Problem Set 4 (Due 2/5/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work. Note: you may not use any results proved in class on 2/2/21 or 2/4/21 to solve these problems (i.e. anything after Section 46 of the textbook cannot be used).

P1. (a) Let $w = \cos^{-1} z$. Show that $\cos^{-1} z = -i \log(z + i(1-z^2)^{1/2})$ by solving the equation

$$z = \frac{e^{iw} + e^{-iw}}{2}.$$

(b) The expression for $\cos^{-1} z$ is multiple-valued. By taking the principal branch of log and the principle branch of the square root[1] (making $\cos^{-1} z$ single-valued and analytic), show that

$$\frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}.$$

P2. Let $z : [a, b] \to \mathbb{C}$ be a parameterization of a smooth arc C and suppose $f(z)$ is analytic at a point $z_0 = z(t_0)$ on C. Show that[2]

$$(f \circ z)'(t) = f'(z(t))z'(t)$$

when $t = t_0$.

P3. Let $\alpha, \beta \in \mathbb{R}$. Evaluate the following integral of real-valued functions

$$\int_0^\pi e^{\alpha x} \cos \beta x \, dx \quad \text{and} \quad \int_0^\pi e^{\alpha x} \sin \beta x \, dx$$

simultaneously by computing a *single* integral of a complex-valued function. [3]

P4. Let $z_1, z_2 \in \mathbb{C}$. Compute the integral

$$\int_C 1 \, dz$$

where C is any contour joining z_1 to z_2.

P5. Let C denote the unit circle with positive orientation. Compute the integral

$$\frac{1}{2\pi i} \int_C \frac{(1+z)^n}{z^{k+1}} \, dz$$

for any integers $0 \le k \le n$.[4]

P6. Integrate the function $f(z) = \overline{z}$ over the following contours:

(a) C_1: the line segment joining 0 to $1 + i$;

(b) C_2: the line segment joining 0 to 1, following by the line segment joining 1 to $1 + i$.

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[1] Evidently, the value of $\frac{d}{dz}\cos^{-1} z$ depends on the branch of square root we choose.

[2] Hint: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$. Then $(f \circ z)(t) = u(x(t), y(t)) + iv(x(t), y(t))$. Use the chain rule from vector calculus to compute $(f \circ z)'$ and then use the Cauchy-Riemann equations.

[3] Hint: consider the complex-valued function of a real variable $f(x) = e^{\alpha x} \cos \beta x + ie^{\alpha x} \sin \beta x$.

[4] Hint: you may assume the binomial theorem holds for complex numbers.