Problem Set 8 (Due 3/12/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work. We will discuss the methods for computing the integrals in P3 & P4 on Tuesday 3/9.

P1. This problem gives another formula for the residue at a pole that is sometimes convenient. Suppose that f has a pole of order m at $z_0 \in \mathbb{C}$. Prove that

$$\mathrm{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)].$$

P2. For each function $f(z)$, classify all of its isolated singularities as either: removable, essential, or a pole. If an isolated singularity is a pole, make sure so specify the order. Compute the residue at every isolated singularity.[1]

(a) $f(z) = \frac{1}{\sin z}$;

(b) $f(z) = \frac{1}{\sin^2 z}$;

(c) $f(z) = \sin\left(\frac{1}{z}\right)$;

(d) $f(z) = \frac{z}{\cos z}$;

(e) $f(z) = \frac{1}{e^z - 1}$;

(f) $f(z) = ze^{\frac{1}{z-2}}$;

(g) $f(z) = \frac{e^z - 1}{z}$.

P3. Compute the following improper integral using complex analysis:

$$\int_0^\infty \frac{x^2}{x^6+1}\, dx.$$

P4. Let $a > 0$. Compute the following improper integral using complex analysis:

$$\int_0^\infty \frac{\cos ax}{x^2+1}\, dx.$$

[1] Note: computing a Laurent series is not always the easiest way to identify the type of singularity.