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Midterm Exam Problems

1. **(9 points)** Determine whether the following statements are **TRUE** or **FALSE**. No justification is required: if you don't know, just guess! Each question is worth **1 point**.

   (a) Two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel if and only if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.
   Planes are parallel if $|n_1 \times n_2| = 0$
   Answer: F

   (b) For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.
   $|u \times v| = |-(v \times u)| = |-1|\,|v \times u| = |v \times u|$
   Answer: T

   (c) If all the level curves of a function $f(x,y)$ are parallel lines, then the graph of $f(x,y)$ is a plane.
   Counterexample: $f(x,y) = x^3$
   Answer: F

   (d) The slope of the tangent line to the $y = b$ trace of a function $f(x,y)$ at the point $(a,b,f(a,b))$ is equal to $\lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$. This limit is the def. of $\frac{\partial f}{\partial y}$.
   Answer: F

   (e) Any two planes which are perpendicular to the same line are parallel.
   Answer: T

   (f) If $\overrightarrow{OP} \cdot (\overrightarrow{OQ} \times \overrightarrow{OR}) = 0$, then the three points $P, Q, R \in \mathbb{R}^3$ are collinear (lie on the same line).
   Answer: F
   counterexample: $P = \langle 0,0,0 \rangle$ $Q = \langle 1,0,0 \rangle$ $R = \langle 0,0,1 \rangle$.

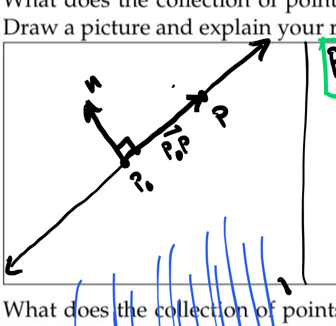
   (g) If $\mathbf{r}(t)$ is a vector-valued function, the $\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$.
   counterexample: $r(t) = \langle t, 1 \rangle$
   Answer: F

   (h) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ satisfy $|\mathbf{u} \times \mathbf{v}| = 0$, then either $\mathbf{u} = 0$ or $\mathbf{v} = 0$.
   Counterexample: $u = v = \langle 1,0,0 \rangle$
   Answer: F

   (i) If $f_x(x,y)$ and $f_y(x,y)$ are constant functions, then the graph of $f(x,y)$ is a plane.
   Not obvious.
   Answer: T

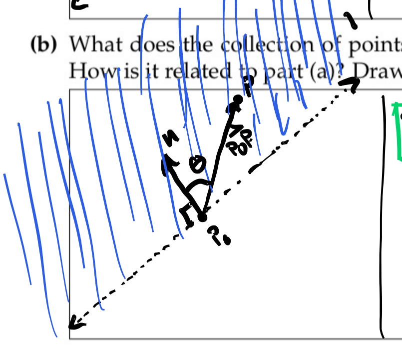
2. **(4 points)** Let $P_0 = (x_0, y_0)$ be a point in $\mathbb{R}^2$ and let $\mathbf{n} = \langle a, b \rangle$ be a vector in $\mathbb{R}^2$. Answer the following questions, each of which is worth **2 points**.

   (a) What does the collection of points $P = (x,y)$ which satisfy the equation $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$ look like geometrically? Draw a picture and explain your reasoning. (Hint: we have seen this equation in 3-dimensions.)



   P belongs to the set $\iff$ $n \cdot \overrightarrow{P_0P} = 0$
   $\iff$ n is perp to $\overrightarrow{P_0P}$
   $\iff$ P lies on the line through P_0 in the direction perp to n.

   (b) What does the collection of points $P = (x,y)$ which satisfy the inequality $\mathbf{n} \cdot \overrightarrow{P_0P} > 0$ look like geometrically? How is it related to part (a)? Draw a picture and explain your reasoning.



   P belongs to the set $\iff$ $n \cdot \overrightarrow{P_0P} > 0$
   $\iff$ angle θ satisfies $0 < \theta \leq \pi/2$
   $\iff$ P lies in the half plane determined by the line from (a) and n, but not on the line.

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3. **(8 points)** Answer the following questions, each of which is worth **2 points**. *Briefly* justify your answer.

   (a) Are the vectors $\mathbf{u} = \langle -1, 2, -12, -4 \rangle$ and $\mathbf{v} = \langle 0, 2, 1, -2 \rangle$ perpendicular?
   $u \cdot v = -1 \cdot 0 + 2 \cdot 2 + -12 \cdot 1 + -4 \cdot -2 = 0$.
   Yes.

   (b) Does the line $\mathbf{r}(t) = \langle 2t-1, -t+1, -t-1 \rangle$ intersect the plane $x + y + z = 1$?
   Direction of line: $\langle 2,-1,-1 \rangle$ Normal: $\langle 1,1,1 \rangle$.
   The line is parallel to the plane since dot product is 0. So line either is contained in plane or does not intersect. Since initial pt $(-1,1,-1)$ does not satisfy plane eq., does not intersect.

   (c) Do the planes $x + 7y + z = 2$ and $-2x - 2y - 2z = 4$ intersect?
   Yes, because the normal vectors $\langle 1,7,1 \rangle$ and $\langle -2,-2,-2 \rangle$ are not parallel.

   (d) Do the vectors $\mathbf{u} = \langle 1, -1, 3 \rangle$, $\mathbf{v} = \langle 1, -1, 1 \rangle$, and $\mathbf{w} = \langle 2, -2, 4 \rangle$ lie in a common plane?
   $$\begin{vmatrix} 1 & -1 & 3 \\ 1 & -1 & 1 \\ 2 & -2 & 4 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ -2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}$$
   $$= (-4 - (-2)) + (4-2) + 3 \cdot 0 = 0.$$
   Yes because volume of parallelepiped is 0.

4. **(6 points)** Consider the function $f(x,y,z) = x\sin(z^2 e^{y+z})$. Complete the following, each of which is worth **2 points**.

   (a) Compute $f_x(x,y,z)$.
   $f_x = \sin(z^2 e^{y+z})$

   (b) Compute $f_y(x,y,z)$.
   $f_y = x \cdot \cos(z^2 e^{y+z}) \cdot z^2 e^{y+z}$

   (c) Compute $f_z(x,y,z)$.
   $f_z = x \cos(z^2 e^{y+z}) \cdot (2z e^{y+z} + z^2 e^{y+z})$

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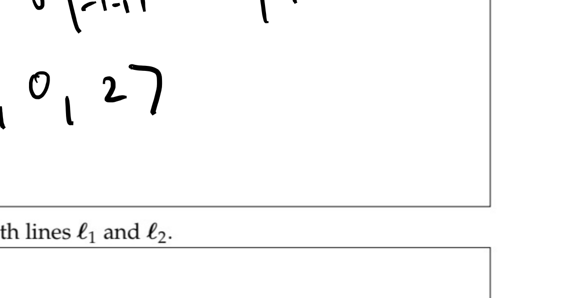
5. **(10 points)** lines $\ell_1(t) = \langle t, t, 1+t \rangle$ and $\ell_2(t) = \langle 2-t, t, 3-t \rangle$. Complete the following tasks.

   (a) **(3 points)** Show that ℓ_1 and ℓ_2 intersect and find the point of intersection.
   Solve $\ell_1(s) = \ell_2(t)$. You will get $s = t = 1$ as the solution.
   The lines intersect at $\ell_1(1) = \ell_2(1) = \langle 1,1,2 \rangle$

   (b) **(1 point)** Show that ℓ_1 and ℓ_2 are not parallel.
   The direction vectors $u = \langle 1,1,1 \rangle$ and $v = \langle -1,1,-1 \rangle$ are not parallel.

   (c) **(1 point)** Draw a picture of the situation and use it to explain why there is a *unique* plane containing ℓ_1 and ℓ_2.
   A plane is determined by a point and a normal vector. Any plane containing both lines contains $(1,1,2)$. The normal vector for a plane containing both lines is $\perp$ to both u and v. Two planes containing a common point w/ parallel normal vectors are equal.



   (d) **(3 points)** Find a vector that is perpendicular to both ℓ_1 and ℓ_2.
   $$n = u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$
   $$= \langle -2, 0, 2 \rangle$$

   (e) **(2 points)** Find a scalar equation for the plane that contains both lines ℓ_1 and ℓ_2.
   $0 = n \cdot \overrightarrow{P_0P}$
   $= \langle -2,0,2 \rangle \cdot \langle x-1, y-1, z-2 \rangle$
   $= -2(x-1) + 2(z-2)$
   or $2x - 2z = -2$

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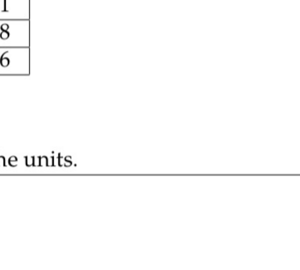
6. **(5 points)** The position of particle in space at time t is given by $\mathbf{r}(t) = \langle \sin(t), \cos(t), t^2 \rangle$.

   (a) **(1 points)** Compute $\mathbf{r}(\pi)$.
   $r(\pi) = \langle 0, -1, \pi^2 \rangle$

   (b) **(2 points)** Compute $\mathbf{r}'(\pi)$ and explain what it means in context.
   $r'(\pi) = \langle \cos(\pi), -\sin(\pi), 2\pi \rangle = \langle -1, 0, 2\pi \rangle$

   (c) **(2 points)** Find a vector-valued function that parametrizes the line that is tangent to the curve $\mathbf{r}(t)$ when $t = \pi$.
   $L(t) = \langle 0, -1, \pi^2 \rangle + t\langle -1, 0, 2\pi \rangle$.



7. **(7 points)** The table below gives the dissolved oxygen concentration $O(T,d)$ in a lake (in milligrams per liter, mg/L) as a function of the water temperature T (in °C) and the depth d (in meters m).

T (°C) \ d (m)	0	5	10	15	20
5	12.8	12.6	12.5	12.4	12.3
10	11.3	11.1	10.9	10.8	10.6
15	10.1	9.8	9.5	9.3	9.1
20	9.1	8.7	8.4	8.1	7.8
25	8.3	7.8	7.4	7.0	6.6

For example, $O(10,5) = 11.1$ mg/L. Answer the following questions.

   (a) **(4 points)** Estimate the partial derivative $O_T(15,10)$. Make sure to include the units.
   $$O_T(15,10) \approx \frac{O(20,10) - O(10,10)}{2 \cdot 5}$$
   $$= \frac{8.4 - 10.9}{10} = -.25\,(\text{mg/L})/{}^\circ\text{C}$$

   (b) **(3 points)** Explain the meaning of the partial derivative that you computed in part (a) in the context of the problem. You can answer this part of the problem even if you don't know how to estimate $O_T(15,10)$.
   When the depth is 10m and the temp is 15°C, the dissolved oxygen concentration will decrease approx. .25mg/L for every 1°C increase in temp.

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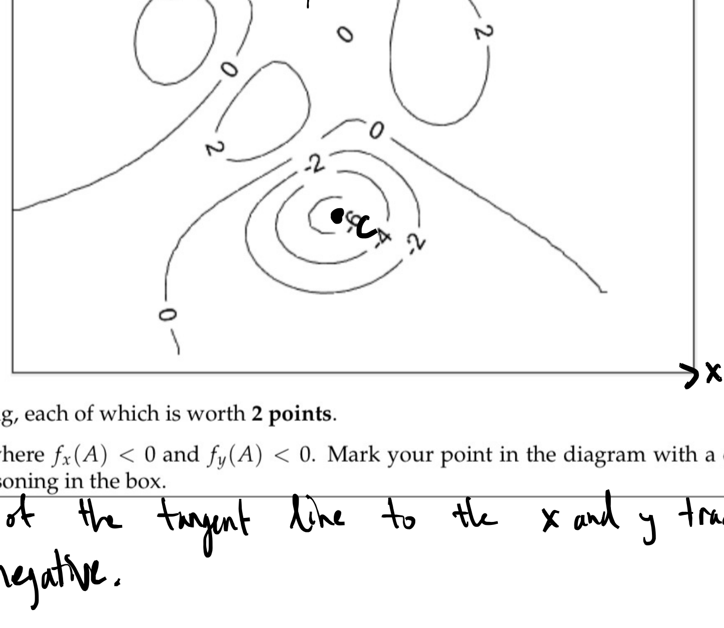
8. **(6 points)** Consider the following contour diagram of a continuously differentiable function $f(x,y)$:



Complete the following, each of which is worth **2 points**.

   (a) Find a point A where $f_x(A) < 0$ and $f_y(A) < 0$. Mark your point in the diagram with a dot and the letter A. Explain your reasoning in the box.
   The slope of the tangent line to the x and y traces through A are negative.

   (b) Find a point B at which $f_x(B) = 0$ and $f_y(B) > 0$. Mark your point in the diagram with a dot and the letter B. Explain your reasoning in the box.
   The slope of the tangent line through B in the y-direction is positive since we are going up hill. The slope in the x-direction is zero since we increase and decrease by the same elevation as we walk through the point. Notice that line thru B in x-direction is tangent to the curve.

   (c) Find a point C at which $f_x(C) = 0$ and $f_y(C) = 0$. Mark your point in the diagram with a dot and the letter C. Explain your reasoning in the box.
   Either point marked C works. In either case, we are at a valley or a peak so the tangent plane is horizontal and both slopes are zero.