Problem Set 2 (Due 1/22/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work.

P1. Compute the following limits and prove your claim by using only the $\varepsilon - \delta$ definition.

(a) $\lim_{z \to i} \overline{z}$

(b) $\lim_{z \to 1+i} z^2$.

P2. Let $M(z) = \frac{z-3}{1-2z}$. Prove that $\lim_{z\to\infty} M(z) = -1/2$ and $\lim_{z\to 1/2} M(z) = \infty$.

P3. Consider the complex valued function

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Show that f is not differentiable at the origin.[1]

P4. Let U be a domain and $f : U \to \mathbb{C}$ a complex valued function that is differentiable at every point in U. Consider the domain $U^* \overset{\text{def}}{=} \{z \in \mathbb{C} : \overline{z} \in U\}$ and the function $f^* : U^* \to \mathbb{C}$ defined by $f^*(z) = \overline{f(\overline{z})}$. Show that f^* is differentiable at every point in U^*.

P5. For each function, determine all points at which the derivative exists. When the derivative exists, find its value. You must justify your claims.

(a) $f(z) = z + i\overline{z}$;

(b) $g(z) = (z + i\overline{z})^2$;

(c) $h(z) = z \operatorname{Im} z$.

P6. (Optional.)[2] By definition, a function $f : U \to \mathbb{C}$ is differentiable at $z_0 \in U$ if the limit $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. Unpacking the limit definition, we see that f is differentiable at z_0 if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \text{ implies } \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon.$$

By appealing only to the definition, show that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}$ is not differentiable anywhere by completing the following steps:

[1] In the lecture, we showed that f satisfies the Cauchy-Riemann equations at the origin. Thus, this will show that the satisfaction of the Cauchy-Riemann equations is not sufficient for the differentiability of f.

[2] This is worth 3 points extra credit, added to your score on this assignment.

(a) Let $z_0 \in \mathbb{C}$ and assume that $f'(z_0)$ exists. Choose $\delta > 0$ according to the definition using $\varepsilon = \frac{1}{2}$.

(b) Consider $z = z_0 + \frac{\delta}{2}$ and conclude from (a) that $|1 - f'(z_0)| < \varepsilon$.

(c) Consider $z = z_0 + \frac{i\delta}{2}$ and conclude from (a) that $|1 + f'(z_0)| < \varepsilon$.

(d) Using the triangle inequality together with (b) and (c), obtain a contradiction.