Midterm (Take-Home) (Due 2/19/21)

Your Name:

Names of Any Collaborators:

This midterm exam is graded out of 27 points and is worth 25% of your overall score in the class. This take-home exam is due by 11:59PM on 2/19/21, submitted via `http://gradescope.com`. Your overall score on the midterm exam is worth 25% of your overall grade.

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

The simple rules for the exam are:

1. You may freely use any theorems or problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. You may freely consult the textbook or my handwritten lecture notes.
2. You cannot use any results that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work. Please name those with whom you have discussed the problems with.

I will pursue anyone suspected of breaking these rules.

You should **turn in this cover page** together with all of the work that you have decided to submit.

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

M1. (**7 points**) Let D be a domain and assume that $f : D \to \mathbb{C}$ is continuous. Determine which statements are true, and which are false. If you think a statement is true, *briefly* explain your reasoning. If you think a statement is false, you must prove it by providing a counterexample. Each statement is worth 1 point: awarded for correctly identifying the truth value (with justification, as described in the preceding two sentences). No points will be awarded if you fail to follow the directions.

(i) If f is analytic on D, then $\int_C f(z)\,dz = 0$ for any closed contour C lying in D.

(ii) If f has an antiderivative on D, then $\int_C f(z)\,dz = 0$ for any closed contour in D.

(iii) Suppose that $f : D \to \mathbb{C}$ is analytic on the domain D except for at a single singularity z_0. Let C_R be a positively oriented circle of radius $R > 0$ (small enough that the circle lies in D) centered at z_0. Then

$$\int_{C_R} f(z)\,dz = \lim_{R\to 0} \int_{C_R} f(z)\,dz.$$

(iv) If f is analytic on D, then there exists an analytic function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$.

(v) Let C be any circle with positive orientation and R the closed disk consisting of C and its interior. If f is entire and constant on C, then f is constant on R.

(vi) If $\int_C f(z)\,dz = 0$ for any closed contour C lying in D, then the real and imaginary parts of f satisfy the Cauchy-Riemann equations on D.

(vii) If f is entire and $n \in \mathbb{N}$, then there exists an entire function F such that $F^{(n)}(z) = f(z)$ for all $z \in \mathbb{C}$ (here $F^{(n)}$ denotes the n-th derivative of F).

M2. (**4 points**) Let C be any positively oriented simple closed contour not crossing through the points i and $-i$. Determine (with proof) all possible values of the integral

$$\int_C \frac{e^{\pi z}}{z^2+1}\,dz.$$

M3. (**4 points**) Let C be a simple closed positively oriented contour whose interior contains -1. Compute the integral

$$\frac{1}{2\pi i}\int_C \frac{z^n}{(z+1)^{k+1}}\,dz$$

for any integers $0 \le k \le n$.

M4. (**4 points**) Suppose that f is analytic everywhere on $\mathbb{C}$ except at a finite number of singular points $z_1, \ldots, z_n$ and let C be any positively oriented simple closed contour whose interior contains the singularities. Describe, in detail, how one can use the Generalized Cauchy-Goursat Theorem to compute the integral $\displaystyle\int_C f(z)\,dz$. You may assume that the value of a contour integral of f is known whenever the contour contains precisely one singularity of f in its interior.

M5. (**4 points**) Suppose that f is an entire function satisfying $|f(z)| \leq \sqrt{1+|z|}$ for all $z \in \mathbb{C}$. Prove that f is constant.[1]

M6. (**4 points**) Suppose that $f(z) = u(x,y) + iv(x,y)$ is an entire function and that there exists $M > 0$ such that $|u(x,y)| \leq M$ for all $z = x+iy \in \mathbb{C}$. Prove that f is constant.[2]

[1] Hint: Try to show $f' = 0$ using Cauchy's Inequality.
[2] Hint: try to apply Liouville's theorem to $e^{f(z)}$.