Weekly Assignment 5 Solutions

Jadyn V. Breland
MATH 117: Advanced Linear Algebra

September 3, 2023

See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

For Problems 1 & 2, $L : V \to V$ is a linear operator on a finite dimensional vector space V over a field F which contains the roots of the characteristic polynomial of L. For any positive integer $i \geq 1$ and eigenvalue $\lambda \in F$ for L, we set $V_i := \ker((L - \lambda \operatorname{Id}_V)^i)$ and $\dim V_i = r_i$. The first two problems were lemmas used in the sketch of the proof of the Jordan basis theorem.

1. (a) Suppose that $i \geq 1$ is a positive integer such that $V_i = V_{i+1}$. Prove that $V_{(\lambda)} = V_i$.

(b) In the lecture, we proved that the multiplicity m of λ in $m_L(x)$ is the smallest integer with the property that $V_{(\lambda)} = V_m$. Prove that the dimensions of the V_i strictly increase until we reach V_m:

$$\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{m-1} \subsetneq V_m = V_{m+1} = \cdots$$

This proves a method for computing m - just compute $\dim(V_i)$ until the first time the dimension does not increase. That is, m is the smallest i such that $\dim(V_i) = \dim(V_{i+1})$.

Proof. (a) It suffices to prove that if $V_i = V_{i+1}$ for some $i \geq 1$, then $V_i = V_j$ for all $j \geq i$. We proceed by induction. The base case is just the hypothesis. Now, let $n \geq 1$ and assume that $V_i = V_{i+k}$ for all $0 \leq k < n$. Let $v \in V_{i+n}$. Setting $K = L - \lambda \operatorname{Id}_V$, we have that

$$K^{i+n-1}(K(v)) = K^{i+n}(v) = 0.$$

Hence, $K(v) \in V_{i+n-1}$. By inductive hypothesis, $K(v) \in V_i$. Thus,

$$K^{i+1}(v) = K^i(K(v)) = K^i(0) = 0.$$

This proves that $v \in V_{i+1}$. But again by inductive hypothesis, $V_{i+1} = V_i$. Thus, $v \in V_i$. This proves the claim. It follows immediately that $V_{(\lambda)} = \cup_{j \geq 1} V_j = \cup_{j=1}^{i} V_j = V_i$.

(b) Suppose not. Then there exists $m - 1 \geq j \geq 1$ such that $V_j = V_{j+1}$. By (a), $V_j = V_{(\lambda)}$, which contradicts the minimality of m.

□

2. Set $K = L - \lambda \operatorname{Id}_V$. Assume $i \geq 2$. Suppose that $v_1, \ldots, v_{r_i} \in V_i$ are chosen such that the cosets $v_1 + V_{i-1}, \ldots, v_{r_i} + V_{i-1}$ form a basis for V_i / V_{i-1}.

(a) Prove that $K(v_1), \ldots, K(v_{r_i}) \in V_{i-1}$.

(b) Prove that the cosets $K(v_1) + V_{i-2}, \ldots, K(v_{r_i}) + V_{i-2}$ are independent in V_{i-1}/V_{i-2}.

Proof. (a) I basically proved this already. We have

$$K^{i-1}(K(v_j)) = K^i(v_j) = 0$$

since $v_j \in V_i$. Thus, $K(v_j) \in V_{i-1}$.

(b) Suppose that

$$\sum_{j=1}^{r_i} \alpha_j (K(v_j) + V_{i-2}) = V_{i-2}.$$

Then $\sum_{j=1}^{r_i} \alpha_j K(v_j) \in V_{i-2} \subseteq V_{i-1}$. By hypothesis, $\alpha_j = 0$ for all $1 \leq j \leq r_i$.

$\square$