Final (Take-home) (Due 3/19/21)

Your Name:

Names of Any Collaborators:

This final exam is graded out of 31 points and is worth 25% of your overall score in the class. This take-home exam is due by 11:59PM on 3/19/21, submitted via http://gradescope.com. Extensions will not be granted under any circumstances.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

The simple rules for the exam are:

1. You may freely use any theorems or problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. You may freely consult the textbook, the problem set, midterm, or quiz solutions, or my handwritten lecture notes.
2. You cannot use any results that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work. Please name those with whom you have discussed the problems with.

If you break the rules, you will receive a zero on the exam. I will pursue anyone suspected of breaking these rules.

You should **turn in this cover page** together with all of the work that you have decided to submit.

To convince me that you have read and understand the instructions, sign (electronically) in the box below.

Signature:

Good luck and have fun!

F1. (**7 points**) For each statement, determine whether it is true or false. If it is true, briefly describe your reasoning (e.g. cite a theorem or sketch a proof). If it is false, provide a counterexample. Each statement is worth 1 point, awarded for correctly identifying the truth value (.5 points) and providing correct reasoning or counterexample (.5 points). No points will be awarded if you fail to correctly identify the truth value of a statement, or fail to follow these directions.

(i) Let $p(z)$ and $q(z)$ be nonzero polynomials with complex coefficient and no factors in common. Assume that $q(z)$ has $n \geq 1$ distinct zeros $q_1, \dots, q_n$, each of which necessarily has order 1. It can be shown that $\frac{p(z)}{q(z)}$ has the partial fraction decomposition

$$\frac{p(z)}{q(z)} = g(z) + \frac{a_1}{z-q_1} + \frac{a_2}{z-q_2} + \cdots + \frac{a_n}{z-q_n}$$

where $g(z)$ is a polynomial with complex coefficients, and $a_1, \dots, a_n \in \mathbb{C}$. The residue of $\frac{p(z)}{q(z)}$ at the isolated singularity $z = q_i$ is given by $\operatorname{Res}_{z=q_i} \frac{p(z)}{q(z)} = a_i$.[1]

(ii) If $f(z)$ is analytic at $z_0 = 0$, then $f\left(\frac{1}{z}\right)$ has an essential singularity at $z_0 = 0$.

(iii) If $f(z)$ is entire and $f\left(\frac{1}{z}\right)$ has a pole at $z_0 = 0$, then $f(z)$ is a polynomial.

(iv) If z_0 is an isolated singularity of $f(z)$ and $\lim_{z \to z_0}(z - z_0)f(z) \neq 0$, then z_0 is a pole.

(v) Let $0 < R < S$. If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to a function $f(z)$ for all $z \in D_R(z_0)$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converges to $f(z)$ for all $z \in D_S(z_0)$, then $a_n = b_n$ for all $n \in \mathbb{N}$.

(vi) If f is analytic and nonzero at every point on and interior to the unit circle C, then $\int_C \frac{f'(z)}{f(z)}\, dz = 0$.

(vii) If a function $f(z)$ has an isolated singularity z_0 that is not removable, then $\operatorname{Res}_{z=z_0} f(z) \neq 0$.

F2. (**4 points**) The function $\frac{ze^{\frac{1}{z}}}{z-1}$ is analytic on the annulus $1 < |z| < \infty$ and the Laurent series there is given by

$$\frac{ze^{\frac{1}{z}}}{z-1} = 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{5}{2z^3} + \frac{8}{3z^4} + \cdots.$$

Can you conclude from this that $\operatorname{Res}_{z=0} \frac{ze^{\frac{1}{z}}}{z-1} = 2$? Briefly explain why or why not.

F3. (**6 points**) Let $f(z) = \frac{z}{(z-i)(z-2i)}$ and consider the partial fraction decomposition

$$\frac{z}{(z-i)(z-2i)} = \frac{2}{z-2i} - \frac{1}{z-i}.$$

Compute the Laurent series for $f(z)$ on each annulus:

[1] The statement to be proved or disproved here is the statement about the residues. You can assume that the partial fraction decomposition is correct.

(a) $0 < |z| < 1$;
(b) $1 < |z| < 2$;
(c) $2 < |z| < \infty$.

F4. (**4 points**) Let $f(z) = \frac{ze^{1/z}}{z^2+5}$. Prove that $f(z)$ has no antiderivative on the domain $D = \{z \in \mathbb{C} : |z| > 5\}$.

F5. (**4 points**) How many zeros (counting orders) does the polynomial

$$p(z) = 2z^5 + 5z^3 + 1$$

have in the annulus $1 \leq |z| < 2$? Justify your computation.

F6. (**6 points**) Let $m \geq 1$ be an integer. Follow the steps outlined in Lecture 19 to prove the integration formula[2]

$$\int_{-\infty}^{\infty} \frac{1}{1+x^{2m}}\,dx = \frac{\pi}{m \sin\left(\frac{\pi}{2m}\right)}.$$

F7. (Optional.)[3] Compute $\mathrm{Res}_{z=0} \frac{e^{\frac{1}{z}}}{1-z}$. Justify your computations.

[2]Hint: to compute the sum of residues, use the formula $\sum_{k=0}^{m-1} w^k = \frac{1-w^m}{1-w}$.
[3]Worth 4 points of extra credit, added to your score on this exam.