Midterm Exam

MATH 22: Calculus of Several Variables

Summer 2025

Overview: This exam consists of 1 cover page and **5 pages of questions**, for a total of 6 pages. There are **8 questions** worth a total of **55 points**. Each question has multiple parts. Your grade on this exam will be calculated as

(**number of points earned**)/50.

This exam is worth **15% of your final grade** in the course.

Directions:

- **Write your NAME and STUDENT ID at the top of each page.**
- Solve as many of the following problems as you can. Try to start with problems that you know how to solve right away, and save the others for last.
- Provide justification and show your work whenever possible. This makes it easier to give you partial credit. I want to give you partial credit!
- Write the work and the answer that you want graded in the provided answer box.
- You may use one two-sided 8.5inch by 11inch sheet of notes. No other resources are allowed. You do not need a calculator.

Good luck, and have fun!

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Midterm Exam Problems

1. **(9 points)** Determine whether the following statements are **TRUE** or **FALSE**. No justification is required: if you don't know, just guess! Each question is worth **1 point**.

   (a) Two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel if and only if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.

   Answer:

   (b) For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.

   Answer:

   (c) If all the level curves of a function $f(x, y)$ are parallel lines, then the graph of $f(x, y)$ is a plane.

   Answer:

   (d) The slope of the tangent line to the $y = b$ trace of a function $f(x, y)$ at the point $(a, b, f(a, b))$ is equal to $\lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$.

   Answer:

   (e) Any two planes which are perpendicular to the same line are parallel.

   Answer:

   (f) If $\overrightarrow{OP} \cdot (\overrightarrow{OQ} \times \overrightarrow{OR}) = 0$, then the three points $P, Q, R \in \mathbb{R}^3$ are collinear (lie on the same line).

   Answer:

   (g) If $\mathbf{r}(t)$ is a vector-valued function, the $\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$.

   Answer:

   (h) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ satisfy $|\mathbf{u} \times \mathbf{v}| = 0$, then either $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

   Answer:

   (i) If $f_x(x, y)$ and $f_y(x, y)$ are constant functions, then the graph of $f(x, y)$ is a plane.

   Answer:

2. **(4 points)** Let $P_0 = (x_0, y_0)$ be a point in $\mathbb{R}^2$ and let $\mathbf{n} = \langle a, b \rangle$ be a vector in $\mathbb{R}^2$. Answer the following questions, each of which is worth **2 points**.

   (a) What does the collection of points $P = (x, y)$ which satisfy the equation $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$ look like geometrically? Draw a picture and explain your reasoning. (Hint: we have seen this equation in 3-dimensions.)

   (b) What does the collection of points $P = (x, y)$ which satisfy the inequality $\mathbf{n} \cdot \overrightarrow{P_0P} > 0$ look like geometrically? How is it related to part (a)? Draw a picture and explain your reasoning.

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3. (8 points) Answer the following questions, each of which is worth **2 points**. *Briefly* justify your answer.

(a) Are the vectors $\mathbf{u} = \langle -1, 2, -12, -4 \rangle$ and $\mathbf{v} = \langle 0, 2, 1, -2 \rangle$ perpendicular?

(b) Does the line $\mathbf{r}(t) = \langle 2t - 1, -t + 1, -t - 1 \rangle$ intersect the plane $x + y + z = 1$?

(c) Do the planes $x + 7y + z = 2$ and $-2x - 2y - 2z = 4$ intersect?

(d) Do the vectors $\mathbf{u} = \langle 1, -1, 3 \rangle$, $\mathbf{v} = \langle 1, -1, 1 \rangle$, and $\mathbf{w} = \langle 2, -2, 4 \rangle$ lie in a common plane?

4. (6 points) Consider the function $f(x, y, z) = x \sin(z^2 e^{y+z})$. Complete the following, each of which is worth **2 points**.

(a) Compute $f_x(x, y, z)$.

(b) Compute $f_y(x, y, z)$.

(c) Compute $f_z(x, y, z)$.

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5. (10 points) lines $\ell_1(t) = \langle t, t, 1+t \rangle$ and $\ell_2(t) = \langle 2-t, t, 3-t \rangle$. Complete the following tasks.

(a) **(3 points)** Show that ℓ_1 and ℓ_2 intersect and find the point of intersection.

(b) **(1 point)** Show that ℓ_1 and ℓ_2 are not parallel.

(c) **(1 point)** Draw a picture of the situation and use it to explain why there is a *unique* plane containing ℓ_1 and ℓ_2.

(d) **(3 points)** Find a vector that is perpendicular to both ℓ_1 and ℓ_2.

(e) **(2 points)** Find a scalar equation for the plane that contains both lines ℓ_1 and ℓ_2.

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6. **(5 points)** The position of particle in space at time t is given by $\mathbf{r}(t) = \langle \sin(t), \cos(t), t^2 \rangle$.

(a) **(1 points)** Compute $\mathbf{r}(\pi)$.

(b) **(2 points)** Compute $\mathbf{r}'(\pi)$ and explain what it means in context.

(c) **(2 points)** Find a vector-valued function that parametrizes the line that is tangent to the curve $\mathbf{r}(t)$ when $t = \pi$.

7. **(7 points)** The table below gives the dissolved oxygen concentration $O(T, d)$ in a lake (in milligrams per liter, mg/L) as a function of the water temperature T (in °C) and the depth d (in meters m).

T (°C) \ d (m)	**0**	**5**	**10**	**15**	**20**
5	12.8	12.6	12.5	12.4	12.3
10	11.3	11.1	10.9	10.8	10.6
15	10.1	9.8	9.5	9.3	9.1
20	9.1	8.7	8.4	8.1	7.8
25	8.3	7.8	7.4	7.0	6.6

For example, $O(10, 5) = 11.1$ mg/L. Answer the following questions.

(a) **(4 points)** Estimate the partial derivative $O_T(15, 10)$. Make sure to include the units.

(b) **(3 points)** Explain the meaning of the partial derivative that you computed in part (a) in the context of the problem. You can answer this part of the problem even if you don't know how to estimate $O_T(15, 10)$.

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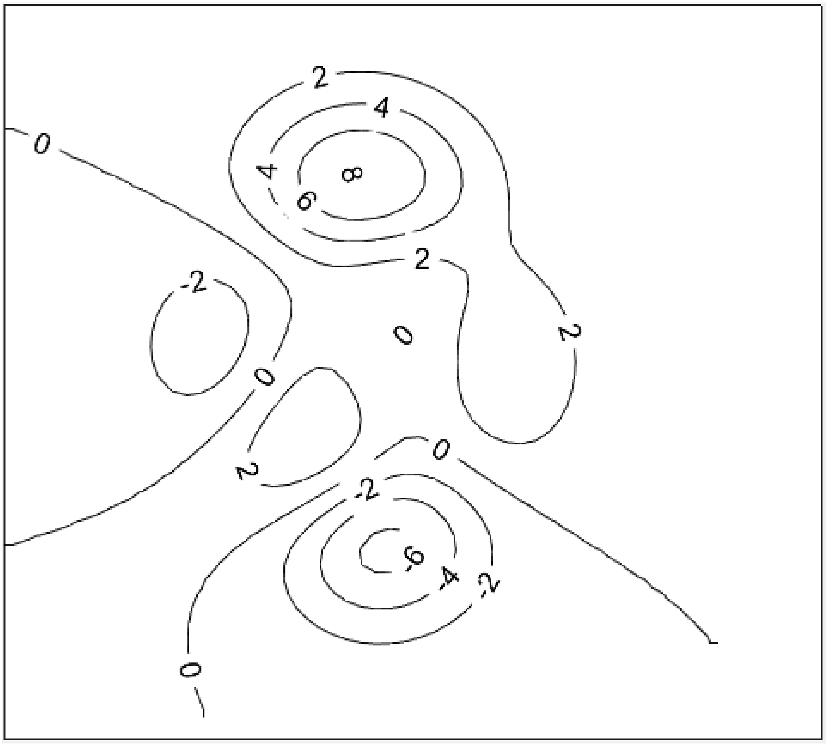
8. (6 points) Consider the following contour diagram of a continuously differentiable function $f(x, y)$:



Complete the following, each of which is worth **2 points**.

(a) Find a point A where $f_x(A) < 0$ and $f_y(A) < 0$. Mark your point in the diagram with a dot and the letter A. Explain your reasoning in the box.

(b) Find a point B at which $f_x(B) = 0$ and $f_y(B) > 0$. Mark your point in the diagram with a dot and the letter B. Explain your reasoning in the box.

(c) Find a point C at which $f_x(C) = 0$ and $f_y(C) = 0$. Mark your point in the diagram with a dot and the letter C. Explain your reasoning in the box.