Midterm Exam Cover Page

Key

Overview: This exam consists of 1 cover page and **4 pages of questions**, for a total of 5 pages. There are **8 questions** worth a total of **52 points**. Each question has multiple parts. Your grade on this exam will be calculated as

(number of points earned)/50.

This exam is worth **15% of your final grade** in the course.

Directions:

- **Write your NAME and STUDENT ID at the top of page 2.**
- Solve as many of the following problems as you can. Try to start with problems that you know how to solve right away, and save the others for last.
- Provide justification and show your work whenever possible. This makes it easier to give you partial credit. I want to give you partial credit!
- Write the work and the answer that you want graded in the provided answer box.
- You may use one two-sided 8.5inch by 11inch sheet of notes. No other resources are allowed. You do not need a calculator.

Good luck, and have fun!

ıme: Answer Key

Student ID:

Midterm Exam Problems

1. **(7 points)** Determine whether the following statements are **TRUE** or **FALSE**. No justification is required: if you don't know, just guess! Each question is worth **1 point**.

(a) Two planes with normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ are parallel if and only if $|\mathbf{n_1} \times \mathbf{n_2}| = 0$.

n_1 and n_2 are parallel iff $|n_1 \times n_2| = 0$.

Answer: T

(b) For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.

$|u \times v| = |-(v \times u)| = |-1|\,|v \times u| = |v \times u|$

Answer: T

(c) If all the level curves of a function $f(x, y)$ are lines, then the graph of $f(x, y)$ is a plane.

Counterexample: $f(x,y) = x^3$.

Answer: F

(d) The slope of the tangent line to the $y = b$ trace of a function $f(x, y)$ at the point $(a, b, f(a, b))$ is equal to $\lim_{h \to 0} \frac{f(a,b+h)-f(a,b)}{h}$.

This limit is $f_y(a,b)$ which is slope of tangent line to $x=a$ trace

Answer: F

(e) Any two lines which are perpendicular to the same plane are parallel.

Clear from the picture:

Answer: T

(f) Any three points $P, Q, R \in \mathbb{R}^3$ are contained in a *unique* plane.

Three collinear points do not uniquely determine a plane

Answer: F

(g) The cross product of two unit vectors is a unit vector.

$|u \times v| = |u||v| \sin(\theta)$ so this is true iff $\theta = \pi/2$

Answer: F

2. **(8 points)** Answer the following questions, each of which is worth **2 points**. Briefly justify your answer.

(a) Are the vectors $\mathbf{u} = \langle 1, 2, 0, -4 \rangle$ and $\mathbf{v} = \langle 0, 2, 1, 2 \rangle$ perpendicular?

$u \cdot v = 1 \cdot 0 + 2 \cdot 2 + 0 \cdot 1 + (-4) \cdot 2 = 4 - 8 = -4 \neq 0$. No, they are not perpendicular.

(b) Does the line $\mathbf{r}(t) = \langle t, t, -2t \rangle$ intersect the plane $x + y + z = 1$?

No. The direction of the line is $\langle 1,1,-2 \rangle$ and the direction of the normal vector is $\langle 1,1,1 \rangle$. Since $\langle 1,1,-2 \rangle \cdot \langle 1,1,1 \rangle = 0$, the plane and the line are parallel. The point $(0,0,0)$ lies on the line, but not the plane. Therefore, they do not intersect.

(c) Are the planes $x + y + z = 2$ and $-2x - 2y - 2z = 4$ parallel?

Yes, the normal vectors $\langle 1,1,1 \rangle$ and $\langle -2,-2,-2 \rangle$ are parallel:
$\langle -2,-2,-2 \rangle = -2 \langle 1,1,1 \rangle$.

(d) Are the vectors $\mathbf{u} = \langle 1, 0, 3 \rangle$, $\mathbf{v} = \langle 1, 1, 1 \rangle$, and $\mathbf{w} = \langle 0, 1, 1 \rangle$ coplanar?

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 3.$$

This is the signed volume of the parallelepiped spanned by u, v, w. It is not zero, so the vectors are not coplanar.

3. (10 points) Consider the lines $\ell_1(t) = \langle t, t, 1+t \rangle$ and $\ell_2(t) = \langle 2-t, t, 3-t \rangle$. Complete the following tasks.

(a) **(3 points)** Show that ℓ_1 and ℓ_2 intersect.

The lines intersect when $t = 1$:

$$\ell_1(1) = \langle 1,1,2 \rangle$$
$$\ell_2(1) = \langle 1,1,2 \rangle$$

(b) **(3 points)** Show that ℓ_1 and ℓ_2 are not parallel.

The direction of ℓ_1 is $u = \langle 1,1,1 \rangle$ and the direction of ℓ_2 is $v = \langle -1,1,-1 \rangle$. These vectors are not parallel since

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = \langle -2,0,2 \rangle \neq \langle 0,0,0 \rangle.$$

(c) **(2 points)** Find a vector that is perpendicular to both ℓ_1 and ℓ_2.

A vector perp. to both $u = \langle 1,1,1 \rangle$ and $v = \langle -1,1,-1 \rangle$ will be perp to both ℓ_1 and ℓ_2. Use the cross product:

$$u \times v = \langle -2,0,2 \rangle$$

(d) **(2 points)** Find a scalar equation for the plane that contains both lines ℓ_1 and ℓ_2.

A normal vector is $n = u \times v = \langle -2,0,2 \rangle$ and a point in the plane is $\langle 1,1,2 \rangle$. A scalar eq. of the plane is

$$-2(x-1) + 0(y-1) + 2(z-2) = 0$$

or

$$2z - 2x = 2.$$

4. (6 points) Consider the curve defined by $\mathbf{r}(t) = \langle \sin(t), \cos(t), t \rangle$.

(a) **(2 points)** Compute $\mathbf{r}(\pi)$.

$$r(\pi) = \langle \sin\pi, \cos\pi, \pi \rangle = \langle 0, -1, \pi \rangle$$

(b) **(2 points)** Compute $\mathbf{r}'(\pi)$.

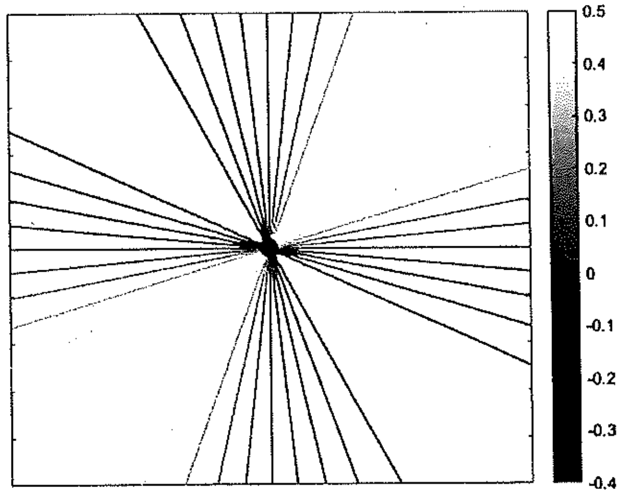
$$r'(t) = \langle \cos t, -\sin t, 1 \rangle$$
$$r'(\pi) = \langle -1, 0, 1 \rangle$$

(c) **(2 points)** Find a vector-valued function that parametrizes the line that is tangent to the curve $\mathbf{r}(t)$ when $t = \pi$.

The direction of the line is $r'(\pi) = \langle -1,0,1 \rangle$ and an initial point is $r(\pi) = \langle 0,-1,\pi \rangle$. A parametrization is

$$\ell(t) = \langle 0,-1,\pi \rangle + t\langle -1,0,1 \rangle = \langle -t, -1, t+\pi \rangle$$

5. (4 points) Consider the following contour diagram of a function $f(x, y)$:



The black dot in the center is the point $(0, 0)$. All of the lines in the contour diagram intersect at $(0, 0)$. Compute the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ or explain why it does not exist.

The limit does not exist. Each level curve represents a path in the plane along which we can approach $(0,0)$. Since there are curves through $(0,0)$ which represent different elevations, the limit cannot exist. For instance, the limit along the lightest colored curve is approx 0.3 while the limit along the darkest colored curve is approx -0.3.

6. (5 points) The table below gives the number $F(d, v)$ of liters (L) of fuel burned by a Ferrari after traveling a distance of d kilometers (km) at an average velocity of v kilometers per hour (km/h).

d \ v	25	50	75	100	125	150
10	25	50	75	100	125	150
20	50	100	150	200	250	300
30	75	150	225	300	375	450
40	100	200	300	400	500	600

For example, $F(20, 125) = 250L$. Answer the following questions.

(a) **(3 points)** Estimate the partial derivative $F_v(30, 75)$. Make sure to include the units.

$$F_v(30,75) \approx \frac{F(30, 75+25) - F(30, 75-25)}{2\cdot 25}$$

$$= F(30,100) - F(30,50)/50 = \frac{300-150}{50} = 3\ L/(km/h)$$

(b) **(2 point)** Explain the meaning of the partial derivative that you computed in part (c).

Instantaneous rate of change of fuel burned with respect to average velocity when traveling at a fixed distance.
If I drive my Ferrari 30 km, I will use approx. 3 L more fuel for every increase of 1 km/h in average velocity.

7. **(6 points)** Consider the function $f(x,y,z) = x^2 \sin(ze^{y+z})$. Complete the following, each of which is worth **2 points.**

(a) Compute $f_x(x,y,z)$.

$f_x(x,y,z) = 2x \sin(ze^{y+z})$

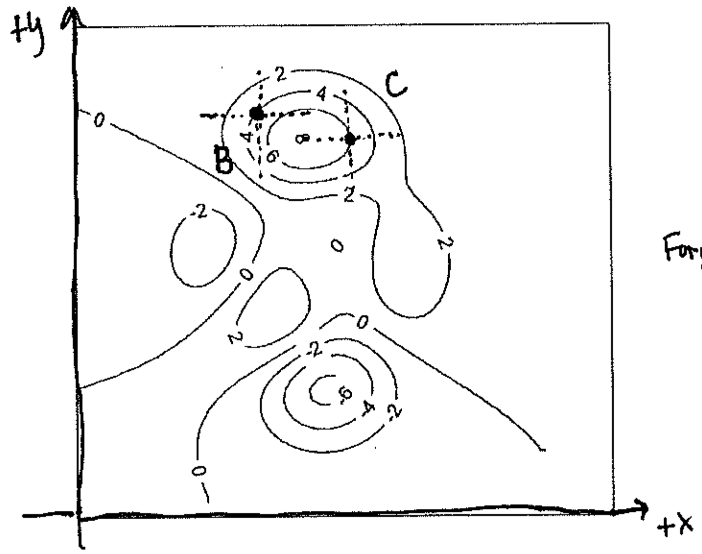
(b) Compute $f_y(x,y,z)$.

$f_y(x,y,z) = x^2 \cos(ze^{y+z}) \cdot z \cdot e^{y+z}$

(c) Compute $f_z(x,y,z)$.

$f_z(x,y,z) = x^2 \cos(ze^{y+z}) \cdot (e^{y+z} + ze^{y+z})$

8. **(6 points)** Consider the following contour diagram of a continuously differentiable function $f(x,y)$:



Forgot to label axes.

Complete the following, each of which is worth **3 points.**

(a) Find a point B where $f_x(B) > 0$ and $f_y(B) < 0$. Mark your point in the diagram with a dot and the letter B. Explain your reasoning.

If I travel from left to right along the dotted line through B, the contours increase in elevation, so I am going uphill and $f_x(B) > 0$.
If I travel from bottom to top along the dotted line through C, the contours decrease in elevation, so I am going downhill and $f_y(B) < 0$.

(b) Find a point C at which $f_x(C) < 0$ and $f_y(C) = 0$. Mark your point in the diagram with a dot and the letter ~~B~~ C. Explain your reasoning.

If I travel left to right through C, I am going downhill and $f_x(C) < 0$.
If I travel bottom to top through C, I go up in elevation and then back down, so I am at a "peak" in the y-direction and $f_y(C) = 0$.