}{ccc} X^{(p)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{F_S} & S \end{array}$$

The scheme $X^{(p)}$ is often referred to as a "twist" of X.

By the functoriality of absolute Frobenius, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{F_S} & S \end{array}$$

This diagram induces a unique morphism $F_{X/S} : X \to X^{(p)}$ called the **relative Frobenius morphism**. The relative Frobenius morphism is an S-morphism that is natural in the structure morphism $X \to S$ and its formation commutes with base change in S (but not in X).

In the special case where X is an S-group scheme, then it follows from functoriality that $X^{(p)}$ is also an S-group scheme and $F_{X/S}$ is an S-group morphism.

Examples:

1. Suppose that $X = \mathbb{G}_m/S$ or $\mathbb{G}_a/S$ is the multiplicative group or the additive group. Then $X^{(p)}$ can be identified with X in such a way that $F_{G/S}$ is the map $t \mapsto t^p$. Here t is the standard coordinate on X.

2. Say that G is a d-dimensional smooth group scheme, locally of finite type over a field k of characteristic p. Then $G^{(p)}$ is a k-group of the same type. The homomorphism $F_{G/k} : G \to G^{(p)}$ is surjective. In fact, it is finite faithfully flat of degree p^d. This is proven in homework 1, exercise 10 (hint: work over $\hat{O}_{G,e}$)

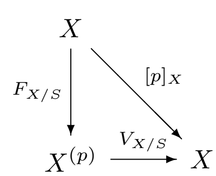
3. Say that X is a finite k-scheme, where k is *any* field of characteristic p. Then $X \to k$ is étale if and only if $F_{X/k}$ is an isomorphism. To prove this, base change to $\bar{k}$. Say $\bar{X} = \mathrm{Spec}(A)$ and $A = k \oplus \mathfrak{m}$ with $\mathfrak{m}$ nilpotent. Study the action of Frobenius on $\mathfrak{m}$.

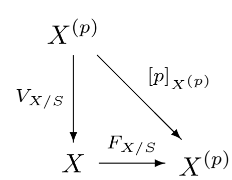
In slightly more generality, we can define the **n-fold relative Frobenius** $F_{X/S,n}$ by the recursive formula $F_{X/S,n} = F_{X^{(p)}/S,n-1} \circ F_{X/S}$.

Example: If G is a finite k-group, then G is connected if and only if $F_{G/k,n} = 0$ for all sufficiently large n. To prove this, base change to $\bar{k}$ and chase components.

2 Verschiebung

Suppose that X is a flat S-scheme. In SGA 3, Exp. XII, sections 4.2-4.4 [1], Lazard's Theorem is used to construct the Verschiebung map $V_{X/S} : X^{(p)} \to X$ with the same good functorial properties as $F_{X/S}$. In the case where X is a flat commutative S-group, the following diagrams commute:





In SGA 3, Exp. XII, section 4.3 [1], it is proven that for X a finite, locally free commutative S-group the canonical identification $(X^\vee)^{(p)} \simeq (X^{(p)})^\vee$ identifies $V_{X/S}$ with $F^\vee_{X/S}$. When the base scheme S is Spec(k), this can be taken to be the definition of $V_{X/S}$.

Examples

1. We have that $F_{\mathbb{G}_m/\mathbb{F}_p} = [p]_{\mathbb{G}_m/\mathbb{F}_p}$. This implies that $V_{\mathbb{G}_m/\mathbb{F}_p} = \mathrm{id}_{\mathbb{G}_m/\mathbb{F}_p}$. Since Verschiebung commutes with base change, we have $V_{\mathbb{G}_m/S} = \mathrm{id}$ for every $\mathbb{F}_p$-scheme S.
2. For $\mathbb{G}_a/\mathbb{F}_p$, multiplication by p is the zero map so $F_{\mathbb{G}_a/\mathbb{F}_p} \circ V_{\mathbb{G}_a/\mathbb{F}_p} = 0$. Since $F_{\mathbb{G}_a/\mathbb{F}_p}$ is faithfully flat, it follows that $V_{\mathbb{G}_a/\mathbb{F}_p} = 0$.
3. For μ_p, we have that $F_{\mu_p/\mathbb{F}_p} = 0$ and $V_{\mu_p/\mathbb{F}_p} = \mathrm{id}$.
4. For $\alpha_p = \ker F_{\mathbb{G}_a/\mathbb{F}_p}$, we have that $F_{\alpha_p/\mathbb{F}_p} = V_{\alpha_p/\mathbb{F}_p} = 0$.
5. For $\underline{\mathbb{Z}/p\mathbb{Z}}$, we have that $F_{\underline{\mathbb{Z}/p\mathbb{Z}}/\mathbb{F}_p} = \mathrm{id}$ and $V_{\underline{\mathbb{Z}/p\mathbb{Z}}/\mathbb{F}_p} = 0$

3 The Dieudonné Ring

Let k be a perfect field. Let $W = W(k)$ denote the **Witt ring of** k. Recall that this is a complete discrete valuation ring of mixed characteristic $(0, p)$ with residue field k and uniformizer p. For k a finite field with p^n elements, this is the valuation ring of the unique degree n unramified extension of $\mathbb{Q}_p$.

On W, there is a distinguished automorphism σ that is the unique lift of the Frobenius map on k. If $[\cdot] : k \to W$ is the Teichmüller map, then every element of W can be written as $\sum [a_n]p^n$. The action of σ can be described by:

$$\sigma(\sum [a_n]p^n) = \sum [a_n^p]p^n$$

Let K denote the fraction field of W. The **Dieudonné ring** D_k is defined to be

$$W(k)\{F, V\}/(FV - p)$$

Here $W(k)\{F, V\}$ is a non-commutative polynomial ring in two commuting variables F and V with the relations:

$$\begin{aligned} VF &= FV \\ Fc &= \sigma(c)F, \text{ “}\sigma\text{-linearity”} \\ Vc &= \sigma^{-1}(c)V, \text{ “}\sigma^{-1}\text{-linearity”} \end{aligned}$$

Every element of $W(k)$ can be written as

$$\sum_{n \geq 0} a_{-n} V^n + a_0 + \sum_{n \geq 0} a_n F^n$$

with $a_n \in W(k), a_n = 0$ for $|n| \gg 0$.

In the special case where $k = \mathbb{F}_p$, the ring D_k is actually commutative and equal to $W[F, V]/(FV - p)$. In general, $W \cap Z(D_k) = \mathbb{Z}_p$.

For any finite commutative p-group scheme G/k, there is a canonical decomposition $G = G^0 \times G^{\text{ét}}$. A similar result hold for D_k modules. Suppose that $\mathbb{D}$ is a (left) D_k-module of finite W-length. We define

$$\mathbb{D}^{\text{ét}} = \cap_{n>0} F^n(\mathbb{D})$$

and

$$\mathbb{D}^0 = \cup_{n>0} \ker(F^n : \mathbb{D} \to \mathbb{D})$$

On $\mathbb{D}^{\text{ét}}$, F acts as an isomorphism, while on $\mathbb{D}^0$ it is a nilpotent operator. Furthermore, $\mathbb{D} = \mathbb{D}^{\text{ét}} \oplus \mathbb{D}^0$.

There is also an analog for Dieudonné modules of the"twisting" operation on p-divisible groups. Suppose that $\mathbb{D}$ is a W-module. The "twist" $\mathbb{D}^{(\sigma)}$ of $\mathbb{D}$ is defined by taking the underlying abelian group structure of $\mathbb{D}$ and adding W-structure by making $c \in W$ act on $\mathbb{D}^{(\sigma)}$ as $\sigma^{-1}(c)$ acts on $\mathbb{D}$. This construction can alternatively be viewed in terms of extending scalars via $\sigma : W(k) \to W(k)$.

For every Dieudonné module $\mathbb{D}$, the semi-linear operators F and V induce W-linear homomorphisms $F^{(\sigma)} : \mathbb{D}^{(\sigma)} \to \mathbb{D}$ and $V^{(\sigma)} : \mathbb{D} \to \mathbb{D}^{(\sigma)}$. We refer to $F^{(\sigma)}$ and $V^{(\sigma)}$ as the **linearization** of F and V respectively.

The process of linearization is reversible. Suppose that $\mathbb{D}$ is a W-module and $F^{(\sigma)} : \mathbb{D}^{(\sigma)} \to \mathbb{D}$ and $V^{(\sigma)} : \mathbb{D} \to \mathbb{D}^{(\sigma)}$ are homomorphisms such that both of the maps $F^{(\sigma)} \circ V^{(\sigma)}$ and $V^{(\sigma)} \circ F^{(\sigma)}$ are equal to the multiplication-by-p map. There is then a unique D_k-module structure on $\mathbb{D}$ such that that linearization of F is $F^{(\sigma)}$ and the linearization of V is $V^{(\sigma)}$. We refer to F and V as the **delinearization** of $F^{(\sigma)}$ and of $V^{(\sigma)}$ respectively.

The basic operations on Dieudonné modules are:

- **Duality:** Suppose that $\mathbb{D}$ is a D_k-module of finite W-length. Set $\mathbb{D}^\vee = \mathrm{Hom}_K(\mathbb{D}, K/W)$. This is a W-module. This module can be given the structure of a (left) D_k-module in a natural manner, as follows. Let $F_{\mathbb{D}^\vee}$ be the operator given by dualizing the linearization of $V_{\mathbb{D}}$ and then delinearizing the resulting operator. The operator $V_{\mathbb{D}^\vee}$ is defined in the analogous manner. In more concrete terms, if $f \in \mathbb{D}^\vee$, then:

$$F_{\mathbb{D}^\vee}(f) = \sigma \circ f \circ V_{\mathbb{D}}$$
$$V_{\mathbb{D}^\vee}(f) = \sigma^{-1} \circ f \circ F_{\mathbb{D}}$$

  The operators $F_{\mathbb{D}^\vee}$ and $V_{\mathbb{D}^\vee}$ endow $\mathbb{D}^\vee$ with the structure of a Dieudonné module. We call this Dieudonné module the **dual** of $\mathbb{D}$.

- **Base Change:** Let $k \to k'$ be an extension of perfect fields. Let $\mathbb{D}$ be a finite-length D_k-module. Consider the $W(k')$-module $W(k') \otimes_{W(k)} \mathbb{D}$ obtatined from $\mathbb{D}$ by extending scalars to $W(k')$. There is a canonical way to extend the Dieudonné structure on $\mathbb{D}$ to $W(k')$-module $W(k') \otimes_{W(k)} \mathbb{D}$. Define $F_{k'}$ to be the operator obtained by linearizing F, extending scalars to $W(k')$, and then delinearizing. In the analogous way, define the operator $V_{k'}$. An explicit formula for these operators is given by:

$$F_{k'}(c \otimes x) = \sigma(c) \otimes F(x)$$
$$V_{k'}(c \otimes x) = \sigma^{-1}(c) \otimes V(x)$$

These operators define the $D_{k'}$ structure on $W(k')$-module $W(k') \otimes_{W(k)} \mathbb{D}$. This Dieudonné module is referred to as the Diuedonné module obtained from $\mathbb{D}$ by **base-changing** to k'.

The duality and base change functors commute with each other in an evident manner.

The significance of Dieudonné modules stems from the following theorem:

Theorem 1 (Fontaine, Ast. 47, Ch. III [2])**.** *There is a natural anti-equivalence of abelian categories*

$$\mathbb{D}_k : \{\textit{finite comm. p-grps } /k\} \to \{\mathbb{D}_k\textit{- modules of finite } W\textit{-length}\}$$

with the following properties:

1. *The homomorphism $\mathbb{D}_k(F_{G/k})$ is equal to the linearization of $F_{\mathbb{D}_k(G)}$ and $\mathbb{D}_k(V_{G/k})$ is equal to the linearization of $V_{\mathbb{D}_k(G)}$.*

2. *We have that $\#G = p^{\ell_W(\mathbb{D}_k(G))}$. Here $\ell_W(\mathbb{D}_k(G))$ denotes the length of the $\mathbb{D}_k(G)$ as a module over W.*

3. *There is a canonical D_k-isomorphism $\mathbb{D}_k(G)^\vee \simeq \mathbb{D}_k(G^\vee)$.*

4. *There exist canonical k-linear isomorphisms $t_G^* \simeq \mathbb{D}_k(G)/F(\mathbb{D}_k(G))$ and $t_{G^\vee} \simeq \ker(\mathbb{D}_k(G) \xrightarrow{V} \mathbb{D}_k(G))$. Here t_G denote the tangent space at the identity.*

5. *There is a canonical $D_{k'}$-isomorphism $W(k') \otimes_W \mathbb{D}_k(G) \simeq \mathbb{D}_{k'}(k' \otimes_k G)$ for any extension k'/k of perfect fields. Furthermore, this isomorphism is transitive with respect to further extension of the perfect base field.*

Properties/Examples:

1. Suppose that G is k-étale. Let W' equal $W(\bar{k})$ and K' denote the field of fractions of W'. Then we have that $\mathbb{D}(G) = \mathrm{Hom}_{\mathrm{Gal}(\bar{k}/k)}(G(\bar{k}), K'/W')$. The operator $F_{\mathbb{D}_k(G)}$ acts as σ, so:

$$(F_{\mathbb{D}_k(G)}\chi)(g) = \chi(\sigma(g)) = \sigma(\chi(g))$$

for $\chi \in \mathrm{Hom}_{\mathrm{Gal}(\bar{k}/k)}(G(\bar{k}), K'/W')$ and $g \in G(\bar{k})$

2. For $\underline{\mathbb{Z}/p\mathbb{Z}}$, we have that $\mathbb{D}_k(\underline{\mathbb{Z}/p\mathbb{Z}}) = (\frac{1}{p}W)/W \simeq k$ with $F = \mathrm{Frob}_k$ and $V = 0$.

3. For μ_p, we have that $\mathbb{D}_k(\mu_p) = k$ with $F = 0$ and $V = \mathrm{Frob}_k^{-1}$.

4. For α_p, we have that $\mathbb{D}_k(\alpha_p) = k$ and $F = V = 0$.

References

[1] M. Artin, J. E. Bertin, M. Demazure, P. Gabriel, A. Grothendieck, M. Raynaud, and J.-P. Serre. *Schémas en groupes. Fasc. 3*, volume 1963/64 of *Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques.* Institut des Hautes Études Scientifiques, Paris, 1964.

[2] Jean-Marc Fontaine. *Groupes p-divisibles sur les corps locaux.* Société Mathématique de France, Paris, 1977. Astérisque, No. 47-48.