Problem Set 1 (Due 1/15/20 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work. Additionally:

(a) you may use freely any results that we discussed during the lecture;

(b) you may use any results that are proved in Chapter 1 of the textbook;

(c) you may use any exercises from Chapter 1 of the textbook if and only if they do not render the problem trivial.

In any case, you should make it clear when a statement in your proof follows from another result, and which result you are using.

P1. Let $p(z) = az^2 + bz + c$ be a polynomial with complex coefficients ($a \neq 0$).

(a) By completing the square, show that the solution to $p(z) = 0$ is[1]

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}.$$

(b) Find the roots of the polynomial $p(z) = iz^2 - 1$ in the form $z = x + iy$.

P2. Consider the set of matrices

$$X \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

It is easy to see that X is closed under matrix addition and matrix multiplication, that is, if $A, B \in X$, then $A + B, AB \in X$.[2]

(a) Let $\mathbb{C}$ denote the set of complex numbers. Show that the map $\varphi : X \to \mathbb{C}$ defined by

$$\varphi \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x + iy$$

is a bijection.

(b) Let $A, B \in X$ and let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the identity matrix. Show that φ has the following properties:

(i) $\varphi(A + B) = \varphi(A) + \varphi(B)$;

[1] Note that the expression $(b^2 - 4ac)^{1/2}$ has multiple values!

[2] Parts (a) and (b) of this problem shows that X and $\mathbb{C}$ are algebraically indistinguishable. In the language of abstract algebra, φ is an *isomorphism of rings*. The subset $\left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \mathbb{R} \right\} \subset X$ is algebraically indistinguishable from the real numbers in the same sense.

(ii) $\varphi(AB) = \varphi(A)\varphi(B)$;

(iii) $\varphi(I) = 1$.

(c) Find a matrix J satisfying $J^2 = -I$ and show that $\varphi(J) = i$.

P3. (a) Let $z \in \mathbb{C}$. Using the principle of mathematical induction, show that the formula holds for all $n \in \mathbb{N}$:

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

(b) If $\omega_1, \ldots, \omega_n$ are the *distinct* n-th roots of unity, show that $\sum_{i=1}^{n} \omega_i = 0$.

(c) Compute the following sum of real numbers:

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{7\pi}{7} + \cos\frac{9\pi}{7} + \cos\frac{11\pi}{7} + \cos\frac{13\pi}{7} + \cos\frac{15\pi}{7}.$$

You must justify your computation.

P4. Let $z, w \in \mathbb{C}$.

(a) Prove the formula

$$|z + w|^2 = |z|^2 + 2\operatorname{Re} z\overline{w} + |w|^2.\text{[3]} \tag{1}$$

(b) From (1), deduce the parallelogram law

$$|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$$

Give a geometric interpretation of this formula.

P5. Let $z_0 \neq z_1 \in \mathbb{C}$ and let $\lambda > 0$. Show that if $\lambda \neq 1$, then the set of points

$$|z - z_0| = \lambda|z - z_1| \tag{2}$$

is a circle of radius $R = \dfrac{\lambda}{|1 - \lambda^2|}|z_0 - z_1|$ centered at $w = \dfrac{z_0 - \lambda^2 z_1}{1 - \lambda^2}$.[4] If $\lambda = 1$, show that (2) defines a line.[5]

Show that every circle in the complex plane can be written in the form of equation (2) for some $\lambda > 0, \lambda \neq 1$ and $z_0 \neq z_1 \in \mathbb{C}$.

P6. (a) Find a necessary and sufficient condition on $z \in \mathbb{C}$ such that $|z - i| = |z + i|$.

(b) Repeat with the condition $|z - 1| = |z + 1|$.

P7. Consider the map $M(z) = \dfrac{z - 3}{1 - 2z}$, $z \in \mathbb{C}$. For which values of $c \in \mathbb{R}$ is the image of the circle $|z - 1| = c$ under M a line? What is the equation of the line when considered as a subset of the plane $\mathbb{R}^2$?

P8. Let $U \subseteq \mathbb{C}$. We say that U is *open* if it does not contain any of its boundary points. Prove that U is open if and only if every $z \in U$ is an interior point.

P9. Let $\varepsilon > 0$. Prove that the open disk $D_\varepsilon(z_0) \overset{\text{def}}{=} \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ is a domain.

[3] Notice that $\operatorname{Re} z\overline{w} = \operatorname{Re} w\overline{z}$.

[4] Suggestion: Try to show that $|z - w|^2 = R^2$. Start by squaring both sides of (2) using (1). Manipulate the resulting equation so that $\frac{\lambda^2|z_1|^2 - |z_0|^2}{1-\lambda^2}$ is isolated on one side. Using (1) again, add something to both sides to complete the square.

[5] Hint: When $\lambda = 1$, argue geometrically!