Problem Set 6 (Due 2/26/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work.

P1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that there exists $M > 0$ and $N \in \mathbb{N}$ such that

$$|f(z)| \leq M(1 + |z|^N)$$

for all $z \in \mathbb{C}$. Prove that f is a polynomial of degree at most N.

P2. (a) Find the Taylor series expansion for $f(w) = \frac{1}{w}$ on the disk $D_1(1) = \{w \in \mathbb{C} : |w - 1| < 1\}$.

(b) Let $\operatorname{Log} z$ be the principal branch of the logarithm. Let $z \in D_1(1)$ and let C be a contour lying interior to $D_1(1)$ and joining $w = 1$ to $w = z$. Find the Taylor series expansion for $\operatorname{Log} z$ on the disk $D_1(1)$ by integrating the Taylor series expansion found in part (a) over the contour C.

P3. For each function, compute the Laurent series on the given annulus.

(a) $f(z) = z^3 \sin\left(\frac{1}{z^3}\right)$, $0 < |z| < \infty$.

(b) $f(z) = \frac{1}{2-z}$, $2 < |z| < \infty$.

(c) $\frac{z}{1+z^2}$, $0 < |z - i| < 2$.

(d) $f(z) = \frac{z}{(z-1)(z-3)}$, $0 < |z - 1| < 2$.

(e) $f(z) = \frac{z}{(z-1)(z-3)}$, $2 < |z - 1| < \infty$.

P4. Prove that the function

$$f(z) = \begin{cases} \frac{1-\cos z}{z^2}, & z \neq 0 \\ \frac{1}{2}, & z = 0 \end{cases}$$

is entire.

P5. **(Optional)**[1] Use complex analysis to prove the formula:[2]

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{7^n} = \sqrt{\frac{7}{3}}$$

(This problem is fantastic, and surprisingly relevant.)

[1] Worth 4 points of extra credit, added to you score on this assignment.

[2] Hint: begin by applying the formula for $\binom{2n}{n}$ from P5 on PSet4.