Notes on Abelian Schemes

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These notes are a summary of some of the fundamental facts concerning abelian schemes. Most proofs have been omitted. Full proofs for most of the results discussed in these notes can be found in *Néron Models* by Bosch, Lütkebohmert, and Raynaud [1] and in chapter 6 of *Geometric Invariant Theory* by Mumford, Fogarty, and Kirwan [3]. Another good source for the material in these notes is Mumford's *Abelian Varieties* [5].

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1 Abelian Schemes

Let S be an arbitrary scheme. An **abelian scheme over S** is an S-group scheme $A \to S$ that is proper, flat, finitely-presented, and has smooth and connected geometric fibers.

When $S = \mathrm{Spec}(k)$ is the spectrum of a field, this is the standard definition of an abelian variety.

Remark. The function $s \to \dim A_s$ is locally constant in the Zariski topology. We typically assume that it is constant and equal to g.

Example. Some basic examples of abelian schemes are as follows:

1. If $C \to S$ is a proper, smooth S-curve with geometrically connected fibers, then $J = \mathrm{Pic}^0_{C/S}$ is an abelian scheme called the **relative Jacobian** of C/S.

2. Suppose that S is a connected Dedekind scheme (for example, the spectrum of a Dedekind ring or a regular curve over a field). Let η be the generic point of S. Given an abelian variety A_η over the generic point, it is a theorem of Néron that if A_η extends to an abelian scheme A_U over a non-empty open subset U of S, then this extension is unique and functorial in A_η. It follows from uniqueness and "denominator-chasing" that A_η extends over a maximal open subset.

> The Weil Extension Lemma combined with the valuative criteria of properness implies that for any abelian scheme $A \to S$ over a base S that is normal, connected, and locally noetherian and a smooth and any separated S-scheme $Z \to S$, the natural map $\mathrm{Hom}_S(Z, A) \to \mathrm{Hom}_\eta(Z_\eta, A_\eta)$ is bijective.

An **isogeny** $f : A' \to A$ of abelian S-schemes is a surjective S-group map that is quasi-finite (has finite fibers). The Miracle Flatness Theorem (see homework 1) together with "proper + quasi-finite $\Rightarrow$ finite" imply that f is finite and locally free.

Theorem 1. *Any abelian scheme A/S is commutative. Any S-scheme map $A \to G$ to a separated S-group scheme that maps the identity to the identity is a S-group homomorphism.*

Proof. Begin by reducing to the case where S is the spectrum of an Artin local ring. To make this reduction, one uses properness, the Krull Intersection Theorem, and $\varinjlim$ formalism from EGA IV_3. Implicitly, we make use of the fact that the identity section of G is cut out by a quasi-coherent sheaf of ideals. This is where the hypothesis that G is separated is necessary.

The case where S is the spectrum of an algebraically closed field is classical (see Mumford [5], Chapter 2). The case where S is the spectrum of a possibly non-algebraically closed field immediately follows since extending scalars from k to $\bar{k}$ defines a faithful functor.

This proves the result on the actual fibers (rather than just on the geometric fibers) over points $s \in S$. By "taking differences", we are reduced to proving:

Lemma 1 (GIT Lemma)**.** *Suppose that S is the spectrum of an Artin local ring with closed point $s \in S$ and that we are given a diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \arrow[dl] \\ S & \end{tikzcd}$$

with π proper and flat. Let $e \in X(S)$. Assume that $H^0(X_s, \mathcal{O}_{X_S}) = k(s)$ (e.g. $X_{\bar{s}}$ is connected and geometrically reduced). If $f(X_s) = \{point\}$, then there exists $\eta \in Y(S)$ such that $f = \eta \circ \pi$.

Proof. By the theory of cohomology and Base Change (see chapter 3 of Hartshorne), it follows that $\mathcal{O}_S \simeq \pi_*(\mathcal{O}_X)$. On the level of topology define the section η to be the continuous map $f \circ e : |S| \to |Y|$. Check that defining $\eta^\#$ to be the composition

$$\mathcal{O}_Y \xrightarrow{f^\#} f_*\mathcal{O}_X = (\eta \circ \pi)_*\mathcal{O}_X = \eta_* \circ \pi_*\mathcal{O}_X \simeq \eta_*\mathcal{O}_X$$

defines a morphism $S \to Y$ satisfying the desired properties. $\square$

□

Corollary 1. *For $N \geq 1$, the morphism $[N]_A : A \to A$ given by multiplication-by-N is an isogeny of degree N^{2g}. Here $g =$ relative dimension of A. In particular, $A[N]$ is a finite, locally free S-group scheme of order N^{2g}. Furthermore, the system $A[\ell^\infty] = \{A[\ell^n]\}_n$ defines an ℓ-divisible group of height $2g$.*

Proof. This is Homework 3, exercise 5. □

Exercise. Some basic facts about the ℓ-divisible group of an abelian scheme:

1. Let A/k be a g-dimensional abelian scheme over a field k. For $\ell \neq \mathrm{char}(k)$, we have that $A[\ell^\infty]$ is an *étale* ℓ-divisible group of height $2g$. In particular, $A[\ell^\infty]$ is the "same" as the Tate module $T_\ell(A) = \varprojlim A[\ell^n](k_s)$ as a finite free $\mathbb{Z}_\ell$-module of rank $2g$ with continuous Galois action.

2. Let k be a perfect field of characteristic $p > 0$. Then the p-divisible group $A[p^\infty]$ is the "same" as the Dieudonne module $\mathbb{D}(A[p^\infty]) = \varprojlim \mathbb{D}(A[p^n])$. This is a module over the Dieudonne ring, D_k, that is finite and free of rank $2g$ over $W(k)$.

3. In characteristic p, the Serre-Tate Equivalence implies that the connected component of $A[p^\infty]$ is isomophic to $\hat{\mathscr{O}}_{A,0}$, the formal group of A. This is in homework set 3.

Suppose that $f : A \to A'$ is an isogeny. Since f is finite and locally free, it follows that $\ker(f)$ is a finite, locally free S-group scheme. The rank of $\ker f$ is locally constant over S. We call it the **degree** of f.

Consider the case where the degree of f is constant and equal to N. By Deligne's Theorem (see problem 1 on homework set 2) the kernel of f, $\ker(f)$, is killed by $[N^{2g}]_A$. One can use considerations with fppf sheaves to prove that there exists an isogeny $f' : A' \to A$ such that $f' \circ f = [N^{2g}]_A$ and $f \circ f' = [N^{2g}]_{A'}$. In particular, there exists an isogeny $f' : A' \to A$. We say that A and A' are **isogenous** if there exists an isogeny $f : A \to A'$. We have just shown that the property of being isogenous is an equivalence relation.

Remark. There is a wonderful Theorem of Raynaud that says that any finite locally free commutative group scheme $G \to S$ is Zariski locally on S a closed subgroup of a relative Jacobian. This provides a foundation for crystalline Dieudonné theory for such group schemes by using Dieudonné theory for abelian schemes.

2 Abelian Varieties

Theorem 2. *All abelian varieties over a field are projective (i.e. admit an ample line bundle).*

Proof. Mumford [5] proves this in the case where the field is algebraically closed. A trick shows that if X is a proper k-scheme and $X_{\bar{k}}$ is projective, then X is projective. □

Theorem 3. *Let A and A' be abelian varieties over a field k. For any prime ℓ, the natural map*

$$t_\ell : \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}_k(A, A') \to \mathrm{Hom}_k(A[\ell^\infty], A'[\ell^\infty])$$

induced by functoriality is injective.

Proof. First reduce to the case where k is algebraically closed. When $\ell \neq \mathrm{char}(k)$, the proof can be found in Mumford's book [5]. For $\ell = \mathrm{char}(k)$, the same argument goes through with the Dieudonné module on the geometric fiber playing the role of the Tate module. □

Corollary 2. *If $f \in \mathrm{End}_k(A)$, set $P_f(n) = \deg([n]_A - f)$. This is understood to be 0 if $[n]_A - f$ is not an isogeny. Then P_f is a polynomial in n with integer coefficients. It is a monic of degree $2g$. Let k'/k be any perfect extension. The polynomial P_f is equal to characteristic polynomial of the induced endmorphism on*

$$\begin{cases} T_\ell(A) & \ell \neq char(k) \\ \mathbb{D}(A_{k'}[p^\infty]) & \ell = char(k) \end{cases}$$

Here we consider $T_\ell(A)$ as a $\mathbb{Z}_\ell$-module and $\mathbb{D}(A_{k'}[p^\infty])$ as a $W(k')$-module.

Proof. First reduce to the case where k is algebraically closed and $k = k'$. For $\ell \neq p$, a proof can be found on pages 180-181 of Mumford's *Abelian Varieties* [5]. For $\ell = p$, the same proof carries over by using Dieudonné modules. □

Using theorem 3 and some cleverness, one can show that $\mathrm{Hom}_{\bar{k}}(A_{\bar{k}}, A'_{\bar{k}})$ is finitely generated over $\mathbb{Z}$ and of rank at most $4 \dim A \dim A'$. It follows that the same holds for $\mathrm{Hom}_k(A, A')$.

Corollary 3 (Riemann Hypothesis)**.** *Suppose k is finite and of cardinality q. If $f = F_{A/k}$ is the relative Frobenius morphism, then all the complex roots of P_f have absolute value $\sqrt{q}$ (i.e. are "Weil q-numbers").*

Proof. See pages 203-207 of Mumford [5]. □

Theorem 4 (Tate's Conjecture)**.** *If k is finitely generated over the prime field, then the map t_ℓ is an isomorphism.*

Proof. This was proven by Tate in the case where k is a finite field. Zahrin extended this result to the case where k is finitely generated over a finite field. Building on Tate's method, Faltings proved the result in the case where k is a number field and later extended his proof to the general case where k is finitely generated over $\mathbb{Q}$. □

3 Duality Theory

We will now discuss the **duality theory** of abelian schemes. First, we define Picard schemes. A Picard scheme is a certain space that parameterizes line bundle on a fixed scheme and is defined for a fairly general class of schemes. For the remainder of this section, we will let $\pi : X \to S$ be a proper, flat, finitely-presented morphism with $\pi_*(\mathcal{O}_X) = \mathcal{O}_S$ holding "universally". Furthermore, let $e : S \to X$ be a section of π. We will be particularly interested in the case where X/S is an abelian variety and e is the identity section.

Given an S-scheme T, we define a functor $\underline{\mathrm{Pic}}_{X/S,e}(\_)$ by:

$$\underline{\mathrm{Pic}}_{X/S,e}(T) = \{(\mathscr{L}, i) : \mathscr{L} \text{ is an invertible sheaf on } X_T, i : e_T^*(\mathscr{L}_T) \simeq \mathcal{O}_T\}/\cong$$

Here $\cong$ indicates that isomorphic pairs $(\mathscr{L}, i)$ are identified. However, the hypotheses on π are set up so that an object $(\mathscr{L}, i)$ has no non-trivial automorphisms. This ensures that no real information is lost in passing from the category of rigidified line bundles over X_T to the set of isomorphism classes. In particular, the fact that a rigidified line bundle $(\mathscr{L}, i)$ has no non-trivial automorphisms implies that assignment $T \mapsto \underline{\mathrm{Pic}}_{X/S,e}(T)$ defines a Zariski sheaf of abelian groups, called the **relative Picard functor**.

Exercise. The **absolute Picard group** of a scheme Y, denoted $\underline{\mathrm{Pic}}(Y)$, is defined to be the group of isomorphism classes of line bundles on Y. Given a line bundle $\mathscr{L}$ over X_T, define $\tau_e(\mathscr{L}) = \mathscr{L} \otimes \pi_T^*(e_T^*(\mathscr{L}^{-1})$. Here π denotes projection onto T. Show that this line bundles carries a canonical trivialization i_{can} over e. Given the existence of this trivialization, the assignment $\mathscr{L} \mapsto (\tau_e(\mathscr{L}), i_{\text{can}})$ defines a functorial homomorphism

$$\underline{\mathrm{Pic}}(X_T) \to \underline{\mathrm{Pic}}_{X/S,e}(T)$$

Prove that the kernal of this homomorphism is $\pi_T^*(\underline{\mathrm{Pic}}(T))$.

There are several general theorems that assert the existence of the Picard scheme of a scheme under suitable hypotheses. For our purposes, the following theorem is more than sufficient.

Theorem 5 (Grothendieck-Oort)**.** *If $S = Spec(k)$ and $X \to S$ satisfies the hypothesis stated at the beginning of this section, then $\underline{\mathrm{Pic}}_{X/k,e}$ is representable by a locally finite type k-group $\mathrm{Pic}_{X/k,e}$. This scheme is a disjoint union of quasi-projective k-schemes.*

We let $\wp$ denote the universal line bundle on $X \times \mathrm{Pic}_{X/k,e}$. This bundle comes equipped with a canonical trivialization $(e \times 1)^*(\wp) \simeq \mathcal{O}_{\mathrm{Pic}_{X/k,e}}$. The connected component of $\mathrm{Pic}_{X/k,e}$ containing the identity is denoted $\mathrm{Pic}^o_{X/k,e}$. The restriction of $\wp$ to the scheme $\mathrm{Pic}^o_{X/k,e}$ is denoted $\wp^o$ and is called the **Poincaré bundle**. In general, $\mathrm{Pic}^o_{X/k,e}$ is geometrically connected and quasi-compact.

Exercise.

1. If $X_{\bar{k}}$ is smooth, then $\mathrm{Pic}^o_{X/k,e}$ is proper (and hence projective over k). Hint: use the valuative criterion.

2. The scheme $\mathrm{Pic}_{X/k,e}$ can be non-smooth even when X/k is smooth. In fact, one can take X to be a surface over a field of characteristic p. Hint: This is highly non-trivial. Examples can be found among the surfaces discussed in Igusa's paper [2]. For more details, see Mumford's book *Lectures on Curves on an Algebraic Surface* [4]

We now specialize to the case of an abelian variety A/k.

Theorem 6. *If A/k is an abelian variety, then the Picard scheme $A^\vee/k = Pic^o_{A/k,e}/k$ is smooth and hence an abelian variety.*

We call $A^\vee$ the **dual abelian variety** of A.

Remark: By construction, the Poincaré sheaf $\wp^o$ has a canonical trivialization over $\{e\} \times A^\vee$. The Poincaré sheaf also has a distinguished trivialization over $A \times \{e^\vee\}$. The argument for this is as follows. By the functorial definition of $\mathrm{Pic}_{X/T,e}$, the restriction of $\wp^o$ to $A \times \{e^\vee\}$ is equal to the identity element of $\mathrm{Pic}^0_{A/k,e}(k)$. Now the identity element of this group is the trivial bundle on A equipped with its canonical trivialization. In particular, the restriction of $\wp^o|_{A\times\{e^\vee\}}$ has a distinguished trivialization.

The bundle $\wp^o$ equipped with this distinguished trivialization over $A \times \{e^\vee\}$ induces a morphism $\kappa_A : A \to (A^\vee)^\vee$. The morphism κ_A of abelian varieties maps the identity section to the identity section. By theorem 1, any such morphism is a homomorphism of abelian varieties.

Theorem 7. *The morphism κ_A is an isomorphism*

Proof. Reduce to the case where k is algebraically closed. This case is covered by Mumford [5]. □

Given $\phi : A \to A^\vee$, there is an induced morphism $(A^\vee)^\vee \to A^\vee$. The homomorphism κ_A can be used to identify A with $(A^\vee)^\vee$. Once this is done, we obtain a morphism $A \to A^\vee$ called the **dual map**, denoted $\phi^\vee$. We say that ϕ is **symmetric** if $\phi = \phi^\vee$.

A **polarization** of A is a symmetric isogeny $\phi : A \to A^\vee$ with the property that $(1, \phi)^*(\wp)$ is ample. A polarization is said to be a **principal polarization** if it is of degree 1.

A symmetric homomorphism of an abelian variety is analogous to a symmetric bilinear form on a finite dimensional real or rational vector space. Under this analogy, a polarization is analogous to a positive-definite quadratic form on such a vector space. The Poincaré line bundle corresponds to the evaluation pairing $V \times V^\vee \to k$ on a vector space. In the complex-analytic theory of abelian varieties, this analogy can be made more precise by relating polarizations to bilinear pairings on $H_1(A(\mathbb{C}), \mathbb{Z})$.

In the general algebraic setting, the homology group $H_1(A(\mathbb{C}), \mathbb{Z})$ is replaced with the Tate module. Fix a prime ℓ not equal to the characteristic. By definition, there is an evaluation pairing $A[\ell^\infty] \times (A[\ell^\infty])^\vee \to \mu_{\ell^\infty}$. Since ℓ is not equal to the characteristic of the ground field, this pairing can be identified with the evaluation pairing $e : T_\ell(A) \times T_\ell(A)^\vee \to \mathbb{Z}_\ell(1)$.

Given a symmetric homomorphism $\phi : A \to A^\vee$, there is an induced map on the ℓ-divisible groups $A[\ell^\infty] \to A^\vee[\ell^\infty]$. One can show that $A^\vee[\ell^\infty]$ can be identified with the Cartier dual $(A[\ell^\infty])^\vee$ of $A[\ell^\infty]$. By universal formalism, giving a homomorphism from an ℓ-divisible group to its Cartier dual is equivalent to giving a bilinear pairing $e_\phi : A[\ell^\infty] \times A[\ell^\infty] \to \mu_{\ell^\infty}$. Since ℓ is not equal to the characteristic, the ℓ-divisible groups $A[\ell^\infty]$ and μ_{ℓ^∞} can be identified with their groups of geometric points with the induced Galois action. After making this identification, we obtain a bilinear pairing $e_\phi : T_\ell(A) \times T_\ell(A) \to \mathbb{Z}_\ell(1)$ called the **Weil pairing**. This pairing is the pairing induced by e via ϕ in the sense that $e_\phi(x, y) = e(x, \phi(y))$. When ϕ is a polarization, it can be shown that e_ϕ is non-degenerate. However, one should beware that the pairing e_ϕ is *skew-symmetric*, not symmetric.

Using the classification of simple, finite, connected, commutative p-divisible groups over an algebraically closed field, one can describe all polarizations ϕ of an abelian variety over $\bar{k}$ and deduce that $\deg \phi$ is a perfect square. In the complex-analytic case, this property follows from the fact that a non-degenerate symplectic space over the integers has square determinant (via Pfaffians).

The notion of the dual abelian scheme can be defined over a fairly general base S. In this generality, one can still define what is meant by a polarization. The fiber-wise degree of a polarization of an abelian scheme is (Zariski-)locally constant on the base S. Given this formalism, one can prove the following theorem:

Theorem 8 (Mumford)**.** *Fix integers $g, d, N \geq 1$. For any $\mathbb{Z}[1/N]$-scheme S, let $M_{g,d,N}(S)$ be the set of isomorphism classes of triples (A, ϕ, i), where A is an abelian scheme over S of relative dimension g, ϕ is a degree d^2 polarization on A, and $i : \underline{(\mathbb{Z}/N\mathbb{Z})}^{2g} \to A[N]$ is an isomorphism of S-groups. Such triples admit no non-trivial automorphisms for $N \geq 3$, and for such N the functor $M_{g,d,N}$ is represented by a quasi-projective $\mathbb{Z}[1/N]$-scheme. In particular, up to isomorphism, over any finite field there are only finitely many g-dimensional abelian varieties equipped with a polarization of degree d^2.*

References

[1] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, pages x+325. Springer-Verlag, Berlin, 1990.

[2] Jun-ichi Igusa. On some problems in abstract algebraic geometry. *Proc. Nat. Acad. Sci. U. S. A.*, 41:964–967, 1955.

[3] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*, pages xiv+292. Springer-Verlag, Berlin, third edition, 1994.

[4] David Mumford. *Lectures on curves on an algebraic surface.* With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59. Princeton University Press, Princeton, N.J., 1966.

[5] David Mumford. *Abelian varieties*, pages viii+242. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.