# Problem Set 7 (Due 3/5/21 11:59PM)

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| **Your Name:** |

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| **Names of Any Collaborators:** |

Please review the problem set guidelines in the syllabus before turning in your work.

**P1.** Use formal division and multiplication of power series to determine the first 4 nonzero terms of following series. It is not necessary to show the steps of the division/multiplication, but you should briefly explain which power series are being dividing/multiplied, and why it is justified.

(a) The Laurent series of $\frac{1}{e^z-1}$ on the annulus $0 < |z| < 2\pi$.

(b) The Taylor series of $e^z \cos z$ about 0.

(c) The Laurent series of $\csc z = \frac{1}{\sin z}$ on the annulus $0 < |z| < \pi$.

**P2.** Let $C$ be the positively oriented unit circle. Compute the following integrals. Justify your computations.

(a) $\int_C \frac{1}{e^z-1}\, dz$;

(b) $\int_C e^z \cos z\, dz$;

(c) $\int_C \csc z\, dz$.

**P3.** For each function $f(z)$, compute $\operatorname*{Res}_{z=0} f(z)$. Justify your computations.

(a) $f(z) = z\cos\left(\frac{1}{z}\right)$;

(b) $f(z) = \frac{\csc z}{z^4}$.

**P4.** For each function $f(z)$, compute $\int_C f(z)\, dz$ where $C$ is the positively oriented circle $|z| = \frac{31}{10}$.[1] Justify your computations.

(a) $f(z) = z\cos\left(\frac{1}{z}\right)$;

(b) $f(z) = \frac{\csc z}{z^4}$;

(c) $f(z) = \frac{1}{2-z}$;

(d) $f(z) = \frac{z}{(z-1)(z-3)}$;

(e) $f(z) = \frac{z}{1+z^2}$.

**P5.** Let $C$ be the unit circle with positive orientation. Let $n \geq 0$ be a nonnegative integer.

(a) The function $f(z) = z^n e^{\frac{1}{z}}$ has an isolated singularity at $z_0 = 0$. Find the Laurent series expansion for $f$ on the annulus $0 < |z| < \infty$.

(b) Use the theory of residues to compute the integral

$$\int_C z^n e^{\frac{1}{z}}\, dz.$$

[1] Hint: several of these functions appeared in PSet 6.

(c) Prove that[2]

$$\int_C e^{z+\frac{1}{z}}\,dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

**P6.** Consider the function

$$f(z) = \frac{2z^3+1}{(z^2-1)(z^2+1)}.$$

Show that $f(z)$ has no antiderivative on the domain $D = \{z \in \mathbb{C} : |z| > 1\}$.[3]

**P7. (Optional.)**[4]

(a) Suppose that $f$ is analytic everyone on $\mathbb{C}$, except at a finite number of (isolated) singular points $z_1, \dots, z_n$. Show that

$$\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

(b) Let

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0$$

be a polynomial of degree $n \geq 2$. Let $C$ be a positively oriented simple closed contour whose interior contains all the zeros of $p(z)$. Show that[5]

$$\int_C \frac{1}{p(z)}\,dz = 0.$$

(c) Using (a) and (b), compute the integral

$$\int_C \frac{1}{(z^{2021}+1)(z-3)}\,dz$$

where $C$ is the positively oriented circle $|z| = 2$.

[2] Hint: write $e^{z+\frac{1}{z}} = e^{\frac{1}{z}} \sum_{n=0}^{\infty} \frac{z^n}{n!}$, then apply the theorem on integrating power series.

[3] Hint: consider a circle $C$ that lies in $D$ and contains all the singularities of $f$ in it's interior. Compute $\int_C f(z)\,dz$ by computing the residue at $\infty$. Use the theorem that says $\operatorname{Res}_{z=\infty} f(z) = \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$.

[4] Worth 4 points of extra credit, added to you score on this assignment.

[5] Hint: show that $\operatorname{Res}_{z=\infty} \frac{1}{p(z)} = 0$.