MODULI OF ABELIAN VARIETIES

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LECTURE 1 (November 6, 2012 - Jesse Kass)

In this lecture we recall the complex analytic construction of the Torelli map and then explain how to construct this map using algebraic geometry. We begin by fixing $g \geq 1$ and working over the field of complex numbers $k = \mathbb{C}$. Define $\mathcal{H}_g$ equal to the Siegel upper half-plane. Recall that is

$$\mathcal{H}_g =: \{\tau \in \mathrm{M}_g(\mathbb{C})\colon \tau^t = \tau, \mathrm{Im}\,\tau > 0\}.$$

Here τ^t denotes the transpose, and we write $\mathrm{Im}\,\tau > 0$ indicate that the symmetric matrix $\mathrm{Im}\,\tau$ is positive definite.

We can construct elements of $\mathcal{H}_g$ as follows. Let C be a compact Riemann surface of genus g. Fix a basis $A_1, \dots, A_g, B_1, \dots, B_g$ for the homology $H_1(C, \mathbf{Z})$ such that the intersection pairing satisfies

$$\begin{aligned} \#A_i \cap A_j =& 0 \\ \#B_i \cap B_j =& 0 \\ \#A_i \cap B_i =& +1 \\ \#A_i \cap B_j =& 0 \text{ if } i \neq j. \end{aligned}$$

Then there is a *unique* basis $\omega_1, \dots, \omega_g$ of holomorphic 1-forms with the property that

$$\int_{B_i} \omega_j = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Consider the matrix formed by the remaining periods

$$\tau_C := (\tau_{i,j} = \int_{A_i} \omega_j).$$

Riemann's Bilinear Relations Theorems states that $\tau_C \in \mathcal{H}_g$, so the rule that assigns to C together with the basis $(A_1, \cdots, A_g, B_1, \cdots, B_g)$ the matrix τ_C defines a set map

$$\{(C; A_1, \dots, A_g, B_1, \dots, B_g)\} \to \mathcal{H}_g \tag{1}$$

from the set of pairs consisting of a compact Riemann surface C and a symplectic basis $(A_1, \dots, A_g, B_1, \dots, B_g)$ for the homology of C. Given a Riemann surface C, the symplectic or modular group

$$\Gamma_g := \mathrm{Sp}(2g, \mathbf{Z})$$

acts transitively on the collection of symplectic bases for $H_1(C, \mathbf{Z})$. We can make Γ_g act on $\mathcal{H}_g$ by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = (a\tau + b)(c\tau + d)^{-1}.$$

With this action, the map (1) is Γ_g-equivariant, so there is an induced quotient map

$$t \colon M_g \to \mathcal{H}_g / \Gamma_g := A_{g,1}$$

out of the set M_g of compact Riemann surfaces of genus g. This map is the Torelli map.

We have constructed the Torelli map using elementary Hodge theory. We would like to interpret this construction in terms of moduli and then construct t over an arbitrary field using tools from algebraic geometry.

To begin, let us interpret $A_{g,1}$ as a (coarse) moduli space. Given $\tau \in \mathcal{H}_g$, define $\Lambda_\tau \subset \mathbb{C}^g$ to be abelian group spanned by the columns $\begin{pmatrix} \tau & \mathrm{id}_g \end{pmatrix}$ and set X_τ equal to the complex torus

$$X_\tau := \mathbb{C}^g / \Lambda_\tau.$$

This torus admits a distinguished divisor defined as follows. The expression

$$\Theta(z, \tau) := \sum_{m \in \mathbf{Z}^g} \exp i\pi(m^t \tau m + 2 m^t z)$$

defines a holomorphic function on $\mathbf{C}^g$ known as the Riemann theta function. The Riemann theta function is not Λ_τ-invariant, but the zero locus is invariant, and we set

$$\Theta_\tau := \{[z] \in X_\tau \colon \Theta(z, \tau) = 0\} \subset X_\tau.$$

One can show that Θ_τ is an ample divisor with $\chi(\mathcal{O}(\Theta_\tau)) = 1$. The reader familiar with the literature on moduli of high dimensional varieties might be tempted to conjecture that $A_{g,1}$ is the coarse moduli space of pairs (X_τ, Θ_τ), but this unfortunately is incorrect. Given $\gamma \in \Gamma_g$ with $\tau_2 = \gamma \cdot \tau_1$ with $\tau_1 \in \mathcal{H}_g$, an isomorphism

$$i_\gamma \colon X_{\tau_1} \to X_{\tau_2}$$

is defined by $z \mapsto ((c\tau_1 + d)^t)^{-1} z$, but this isomorphism does NOT satisfy $i_\gamma^{-1}(\Theta_{\tau_2}) = \Theta_{\tau_1}$. Rather, we have

$$i_\gamma^{-1}(\Theta_{\tau_2}) = \Theta_{\tau_1} + z_\gamma$$

for a certain $z_\gamma \in X_{\tau_1}$.

What is unique is the polarization determined by Θ_τ. The polarization is the cohomology class $c_1(\mathcal{O}(\Theta_\tau)) \in H^{1,1}(X_\tau)$. This cohomology class can be described very explicitly. We have

$$H^{1,1}(X_\tau) = \{\text{Hermitian forms } H \text{ on } \mathbb{C}^g \text{ with } \mathrm{Im}(\Lambda_\tau, \Lambda_\tau) \subset \mathbf{Z}\ \},$$

and under this identification $c_1(\mathcal{O}(\Theta_\tau))$ is the Hermitian form H_τ whose matrix with respect to the standard basis is $(\mathrm{Im}\,\tau)^{-1}$. The reader may check that $\mathrm{Im}\, H_\tau$ has integer values on Λ_τ, and the form is independent of τ in the sense that if τ_1 and τ_2 are as before, then

$$z_1^t (\mathrm{Im}\,\tau_1)^{-1} \overline{z}_2 = w_1^t (\mathrm{Im}\,\tau_2)^{-1} \overline{w}_2$$

for

$$w_i := ((c\tau_1 + d)^t)^{-1} w_i.$$

The form H_τ is not an arbitrary element of $H^{1,1}(X_\tau)$. The form is positive defined and the imaginary part $\operatorname{Im} H_\tau$, which is skew-symmetric form, is unimodular. An element $H \in H^{1,1}(X_\tau)$ such that $\operatorname{Im} H_\tau$ is unimodular is called a principal polarization. The quotient $A_{g,1}$ is the moduli space of principally polarized complex tori of dimension g. The Torelli map $M_g \to A_{g,1}$ is the map that associates to a Riemann surface C the principally polarized complex torus defined by the period matrix. This is a moduli-theoretic description of the Torelli map, but it is still not algebraic description.

Mumford gave an algebraic description and construction of $A_{g,1}$ in his book **cite book**. We now replace $\mathbb{C}$ with an arbitrary algebraically closed field k. The algebraic analogue of a complex torus is an **abelian variety**, which we defined to be a proper smooth connected k-group scheme of finite type. The algebraic definition of a polarization is a bit more complicated. To define it, we ned to review some facts about line bundles on an abelian variety.

Associated to an abelian variety X is the Picard scheme $\operatorname{Pic}_{X/k} = \operatorname{Pic}(X/k)$, which is defined as follows. Write 0 for the identity of X. The Picard scheme is the k-scheme that represents the functor $\mathcal{P}ic_(X/k)$ that assigns to a k-scheme T the set of isomorphism classes of pairs (L, t) consisting of a line bundle L on $X \times_k T$ and isomorphism (or trivialization)

$$t \colon L|_{0 \times_k T} \cong \mathcal{O}_T.$$

The reader may verify that this rule defines a functor, and it is a difficult theorem of Grothendieck that $\operatorname{Pic}_{X/k}$ exists as a locally finite type k-scheme. Because $\operatorname{Pic}_{X/k}$ is locally of finite type, it makes sense to talk about the connected components of $\operatorname{Pic}_{X/k}$, and we define the **dual abelian variety** $X^\vee$ to be the connected component containing the trivial line bundle. The name dual abelian variety is not a misnomer.

Theorem 0.0.1. *The dual abelian variety $X^\vee$ is an abelian variety.*

Proof. Tensor product (of line bundles and trivializations) defines a group law on $X^\vee$, so the content of the theorem is that $X^\vee$ is smooth and proper. Properness can be verified using the valuative criteria, and smoothness can be verifying by an infinitesimal computation (though one must take care: the relevant obstruction group $H^2(X, \mathcal{O}_X)$ is nonzero). Alternatively, one can construct $X^\vee$ as Mumford does in **BOOK** by fixing an ample line bundle L and then defining $X^\vee$ to be the quotient of X but the finite subgroup scheme $K(L)$ that we defined below. □

Now suppose that we are given a line bundle L on X. Define $m \colon X \times X \to X$ to be the addition morphism (coming from the k-group scheme structure) and $p_1, p_2 \colon X \times X \to X$ to be the projection morphisms. Consider the line bundle $m^* L \otimes p_1 L^{-1} \otimes p_2 L^{-1}$ on $X \times X$. This line bundle has the property that the restriction to $0 \times X = X$ is canonically isomorphic to $L \otimes \mathcal{O}_X \otimes L^{-1}$, and in particular, there is a canonical trivialization t_{can} of $m^* L \otimes p_1 L^{-1} \otimes p_2 L^{-1}$. The pair $(m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}, t_{\text{can}})$ is an element of $\mathcal{P}ic_{X/k}(X) = \operatorname{Hom}(X, \operatorname{Pic}_{X/k})$, so it determines a morphism $X \to \operatorname{Pic}_{X/k}$. In fact, the image lies in $X^\vee \subset \operatorname{Pic}_{X/k}$ because the image is connected and contains $\mathcal{O}_X$ (which is the image of $0 \in X$). We define

$$\phi_L \colon X \to X^\vee$$

to be the associated morphism. Informally, this is the morphism that sends a point $x \in X$ to the line bundle $T_x^* L \otimes L^{-1}$, where $T_x \colon X \to X$ is the map given by translation by x.

LECTURE 2 (November 13, 2012 - Matteo Tommasini)

Theorem 0.0.2. *The morphism ϕ_L has the following properties:*

(1) for every line bundle L on X, ϕ_L is a group homomorphism;
(2) $\phi_L = 0$ if and only if $L \in X^\vee$;
(3) $\phi_{L\otimes M} = \phi_L + \phi_M$ and $\phi_{L^{-1}} = -\phi_L$; this means that we have a group homomorphism

$$\Phi : \mathsf{Pic}(X) \to \mathit{Hom}(X, X^\vee)$$

given by $L \mapsto \phi_L$.
(4) $\deg(\phi_L) = \chi(L)^2$.
(5) provided $h^0(X, L) \neq 0$, $K(L) := \phi_L^{-1}(0)$ is finite if and only if L is ample.

By the above theorem, if L is an ample line bundle, then ϕ_L is surjective with finite kernel $K(L)$. Furthermore, two ample line bundles L and M define the same surjection precisely when $L = M \otimes N$ for $N \in X^\vee$. When $k = \mathbb{C}$, one can show that the condition $L = M \otimes N$ if an only if $c_1(L) = c_1(M)$ or, equivalently, the two associated Hermitian forms are equal. We have thus found our algebraic definition of a polarization.

Definition 0.0.3. *A **polarization of** X is a surjective homomorphism $\phi \colon X \to X^\vee$ with the property that there exists an ample line bundle L with $\phi = \phi_L$. We say that a polarization ϕ has degree d if the degree of the finite k-scheme $K(\phi) := \ker(\phi)$ is d^2. A **principal polarization** is a polarization of degree 1.*

We would now like to give an algebraic definition of the moduli space $A_{g,1}$ of principally polarized abelian varieties. In order to do that, first we need to set the definitions of "family" of abelian variety (namely abelian schemes), dual of any such family and polarizations.

First of all, we need to recall the definition of the relative functor $\mathcal{P}ic$.

Definition 0.0.4. *Let us fix any flat projective morphism $\pi : X \to T$. Then for all locally noetherian T-schemes $f : S \to T$ we set (see GIT, pp.22-23)*

$$\mathcal{P}ic_{X/T}(S) := \frac{\{\text{group of invertible sheaves on } X_S := X \times_T S\}}{\{\text{subgroup of sheaves of the form } p_2^*(M) \text{ for } M \in \mathsf{Pic}(S)\}}, \tag{2}$$

where we are considering a cartesian diagram:

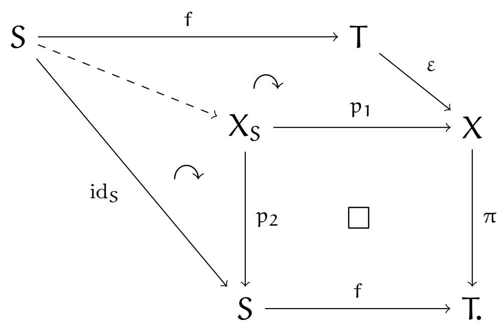
$$\begin{array}{ccc} X_S & \xrightarrow{p_1} & X \\ \downarrow{\scriptstyle p_2} & \square & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{f} & T. \end{array}$$

If $\pi : X \to T$ has a section $\varepsilon : T \to X$, then we can prove that

$$\mathcal{P}ic_{X/T}(S) = \{\text{group of pairs of isomorphism classes of pairs } (L, t)$$

(3) $$\text{where } L \text{ is any invertible sheaf on } X_S \text{ and } t : (\varepsilon \circ f, \mathrm{id}_S)^* L \tilde{\to} \mathcal{O}_S\}.$$

Here we the map $(\varepsilon \circ f, \mathrm{id}_S)$ is the unique morphism induced by the following diagram:



Even without the existence of ε, $\mathcal{P}ic_{X/T}$ is a contravariant functor from locally noetherian T-schemes to abelina groups. A known fact is that there exists a scheme $\mathrm{Pic}_{X/T}$ over T and a natural transformation of functors

$$\delta : \mathcal{P}ic_{X/T} \to \underline{\mathrm{Pic}_{X/T}}$$

i.e. a family of compatible set maps

$$\delta_S : \mathcal{P}ic_{X/T}(S) \to \mathrm{Hom}_T(S, \mathrm{Pic}_{X/T})$$

for S varying in the category of S-schemes, such that

(a) δ_S is always injective;
(b) δ_S is surjective whenever $X_S \to S$ admits a section.

In particular, if $X \to T$ has a section, then $X_S \to S$ has a section for every T-scheme $S \to T$. Therefore in this case δ is a natural equivalence of functors, so the functor $\mathcal{P}ic_{X/T}$ is represented by $\mathrm{Pic}_{X/T}$.

Also when $\pi : X \to T$ has no sections, we can write $\mathrm{Pic}_{X/T}$ as a disjoint union of components $\mathrm{Pic}^P_{X/T}$ defined as follows. Take any sheaf $\mathcal{O}_X(1)$, relatively ample for π. For all invertible sheaves L on X_S and for all $s \in S$, let L_s be the sheaf induced by restriction on $X \times_T \mathrm{Spec}(k(s))$. Then for every polynomial $P \in \mathbb{Z}[n]$, $\mathrm{Pic}^P_{X/T}$ is the connected component of $\mathrm{Pic}_{X/T}$ associated to the functor $\mathcal{P}ic^P_{X/T}$ defined for every T-scheme S by

$$\mathcal{P}ic^P_{X/T}(S) := \{[L] \text{ for } L \text{ invertible sheaf on } X_S \text{ such that}$$

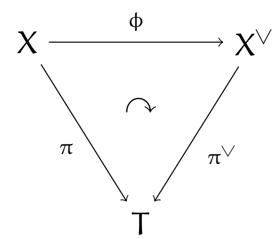
$$\chi(L_s(n)) = P(n) \text{ for } n >> 0\} \subset \mathcal{P}ic_{X/T}(S).$$

Every $\mathsf{Pic}_{X/T}$ is a quasi-projective scheme over T.

Definition 0.0.5. *Given a* k*-scheme* T*, we define an* ***abelian scheme*** $\pi : X \to T$ *of dimension* g *to be any finitely presented, smooth, proper* T*-group scheme with connected geometric fibers of dimension* g*. Given an abelian scheme* X/T*, we define the dual abelian scheme* $X^\vee/T$ *to be the connected component of the relative Picard scheme* $\mathrm{Pic}_{X/T}/T$ *containing the trivial line bundle. We denote by* $\pi^\vee$ *the structure morphism* $X^\vee \to T$*.*

Note that since $X \to T$ is a T-group, we have a section of π, namely the identity ε of the T-group, so we can use either (2) or (3).

Definition 0.0.6. *A* ***polarization*** *of an abelian scheme* X/T *is a* T*-group scheme homomorphism*



such that for all geometric points t *of* T *the induced homomorphism*

$$\phi_t \colon X_t \to (X^\vee)_t = (X_t)^\vee$$

is a polarization in the sense defined earlier. We say that the degree of ϕ *is* d *is the degree of every* ϕ_t *is so. A principal polarization is a polarization* ϕ *of degree* 1*.*

In other terms, we are imposing that for every t of T there exists an ample line bundle L_t on X_t such that $\phi_t = \phi_{L_t}$. We would like to prove that there is a global L over X such that $\phi = \phi_L$. Actually, this will not always be the case (see below).

Since $\pi : X \to T$ has a section ε, then by the previous description of the functor $\mathcal{P}ic_{X/T}$ we get that such a functor is represented by $\mathsf{Pic}_{X/T}$. In particular, there exists a universal object

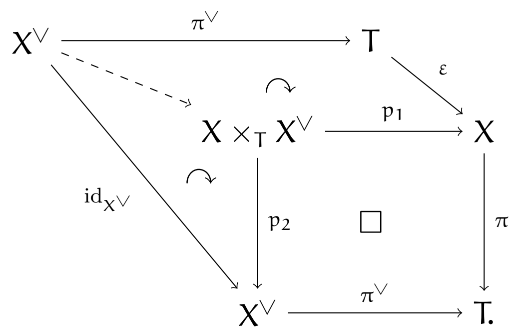
$$\mathbf{univ} \in \mathcal{P}ic_{X/T}(\mathsf{Pic}_{X/T}).$$

The inclusion $X^\vee \hookrightarrow \mathsf{Pic}_{X/T}$ induces a map

$$\mathcal{P}ic_{X/T}(\mathsf{Pic}_{X/T}) \to \mathcal{P}ic_{X/T}(X^\vee).$$

We denote by $\mathbf{univ}'$ the image of $\mathbf{univ}$ by this map. By (2) $\mathbf{univ}'$ is the class of an invertible sheaf $\mathcal{L}$ over $X \times_T X^\vee$, modulo tensor product with objects of the form $p_2^*(M)$ for

M invertible sheaf on $X^\vee$. Now let us denote by $\varepsilon \circ \pi^\vee \times \mathrm{id}_{X^\vee}$ the unique morphism induced by the following diagram



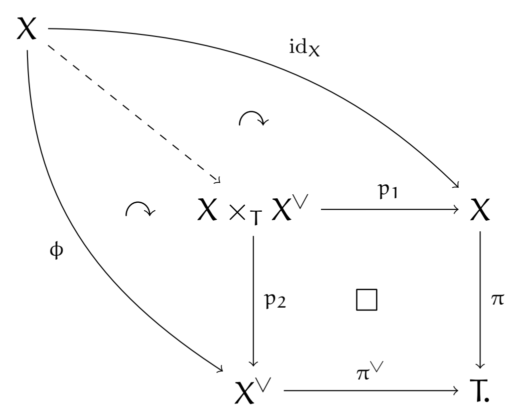
Then there exists a unique $\mathcal{L}_{\text{univ}}$ in the class **univ**′ such that

$$(\varepsilon \circ \pi^\vee) \times \mathrm{id}_{X^\vee})^* \mathcal{L}_{\text{univ}} = \mathcal{O}_{X^\vee}$$

(see pp. 120-121 GIT). Now for every morphism of T-schemes $\phi : X \to X^\vee$ we set

$$L^\Delta(\phi) := (\mathrm{id}_X, \phi)^*(\mathcal{L}_{\text{univ}}).$$

Here (id_X, ϕ) is the unique morphism induced by the following diagram (we recall that ϕ is a morphism of T-schemes, so the external part of the diagram commutes)



So we have associated to every morphism of T-schemes $\phi : X/T \to X^\vee$ an invertible sheaf $L^\Delta(\phi)$ on X.

Theorem 0.0.7. *(GIT, proposition 6.10) If* $\overline{\phi}$ *is any polarization for* X/T, *then* $\phi_{L^\Delta(\overline{\phi})} = 2\overline{\phi}$.

It turns out (GIT proposition 6.11) that the objects of the form $L^\Delta(\phi)$ are relatively ample with respect to $\pi : X \to T$. Moreover, any relatively ample L can be obtained in at most one way as $L = L^\Delta(\phi)$ for a polarization ϕ of X/T.

In addition, for every polarization ϕ of X/T, the sheaf $(L^{\Delta}(\phi))^{\otimes 3}$ is very ample with respect to π. For any polarization ϕ, let us consider the $\mathcal{O}_T$-module

$$\mathcal{E}(\phi) := \pi_*((L^{\Delta}(\phi))^{\otimes 3}).$$

It turns out that $\mathcal{E}$ is locally free and that

$$\mathrm{rk}(\mathcal{E}(\phi)) = 6^g \cdot d$$

where d is the degree of ϕ. In particular, if ϕ is a principal polarization, we have that such rank is equal to 6^g.

Now for every k-scheme T over k we set

$$\mathcal{A}_{g,1}(T) := \{\text{all pairs } (X/T, \phi) \text{ s.t. } X/T \text{ is an abelian scheme and } \\ \phi : X \to X^{\vee} \text{ is a principal polarization of } X/T\}.$$

It turns out that $\mathcal{A}_{g,1}$ is a contravariant functor from the category of k-schemes to the category of sets.

Theorem 0.0.8. *(Mumford)* $\mathcal{A}_{g,1}$ *has a* coarse *moduli space* $A_{g,1}$ *that is a quasi-projective* k*-scheme.*

Proof. (sketch) For every pair $(X/T, \phi)$ the sheaf $(L^{\Delta}(\phi))^{\otimes 3}$ is very ample with respect to $\pi : X \to T$, so it induces an embedding $X \hookrightarrow \mathbb{P}^m \times T$, where $m := 6^g - 1$. To be more precise, there exists an isomorphism θ and a diagram as follows:

$$\begin{array}{ccccc} X & \hookrightarrow & \mathbb{P}(\mathcal{E}(\phi)) & \xrightarrow[\sim]{\theta} & \mathbb{P}^m \times T. \\ & \searrow^{\pi} \ \curvearrowright & \downarrow & \curvearrowright \ \swarrow_{p_2} & \\ & & T & & \end{array}$$

We say that any θ as before is a *linear rigidification* of the pair $(X/T, \phi)$.

In general linear rigidifications are not unique because one can always replace θ with $(\alpha \times \mathrm{id}_T) \circ \theta$ for any $\alpha \in \mathrm{Aut}(\mathbb{P}^m) = PGL(m+1)$. Therefore Mumford defines also another contravariant functor $\mathcal{H}_{g,1}$ on the category of k-schemes by setting

$$\mathcal{H}_{g,1}(T) := \{\text{all triples } (X/T, \phi, \theta) \text{ s.t. } X/T \text{ is an abelian scheme,}$$

$$\phi : X \to X^\vee \text{ is a principal polarization of } X/T$$

$$\text{and } \theta \text{ is a linear rigidification of } (X/T, \phi)\}$$

for each k-scheme T. First of all, Mumford proves that this functor is represented by a quasi-projective scheme $H_{g,1}$ that is obtained as a suitable locally closed subscheme of an Hilber scheme $\mathtt{Hilb}^P_{\mathbb{P}^m}$. The group PGL$(m+1)$ acts on $H_{g,1}$ by change of basis and $A_{g,1}$ should be the quotient of $H_{g,1}$ by PGL$(m+1)$. Mumford shows that this quotient exists bt using the method of the covariant.

Mumford fixes a large integer M and then shows that the rule that assigns to a linearly rigidified abelian variety the M-torsion on the abelian variety defines a PGL$(m+1)$ morphism $H_{g,1} \to \mathbb{P}^m \times \cdots \times \mathbb{P}^m$ whose image is contained in the open locus $(\mathbb{P}^m \times \cdots \times \mathbb{P}^m)_{st}$ of GIT stable points. The geometric quotient of this stable locus exists by the main theorem of GIT, and the main theorem of the covariant asserts that we can conclude that the geometric quotient of $H_{g,1}$ exists. This completes the proof sketch. □

LECTURE 3 (November 20, 2012 - Nicola Tarasca) - THINGS NOT SAID IN THE SECOND LECTURE

We can now algebraically construct the Torelli map $t \colon M_g \to A_{g,1}$. Recall that M_g is the coarse moduli space of smooth curves. That is, M_g is the quasi-projective k-scheme that co-represents the functor $\mathcal{M}_g$ that assigns to a scheme T the set of isomorphism classes of proper smooth T-curve $C \to T$ with geometrically connected genus g smooth fibers.

Definition 0.0.9. *Given a T-curve $C \to T$, for any d we denote by J^d be the subscheme of* Pic$_{C/T}$ *that parameterizes degree d line bundles. We set $J(C/T) := J^0$.*

Note: we don't write $C^\vee$ for $J(C/T)$ since in general $C \to T$ is not an abelian scheme. Nonetheless, we have:

Theorem 0.0.10. $J(C/T) \to T$ *is an abelian scheme.*

Now we want to show that $J(C/T) \to T$ admits a distinguished principal polarization. If we are able to do that, then we can associate to any element $(C \to T) \in \mathcal{M}_g(T)$ a point in $\mathcal{A}_{g,1}(T)$. This will give a natural transformation of functors

$$\alpha : \mathcal{M}_g \to \mathcal{A}_{g,1}$$

and this will correspond to the Torelli map $\alpha : M_g \to A_{g,1}$.

In order to do that, first of all let us consider the diagonal $\Delta \subset C \times_T C$. This is a relative Cartier divisor over C; the associated line bundle $\mathcal{O}_{C \times_T C}(\Delta)$ is such that its restriction to every point of C gives a line bundle on C of degree 1. Therefore $\mathcal{O}_{C \times_T C}(\Delta)$ induces a unique morphism

$$\delta : C \to J^1.$$

Suppose that $\pi : C \to T$ has a section ε and define the morphism of T-schemes

$$\theta_\varepsilon := \delta - \delta \circ \varepsilon \circ \pi : C \to J(C/T).$$

Set theoretically, this morphism is given by

$$p \in C \mapsto \mathcal{O}_C(p - \varepsilon(\pi(p)))$$

By pullbacks, θ_ε induces a morphism

$$\mathsf{Pic}_{J(C/T)} \to \mathsf{Pic}_{C/T}$$

and by restriction to line bundles of degree zero also a morphism:

$$\psi_\varepsilon : J^\vee(C/T) \to J(C/T).$$

Lemma 0.0.11. *Suppose that $\pi : C \to T$ has a section ε. Then the morphism ψ_ε is an isomorphism.*

Lemma 0.0.12. *If there are 2 sections $\varepsilon_i : T \to X$ for $i = 1, 2$, then*

$$\psi_{\varepsilon_1} = \psi_{\varepsilon_2}.$$

Now by base change $S \to T$ we can always assume that $\pi : X \to T$ has a section, at least étale locally. Since any 2 sections as before give rise to the same morphism we concllude that there is a unique

$$\psi : J^\vee(C/T) \to J(C/T)$$

that is induced by the local ψ_ε (note: we are not claiming that there exists a global section ε, but only that there exists a global ψ). Since ψ_ε is always an isomorphism, so is ψ.

Theorem 0.0.13. $\psi^{-1} : J(C/T) \to J^\vee(C/T)$ *is a polarization of the abelian scheme* $J(C/T)/T$.

LECTURE 3 (November 20, 2012 - Nicola Tarasca) - THETA DIVISOR

Definition 0.0.14. *Given an Abelian variety A, a Poincaré sheaf $\mathcal{P}$ is a sheaf on $A \times A^\vee$ such that (i) $\mathcal{P}|_{A\times\{b\}} \in \mathrm{Pic}^0(A \times \{b\})$ for every $b \in A^\vee$, and (ii) $\mathcal{P}|_{\{0_A\}\times A^\vee}$ is trivial.*

Given an Abelian variety A, a pair $(A^\vee, \mathcal{P})$, where $A^\vee$ is the dual Abelian variety and $\mathcal{P}$ is a Poincaré sheaf, satisfies the following universal property: for every pair $(B, \mathcal{L})$ with B an algebraic variety and $\mathcal{L}$ a sheaf on $A \times B$ verifying the following two properties: (i) $\mathcal{L}|_{A\times\{b\}} \in \mathrm{Pic}^0(A \times \{b\})$ for every $b \in B$, and (ii) $\mathcal{L}|_{\{0_A\}\times B}$ is trivial, there exists a unique regular map $\alpha\colon B \to A^\vee$ such that $(1 \times \alpha)^*\mathcal{P} \cong \mathcal{L}$.

Let C be a complete nonsingular curve over a field k, with $\mathrm{char}(k) = 0$. Let P^0_C be the functor defined as follows: given T an algebraic space over k,

$$P^0_C(T) := \{\mathcal{L} \in \mathrm{Pic}(C \times T) \mid \deg(\mathcal{L}|_t) = 0 \,\forall\, t\}/\pi_2^*\mathrm{Pic}(T).$$

Theorem 0.0.15. *There exists an Abelian variety J defined over k that represents the functor P^0_C.*

Proposition 0.0.16. *The tangent space to J at 0 is canonically isomorphic to $H^1(C, \mathcal{O}_C)$. In particular,* $\dim J = \mathrm{genus}(C) =: g$.

When $g = 0$, one has $J = 0$. In the following we will assume $g > 0$.

Definition 0.0.17. *Given $P \in C$, let us define $f_P\colon C \to J$ so that $Q \mapsto \mathcal{L}(Q) \otimes \mathcal{L}(p)^{-1}$, or equivalently $Q \mapsto [Q - P]$, the linear equivalence class of $Q - P$.*

Remark 1. *(i) The induced map $\mathrm{Div}^0(C) \to J$, $\sum_Q n_Q \cdot Q \mapsto \sum_Q n_Q f_P(Q) = [\sum_Q n_Q \cdot Q]$ is independent of P, is surjective, and its kernel coincides with principal divisors.*

(ii) If $P' \in C$ is a different point, then $f_{P'} = t_{[P-P']} \circ f_P$.

In the following, set $f := f_P$. Let $f^r\colon C^r \to J$, $(P_1, \ldots, P_r) \mapsto [P_1 + \cdots + P_r - rP]$ be the induced map on C^r. The map f^r is symmetric, hence it induces a map $f^{(r)}\colon C^{(r)} := C^r/S_r = \mathrm{Div}^r(C) \to J$ such that $D \mapsto [D - rP]$ for $D \in \mathrm{Div}^r(C)$. The fiber of $f^{(r)}$ containing D is the complete linear system $|D|$.

Definition 0.0.18. $W^r := \mathrm{Im}(f^{(r)})$.

Theorem 0.0.19. *For every $r \leq g$, the morphism $f^{(r)}\colon C^{(r)} \to W^r$ is birational, in particular $f^{(g)}$ is a birational map from $C^{(g)}$ onto J.*

Corollary 0.0.20. *For every $r \leq g$, the morphism $f^r\colon C^r \to W^r$ has degree $r!$*

Proof. The morphism f^r is the composition of the followings $C^r \to C^{(r)} \to W^r$. □

Definition 0.0.21. *The theta divisor is defined as $\Theta := W^{g-1}$ in J.*

Remark 2. *(i) For every effective divisor D in J, consider $m^*\mathcal{L}(D) \otimes \pi_1^*\mathcal{L}(D)^{-1} \otimes \pi_2^*\mathcal{L}(D)^{-1} := \mathcal{L}'(D)$, where $m\colon J \times J \to J$ is the addition map, and $\pi_i\colon J \times J \to J$ is the projection on the i-th factor. Since $\mathcal{L}'(D)$ admits a trivialization on $\{0_J\} \times J$ and $J \times \{0_J\}$, it induces a morphism $\varphi_{\mathcal{L}(D)}\colon J \to J^\vee$.*

(ii) A divisor D is ample if and only if $\varphi_{\mathcal{L}(D)}$ is an isogeny. In this case $(1 \times \varphi_{\mathcal{L}(D)})^*\mathcal{P} = \mathcal{L}'(D)$.

Definition 0.0.22. *(i)* $\Theta^- = (-1)^*\Theta$, *where* $(-1)\colon J \to J$ *is the inverse map.*

(ii) $\Theta_a := \Theta + a$, *for* $a \in J$.

(iii) $\Theta_a^- := (\Theta^-)_a$.

Remark 3. $\varphi_{\mathcal{L}(\Theta^-)} = \varphi_{\mathcal{L}(\Theta)} = \varphi_{\mathcal{L}(\Theta_a)}$.

Theorem 0.0.23. *The morphism* $\varphi_{\mathcal{L}(\Theta)}\colon J \to J^\vee$ *is an isomorphism.*

In the remaining part of this section we will sketch a proof of this theorem. The treatment is loosely based on the proof from Milne's book "Abelian Varieties".

The proof can be divided into 4 steps.

Lemma 0.0.24. *Let* U *be the largest open subset of* J *such that: (i) the fibre of* $f^{(g)}\colon C^{(g)} \to J$ *at any point of* U *has dimension* 0*, and (ii) if* $a \in U$ *and* $D(a)$ *is the unique element in* $(f^{(g)})^{(-1)}(a)$*, then* $D(a) = \sum_{i=1}^g P_i$*, where* $P_i \neq P_j$ *for* $i \neq j$.

Then $f^{-1}(\Theta_a^-) = D(a)$ *for every* $a \in U$*, where* $f := f_P\colon C \to J$.

Proof. Let $a \in U$ and $D := D(a) = \sum_{i=1}^g P_i$ be the unique element in $(f^{(g)})^{(-1)}(a)$. Let $Q_1 \in C$. Then $f(Q_1) \in \Theta_a^-$ iff there exist $Q_2, \dots, Q_g$ such that $f(Q_1) = -\sum_{i=2}^g f(Q_i) + a$. That is, $\sum_{i=1}^g Q_i$ is linear equivalent to D. Since $a \in U$, it follows that $\sum_{i=1}^g Q_i = D$. Hence the support of $f^{-1}(\Theta_a^-)$ is $\{P_1, \dots, P_g\}$. Furthermore, one shows that $\deg f^{-1}(\Theta_a^-) \leq g$, hence the statement. □

Lemma 0.0.25. *Let* $a \in J$ *and* $D \in C^{(g)}$ *such that* $f^{(g)}(D) = a$*. Then* $f^*\mathcal{L}(\Theta_a^-) \cong \mathcal{L}(D)$.

Proof. The Lemma is true on a dense open subset of J by Lemma 1.0.24. □

Let $\mathcal{M}$ be a universal sheaf on $C \times J$, that is, (i) $\mathcal{M}|_{C\times\{a\}} \cong \mathcal{L}(D - gP)$, for D such that $f^{(g)}(D) = a$, and (ii) $\mathcal{M}|_{\{P\}\times J}$ is trivial.

Lemma 0.0.26. *On* $C \times J$ *we have* $(f \times (-1)_J)^*\mathcal{L}'(\Theta^-) \cong \mathcal{M}$.

Remark 4. *Note that the composition of the following maps* $C \to C \times \{a\} \xrightarrow{f\times(-1)} J \times J \xrightarrow{m} J$ *is* $t_{-a} \circ f$.

(i) $(f \times (-1))^* m^* \mathcal{L}(\Theta^-)|_{C\times\{a\}} \cong \mathcal{L}(t_{-a}^{-1}\Theta^-)|_{f(C)} \cong \mathcal{L}(\Theta_a^-)|_{f(C)} \cong f^*\mathcal{L}(\Theta_a^-)$.

(ii) $(f \times (-1))^* \pi_1^* \mathcal{L}(\Theta^-)|_{C\times\{a\}} \cong f^*\mathcal{L}(\Theta^-)$.

(iii) $(f \times (-1))^* \pi_2^* \mathcal{L}(\Theta^-)|_{C\times\{a\}} \cong \mathcal{O}$.

Proof of Lemma 1.0.26. From Remark 4 (i) and Lemma 1.0.25 we have

$$(f \times (-1))^* m^* \mathcal{L}(\Theta^-)|_{C\times\{a\}} \cong f^*\mathcal{L}(\Theta_a^-) \cong \mathcal{M} \otimes \pi_1^*\mathcal{L}(gP)|_{C\times\{a\}}.$$

When $a = 0$, one has $f^*\mathcal{L}(\Theta^-) \cong \mathcal{L}(gP)$. It follows that $(f \times (-1))^*\pi_1^*\mathcal{L}(\Theta^-) \cong \pi_1^*\mathcal{L}(gP)$. Finally

$$(f \times (-1))^*(m^*\mathcal{L}(\Theta^-) \otimes \pi_1^*\mathcal{L}(\Theta^-)^{-1}) \cong \mathcal{M} \otimes \pi_2^*\mathcal{N}$$

for some sheaf $\mathcal{N}$ on J. Computing the restriction of the sheaves to $\{P\} \times J$, one finds $\mathcal{N} \cong (-1)^*\mathcal{L}(\Theta^-)$. □

Consider the sheaf $(f \times 1)^*\mathcal{P}$ on $C \times J^\vee$. This sheaf comes with a trivialization on $\{P\} \times J^\vee$ and $C \times \{0_{J^\vee}\}$, hence there exists a unique homomorphism $f^\vee \colon J^\vee \to J$ such that $(1 \times f^\vee)^*\mathcal{M} \cong (f \times 1)^*\mathcal{P}$.

Lemma 0.0.27. *The morphism $\varphi_{\mathcal{L}(\Theta^-)}$ is injective, hence it is an isomorphism.*

Proof. We have the following isomorphisms

$$\begin{aligned}
(1 \times f^\vee \circ (-\varphi_{\mathcal{L}(\Theta^-)}))^*\mathcal{M} &\cong (1 \times (-\varphi_{\mathcal{L}(\Theta^-)}))^*(1 \times f^\vee)^*\mathcal{M} \\
&\cong (1 \times (-\varphi_{\mathcal{L}(\Theta^-)}))^*(f \times 1)^*\mathcal{P} \\
&\cong (f \times (-\varphi_{\mathcal{L}(\Theta^-)}))^*\mathcal{P} \\
&\cong (f \times (-1))^*(1 \times \varphi_{\mathcal{L}(\Theta^-)})^*\mathcal{P} \\
&\cong (f \times (-1))^*\mathcal{L}'(\Theta^-) \\
&\cong \mathcal{M}.
\end{aligned}$$

It follows that $f^\vee \circ (-\varphi_{\mathcal{L}(\Theta^-)}) \colon J \to J$ is such that $(1 \times f^\vee \circ (-\varphi_{\mathcal{L}(\Theta^-)}))^*\mathcal{M} \cong \mathcal{M}$, hence $f^\vee \circ (-\varphi_{\mathcal{L}(\Theta^-)})$ is the identity and $\varphi_{\mathcal{L}(\Theta^-)}$ is injective. □